IMKH California

ALGEBRA 1

Student Edition

UNITS 7-8

Illustrative Mathematics
LEARN MATH FOR LIFE

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Book 3

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UNIT

7

Introduction to Quadratic Functions

Content Connections

In this unit you will look at the patterns that grow quadratically and contrast the growth with linear and exponential growth. You will make connections by:

- **Reasoning with Data** while using tables, graphs, and equations to describe the movement of free-falling objects whose height over time can be modeled with quadratic functions.
- **Discovering Shape and Space** while generalizing ideas of how horizontal and vertical translation and vertical and horizontal stretching of graphs relates to modifying the equation of a function.
- **Taking Wholes Apart, Putting Parts Together** while investigating situations that involve linear, quadratic, and exponential models and use the models to solve problems.
- **Exploring Changing Quantities** while developing an understanding of the meaning of functions and how it represents the data that it is modeling.

Addressing the Standards

As you work your way through **Unit 7 Introduction to Quadratic Functions,** you will use some mathematical practices that you may have started using in kindergarten and have continued strengthening over your school career. These practices describe types of thinking or behaviors that you might use to solve specific math problems.

Mathematical Practices	Where You Use These MPs
MP1 Make sense of problems and persevere in solving them.	Lessons 4, 14, and 17
MP2 Reason abstractly and quantitatively.	Lessons 1, 3, 6, 7, 12, and 14
MP3 Construct viable arguments and critique the reasoning of others.	Lessons 2, 3, 12, 15, 16, and 17
MP4 Model with mathematics.	Lessons 7 and 17
MP5 Use appropriate tools strategically.	Lessons 1, 4, 6, and 12
MP6 Attend to precision.	Lessons 7, 9, 11, 14, and 15
MP7 Look for and make use of structure.	Lessons 1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 15, 16, and 17
MP8 Look for and express regularity in repeated reasoning.	Lessons 2, 5, 6, 7, 9, 10, 11, 12, 13, and 15

The California Common Core State Standards for Mathematics (CA CCSSM) describe the topics you will learn in this unit. Many of these topics build upon knowledge you already have and challenge you to expand upon that knowledge. The table below shows the standards being addressed in this unit.

Big Ideas You Are Studying	California Content Standards	Lessons Where You Learn This
• Model with Functions • Function Investigations • Growth and Decay	**F-BF.1** Write a function that describes a relationship between two quantities. a. Determine an explicit expression, a recursive process, or steps for calculation from a context. b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*	Lesson 6
• Model with Functions • Function Investigations • Growth and Decay	**F-BF.1a** Write a function that describes a relationship between two quantities. a. Determine an explicit expression, a recursive process, or steps for calculation from a context.	Lessons 1, 2, 3, 4, 5, 6, and 7

Big Ideas You Are Studying	California Content Standards	Lessons Where You Learn This
• Growth and Decay	**F.BF.3** Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*	Lessons 12, 13, 15, and 17
• Systems of Equations	**A-SSE.1** Interpret expressions that represent a quantity in terms of its context. a. Interpret parts of an expression, such as terms, factors, and coefficients. b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)n$ as the product of P and a factor not depending on P.*	Lessons 2, 3, and 11
	A-SSE.2 Use the structure of an expression to identify ways to rewrite it.	Lessons 8, 9, and 11
• Features of Functions	**A-SSE.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. a. Factor a quadratic expression to reveal the zeros of the function it defines. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.	Lessons 2, 8, 9, 10, and 13
• Model with Functions • Function Investigations	**F-IF.2** Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.	Lessons 3, 5, and 14
• Features of Functions	**F-IF.3** Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ for $n \geq 1$.*	Lesson 2
• Model with Functions • Function Investigations • Features of Functions • Growth and Decay	**F-IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*	Lesson 14

Big Ideas You Are Studying	California Content Standards	Lessons Where You Learn This
• Model with Functions • Function Investigations • Growth and Decay	**F-IF.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*	Lessons 6 and 7
• Model with Functions • Function Investigations	**F-IF.7** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. a. Graph linear and quadratic functions and show intercepts, maxima, and minima. b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.	Lessons 12 and 13
• Model with Functions • Function Investigations	**F-IF.7a** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. Graph linear and quadratic functions and show intercepts, maxima, and minima.	Lessons 4, 6, 7, 11, 12, 13, 14, 15, 16, and 17
• Model with Functions • Function Investigations	**F-IF.8** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as y = (1.02) t, y = (0.97)t, y = (1.01)12t, and y = (1.2)t/10, and classify them as representing exponential growth or decay.*	Lessons 14 and 16
• Model with Functions • Function Investigations	**F-IF.8b** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as y = (1.02) t, y = (0.97)t, y = (1.01)12t, and y = (1.2)t/10, and classify them as representing exponential growth or decay.*	Lesson 4

Big Ideas You Are Studying	California Content Standards	Lessons Where You Learn This
• Model with Functions • Function Investigations • Growth and Decay	**F-IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*	Lessons 12, 14, and 17
• Model with Functions • Systems of Equations • Function Investigations • Features of Functions • Growth and Decay	**F-LE.2** Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).	Lesson 12
• Growth and Decay	**F-LE.3** Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.	Lesson 4
• Features of Functions • Growth and Decay	**F-LE.6** Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity.	Lessons 5, 6, 10, and 14
• Features of Functions	**A-APR.1** Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.	Lessons 8 and 9

Note: For a full explanation of the California Common Core State Standards for Mathematics (CA CCSSM) refer to the standards section at the end of this book.

Addressing CA CCSSM F-BF.1a; building on F-LE.1; building toward F-LE.2-3; practicing MP2, MP5, MP7

A Different Kind of Change

Let's find the rectangle with the greatest area.

1.1 Notice and Wonder: Three Tables

Look at the patterns in the three tables. What do you notice? What do you wonder?

x	y
1	0
2	5
3	10
4	15
5	20

x	y
1	3
2	6
3	12
4	24
5	48

x	y
1	8
2	11
3	10
4	5
5	-4

1.2 Measuring a Garden

Noah has 50 meters of fencing to completely enclose a rectangular garden in the backyard.

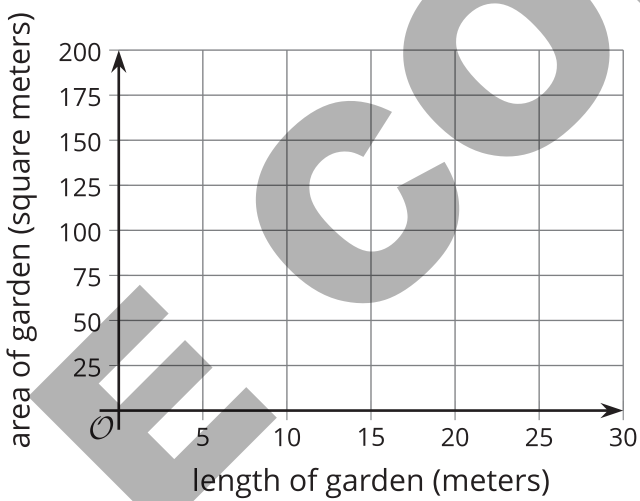
1. Draw some possible diagrams of Noah's garden. Label the length and width of each rectangle.

2. Find the length and width of such a rectangle that would produce the largest possible area. Explain or show why you think that pair of length and width gives the largest possible area.

Sec A

1.3 Plotting the Measurements of the Garden

1. Plot some values for the length and area of Noah's garden, with a perimeter of 50 meters, on the coordinate plane.



2. What do you notice about the plotted points?

3. The points $(3, 66)$ and $(22, 66)$ each represent the length and area of the garden. Plot these 2 points on the coordinate plane, if you haven't already done so. What do these points mean in this situation?

4. Could the point $(1, 25)$ represent the length and area of the garden? Explain how you know.

Are you ready for more?

What does the graph look like if you use width and area instead of length and area? What do you notice about how this graph compares to the graph using length?

Lesson 1 Summary

In this lesson, we looked at the relationship between the side lengths and the area of a rectangle when the perimeter is unchanged.

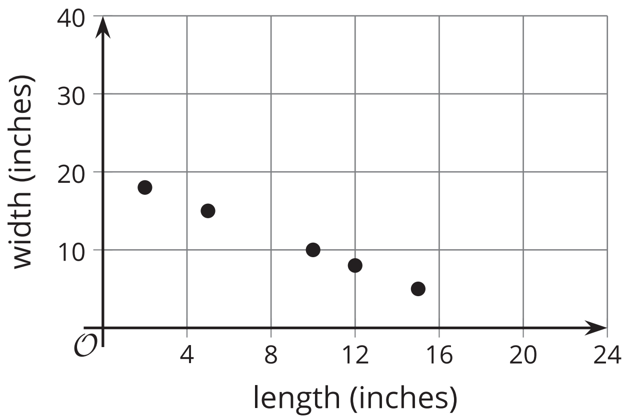
If a rectangle has a perimeter of 40 inches, we can represent some of the possible lengths and widths as shown in the table.

length (inches)	width (inches)
2	18
5	15
10	10
12	8
15	5

We know that twice the length and twice the width must equal 40, which means that the length plus width must equal 20, or $\ell + w = 20$.

To find the width given a length ℓ, we can write: $w = 20 - \ell$.

The relationship between the length and the width is linear. If we plot the points from the table representing the length and the width, they form a line.

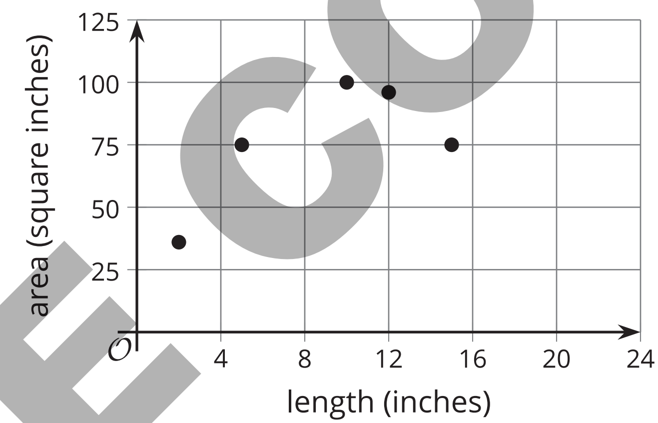


What about the relationship between the side lengths and the area of rectangles with a perimeter of 40 inches?

Here are some possible areas of different rectangles that have a perimeter of 40 inches.

length (inches)	width (inches)	area (square inches)
2	18	36
5	15	75
10	10	100
12	8	96
15	5	75

Here is a graph of the lengths and areas from the table:



Notice that, initially, as the length of the rectangle increases (for example, from 5 to 10 inches), the area also increases (from 75 to 100 square inches). Later, however, as the length increases (for example, from 12 to 15), the area decreases (from 96 to 75).

We have not studied relationships like this yet and will investigate them further in this unit.

Lesson 1 Practice Problems

1 Here are a few pairs of positive numbers whose sum is 50.

a. Find the product of each pair of numbers.

b. Find a pair of numbers that have a sum of 50 and will produce the largest possible product.

c. Explain how you determined which pair of numbers have the largest product.

first number	second number	product
1	49	
2	48	
10	40	

2 Here are some lengths and widths of a rectangle whose perimeter is 20 meters.

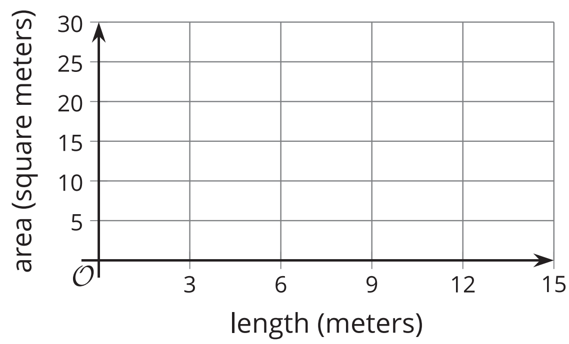
a. Complete the table. What do you notice about the areas?

length (meters)	width (meters)	area (square meters)
1	9	
3	7	
5		
7		
9		

b. Without calculating, predict whether the area of the rectangle will be greater or less than 25 square meters if the length is 5.25 meters.

c. On the coordinate plane, plot the points for length and area from your table.

Do the values change in a linear way? Do they change in an exponential way?



3 The table shows the relationship between x and y, the side lengths of a rectangle, and the area of the rectangle.

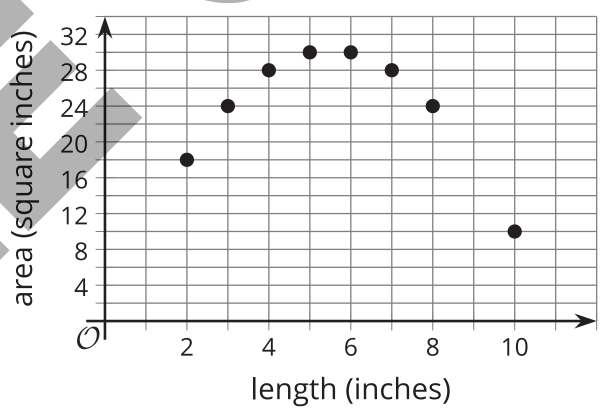
a. Explain why the relationship between the side lengths is linear.

x (cm)	y (cm)	area (sq cm)
2	4	8
4	8	32
6	12	72
8	16	128

Sec A

b. Explain why the relationship between x and the area is neither linear nor exponential.

4 Which statement best describes the relationship between a rectangle's side length and area, as represented by the graph?



A. As the side length increases by 1, the area increases and then decreases by an equal amount.

B. As the side length increases by 1, the area increases and then decreases by an equal factor.

C. As the side length increases by 1, the area does not increase or decrease by an equal amount.

D. As the side length increases by 1, the area does not change.

5 from Unit 6, Lesson 2

Copies of a book are arranged in a stack. Each copy of a book is 2.1 cm thick.

a. Complete the table.

copies of book	stack height in cm
0	
1	
2	
3	
4	

b. What do you notice about the differences in the height of the stack of books when a new copy of the book is added?

c. What do you notice about the factor by which the height of the stack of books changes when a new copy is added?

d. How high is a stack of b books?

6 from Unit 6, Lesson 5

The value of a phone when it was purchased was $500. It loses $\frac{1}{5}$ of its value a year.

a. What is the value of the phone after 1 year? What about after 2 years? 3 years?

b. Tyler says that the value of the phone decreases by $100 each year because $\frac{1}{5}$ of 500 is 100. Do you agree with Tyler? Explain your reasoning.

7 from Unit 3, Lesson 5

Technology required. The data in the table represents the price of one gallon of milk in different years.

Use graphing technology to create a scatter plot of the data.

x, time (years)	price per gallon of milk (dollars)
1930	0.26
1935	0.47
1940	0.52
1940	0.50
1945	0.63
1950	0.83
1955	0.93
1960	1.00
1965	1.05
1970	1.32
1970	1.25
1975	1.57
1985	2.20
1995	2.50
2005	3.20
2018	2.90
2018	3.25

a. Does a linear model seem appropriate for this data? Why or why not?

b. If it seems appropriate for the data, create the line of best fit. Round to two decimal places.

c. What is the slope of the line of best fit, and what does it mean in this context? Is it realistic?

d. What is the y-intercept of the line of best fit, and what does it mean in this context? Is it realistic?

8 from Unit 3, Lesson 7

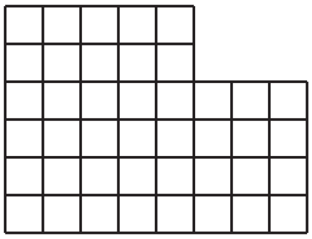
Give a value for r that indicates that a line of best fit has a negative slope and the variables are strongly correlated.

Unit 7, Lesson 2
Addressing CA CCSSM A-SSE.1, A-SSE.3, F-BF.1a, F-IF.3; practicing MP3, MP7, MP8

How Does It Change?

Let's describe some patterns of change.

2.1 Squares in a Figure

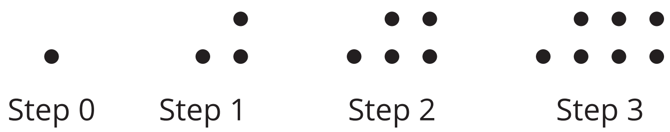


- Expression A: $6 \cdot 8 - 2 \cdot 3$
- Expression B: $4 \cdot 8 + 2 \cdot 5$
- Expression C: $8 + 8 + 8 + 8 + 5 + 5$
- Expression D: $5 \cdot 6 + 3 \cdot 4$

1. Which expression best matches how you would describe the number of small squares in the figure?
2. Select at least one other expression and explain how it represents the number of small squares in the figure.

2.2 Patterns of Dots

Pattern 1



Pattern 2

1. Study the two patterns of dots.
   a. How are the number of dots in each pattern changing?

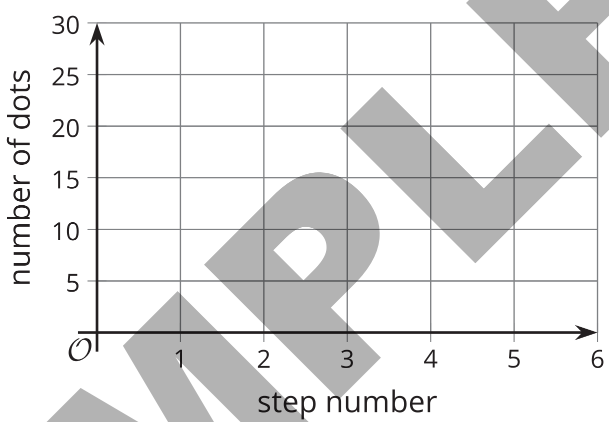
   b. How would you find the number of dots in the 5th step in each pattern?

2. Complete the table with the number of dots in each pattern.

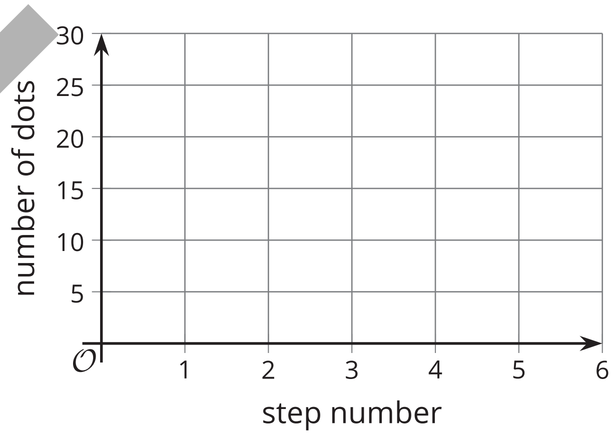
step	number of dots in Pattern 1	number of dots in Pattern 2
0		
1		
2		
3		
4		
5		
10		
n		

3. Plot the number of dots at each step number.

Pattern 1

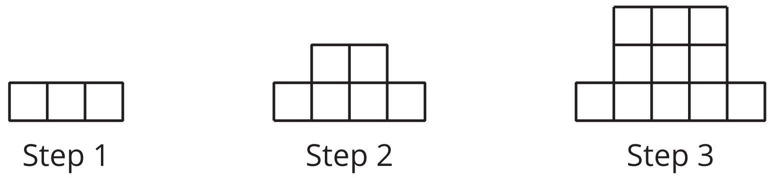


Pattern 2



4. Explain why the graphs of the two patterns look the way they do.

2.3 Expressing a Growth Pattern



Here is a pattern of squares.

1. Is the number of small squares growing linearly? Explain how you know.

2. Complete the table.

step	number of small squares
1	
2	
3	
4	
5	
10	
12	
n	

3. Is the number of small squares growing exponentially? Explain how you know.

Are you ready for more?

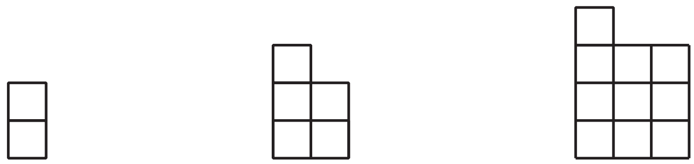
Han wrote $n(n+2) - 2(n-1)$ for the number of small squares in the design at Step n.

1. Explain why Han is correct.

2. Label the picture in a way that shows how Han saw the pattern when writing his expression.

Lesson 2 Summary

In this lesson, we saw some quantities that change in a particular way, but the change is neither linear nor exponential. Here is a pattern of shapes, followed by a table showing the relationship between the step number and the number of small squares.



Step 1 Step 2 Step 3

step	total number of small squares
1	2
2	5
3	10
n	$n^2 + 1$

The number of small squares increases by 3, and then by 5, so we know that the growth is not linear. It is also not exponential because it is not changing by the same factor each time. From Step 1 to Step 2, the number of small squares grows by a factor of $\frac{5}{2}$, while from Step 2 to Step 3, it grows by a factor of 2.

From the diagram, we can see that in Step 2, there is a 2-by-2 square plus 1 small square added on top. Likewise, in Step 3, there is a 3-by-3 square with 1 small square added. We can reason that the nth step is an n-by-n arrangement of small squares with an additional small square on top, giving the expression $n^2 + 1$ for the number of small squares.

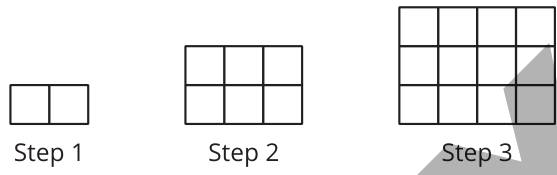
The relationship between the step number and the number of small squares is a quadratic relationship, because it is given by the expression $n^2 + 1$, which is an example of a **quadratic expression**. We will investigate quadratic expressions in depth in future lessons.

Glossary

- quadratic expression

Lesson 2 Practice Problems

1 How many small squares are in Step 10?



A. 10

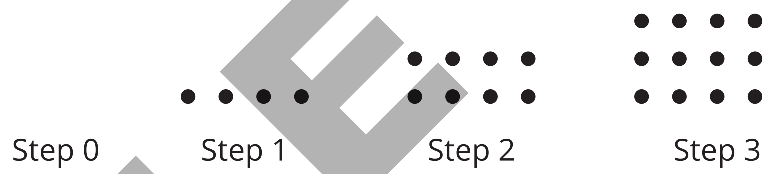
B. 11

C. 90

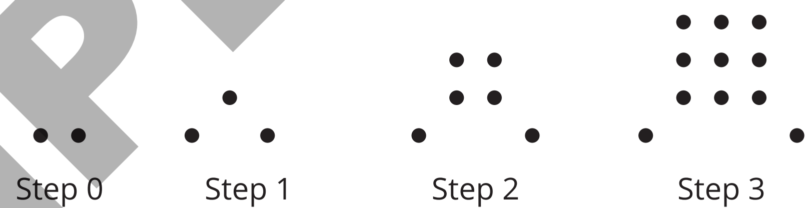
D. 110

2 Here are two patterns of dots.

Pattern A



Pattern B



a. How many dots will there be in Step 4 of each pattern?

b. Which pattern shows a quadratic relationship between the step number and the number of dots? Explain how you know.

Sec A

3 Here are descriptions for how two dot patterns are growing.

- Pattern A: Step 2 has 10 dots. It grows by 3 dots at each additional step.
- Pattern B: The total number of dots can be expressed by $2n^2 + 1$, where n is the step number.

For each pattern, draw a diagram of Step 0 to Step 3.

4 Each expression represents the total number of dots in a pattern, where n represents the step number.

Select **all** the expressions that represent a quadratic relationship between the step number and the total number of dots. (If you get stuck, consider sketching the first few steps of each pattern as described by the expression.)

A. n^2

B. $2n$

C. $n \cdot n$

D. $n + n$

E. $n + 2$

F. $n \div 2$

5 from Unit 5, Lesson 3

The function, C, gives the percentage of homes using only cell phone service x years after 2004. Explain the meaning of each statement.

a. $C(10) = 35$

b. $C(x) = 10$

c. How is $C(10)$ different from $C(x) = 10$?

6 from Unit 7, Lesson 1

Here are some lengths, widths, and areas of a garden whose perimeter is 40 feet.

a. Complete the table with the missing measurements.

b. What lengths and widths do you think will produce the largest possible area? Explain how you know.

length (ft)	width (ft)	area (sq ft)
4	16	64
8	12	
10		
12		96
14		
16		64

Sec A

7 from Unit 6, Lesson 3

A bacteria population is 10,000 when it is first measured and then doubles each day.

a. Use this information to complete the table.

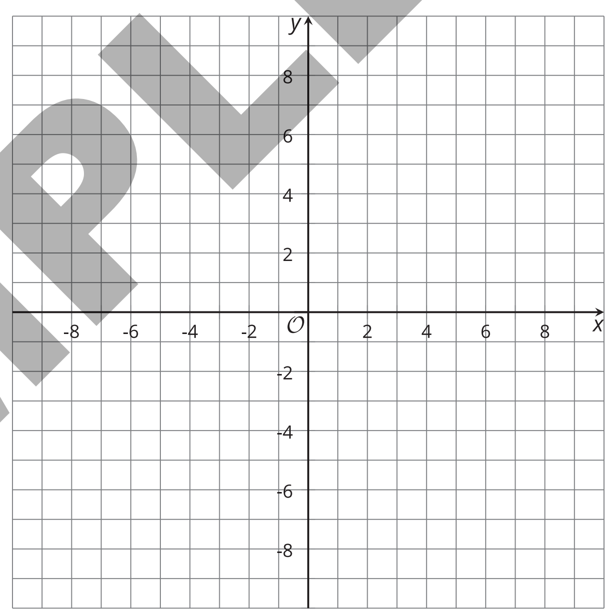
b. Which is the first day, after the population was originally measured, that the bacteria population is more than 1,000,000?

c. Write an equation relating p, the bacteria population, to d, the number of days since it was first measured.

d, time (days)	p, population (thousands)
0	
1	
2	
5	
10	
d	

8 from Unit 4, Lesson 4

Graph the solutions to the inequality $7x - 3y \geq 21$.



Addressing CA CCSSM A-SSE.1, F-BF.1a, F-IF.2; building on A-CED.2, A-SSE.1, F-LE.1; practicing MP2, MP3, MP7

Building Quadratic Functions from Geometric Patterns

Let's describe some other geometric patterns.

3.1 Quadratic Expressions and Area

Figure A is a large square. Figure B is a large square with a smaller square removed. Figure C is composed of two large squares with one smaller square added.

Figure A

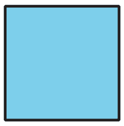


Figure B

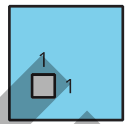
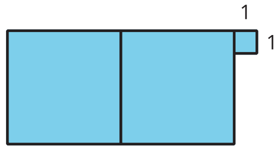


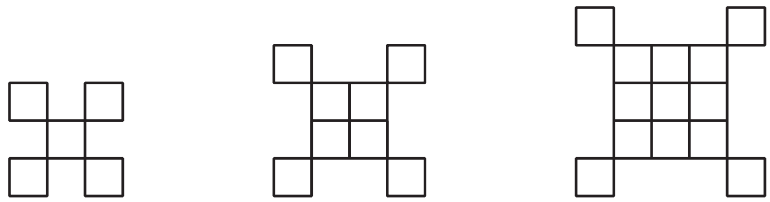
Figure C



Write an expression to represent the area of each shaded figure when the side length of the large square is as shown in the first column.

side length of large square	area of A	area of B	area of C
4			
x			
$4x$			
$(x + 3)$			

3.2 Expanding Squares



Step 1 Step 2 Step 3

Sec B

1. If the pattern continues, what will we see in Step 5 and Step 18?
   a. Sketch or describe the figure in each of these steps.
   b. How many small squares are in each of these steps? Explain how you know.

2. Write an equation to represent the relationship between the step number, n, and the number of squares, y. Be prepared to explain how each part of your equation relates to the pattern. (If you get stuck, try making a table.)

3. Sketch the first 3 steps of a pattern that can be represented by the equation $y = n^2 - 1$.

Are you ready for more?

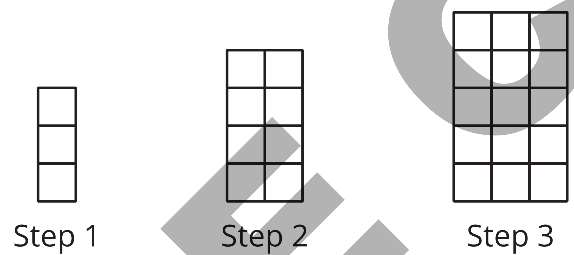
1. For the original step pattern in the statement, write an equation to represent the relationship between the step number, n, and the perimeter, P.

2. For the step pattern you created in part 3 of the activity, write an equation to represent the relationship between the step number, n, and the perimeter, P.

3. Are these linear functions?

Sec B

3.3 Growing Steps

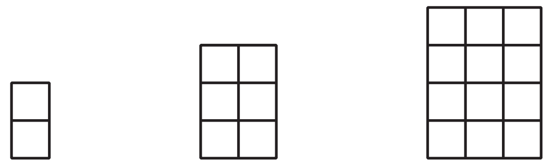


1. Sketch the next step in the pattern.
2. Kiran says that the pattern is growing linearly because as the step number goes up by 1, the number of rows and the number of columns also increase by 1. Do you agree? Explain your reasoning.

3. To represent the number of squares after n steps, Diego and Jada wrote different equations. Diego wrote the equation $f(n) = n(n+2)$. Jada wrote the equation $f(n) = n^2 + 2n$. Are either Diego or Jada correct? Explain your reasoning.

Lesson 3 Summary

Sometimes a quadratic relationship can be expressed without writing a squared term that appears as a variable raised to the second power (like n^2 or x^2). Let's take this pattern of squares, for example.

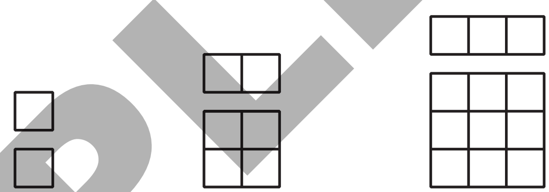


Step 1 Step 2 Step 3

From the first 3 steps, we can see that both the length and the width of the rectangle increase by 1 at each step. Step 1 is a 1-by-2 rectangle, Step 2 is a 2-by-3 rectangle, and Step 3 is a 3-by-4 rectangle. This suggests that Step n is a rectangle with side lengths of n and $n + 1$, so the number of squares at Step n is $n(n + 1)$.

This expression may not look like quadratic expressions with a squared term, which we saw in earlier lessons, but if we apply the distributive property, we can see that $n(n + 1)$ is equivalent to $n^2 + n$.

We can also visually show that these expressions are the equivalent by breaking each rectangle into an n-by-n square (the n^2 in the expression) and an n-by-1 rectangle (the n in the expression).



Step 1 Step 2 Step 3

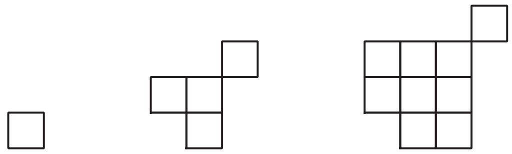
The relationship between the step number and the number of squares can be described by a **quadratic function**, f, whose input is n and whose output is the number of squares at Step n. We can define f with $f(n) = n(n + 1)$ or with $f(n) = n^2 + n$.

Glossary

- quadratic function

Lesson 3 Practice Problems

1 a. Sketch or describe the figure in Step 4 and Step 15.



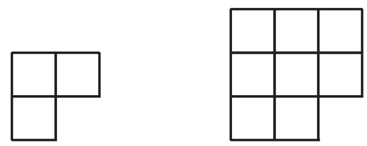
Step 1 Step 2 Step 3

b. How many small squares will there be in each of these steps?

c. Write an equation to represent the relationship between the step number, n, and the number of small squares, y, in each step.

d. Explain how your equation relates to the pattern.

2 Which expression represents the relationship between the step number, n, and the total number of small squares in the pattern?



Step 1 Step 2 Step 3

A. $n^2 + 1$

B. $n^2 - 1$

C. $n^2 - n$

D. $n^2 + n$

3 Each figure is composed of one or more large squares and some small squares. The side length of the large square is x. Write an expression for the area of the shaded part of each figure.

Figure A

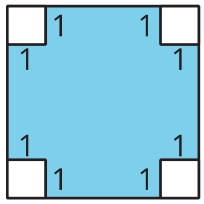
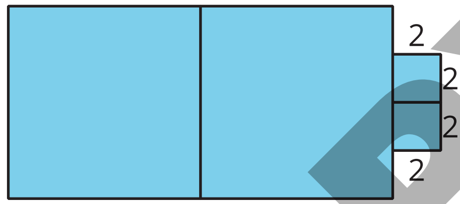


Figure B

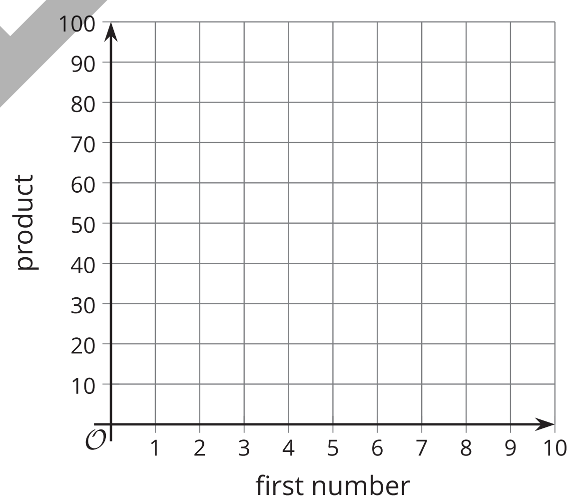


4 from Unit 7, Lesson 1

Here are a few pairs of positive numbers whose difference is 5.

a. Find the product of each pair of numbers. Then, plot some points to show the relationship between the first number and the product.

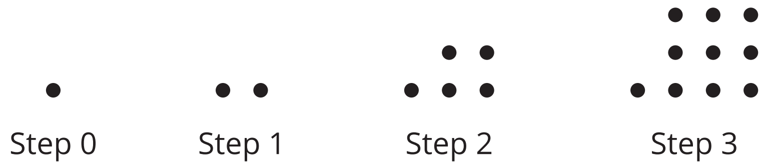
first number	second number	product
1	6	
2	7	
3	8	
5	10	
7	12	



b. Is the relationship between the first number and the product exponential? Explain how you know.

5 from Unit 7, Lesson 2

Here is a pattern of dots.



step	total number of dots
0	
1	
2	
3	

a. Complete the table.

b. How many dots will there be in Step 10?

c. How many dots will there be in Step n?

6 from Unit 4, Lesson 8

Mai has a jar of quarters and dimes. She takes at least 10 coins out of the jar and finds that she has taken out less than $2.00 total.

a. Write a system of inequalities that represents the number of quarters, q, and the number of dimes, d, that Mai could have taken out of the jar.

Sec B

b. Is it possible that Mai has each of these combinations of coins? If so, explain or show how you know. If not, state which constraint—the amount of money or the number of coins—it does not meet.

i. 3 quarters and 12 dimes

ii. 4 quarters and 10 dimes

iii. 2 quarters and 5 dimes

7 from Unit 5, Lesson 10

A stadium can seat 63,026 people. For each game, the amount of money that the organization brings in through ticket sales is a function of the number of people, n, in attendance.

If each ticket costs $30.00, find the domain and range of this function.

Unit 7, Lesson 4
Addressing CA CCSSM F-BF.1a, F.IF.7a, F-IF.8b, F-LE.3; building on 6.EE.1; building toward F-LE.3; practicing MP1, MP5, MP7

Comparing Quadratic and Exponential Functions

Let’s compare quadratic and exponential changes and see which one grows faster.

4.1 From Least to Greatest

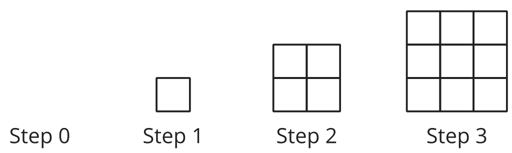
List these quantities in order, from least to greatest, without evaluating each expression. Be prepared to explain your reasoning.

2^{10} 10^2 2^9 9^2

4.2 Which One Grows Faster?

- In Pattern A, the length and width of the rectangle grow by one small square from each step to the next.
- In Pattern B, the number of small squares doubles from each step to the next.
- In each pattern, the number of small squares is a function of the step number, n.

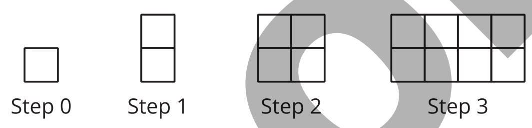
Pattern A



1. Write an equation to represent the number of small squares at Step n in Pattern A.
2. Is the function linear, quadratic, or exponential?
3. Complete the table:

n, step number	$f(n)$, number of small squares
0	
1	
2	
3	
4	
5	
6	
7	
8	

Pattern B



1. Write an equation to represent the number of small squares at Step n in Pattern B.
2. Is the function linear, quadratic, or exponential?
3. Complete the table:

n, step number	$g(n)$, number of small squares
0	
1	
2	
3	
4	
5	
6	
7	
8	

How would the two patterns compare if they continue to grow? Make 1–2 observations.

4.3 Comparing Two More Functions

Here are two functions: $p(x) = 6x^2$ and $q(x) = 3^x$.

Investigate the output of p and q for different values of x. For large enough values of x, one function will have a greater value than the other. Which function will have a greater value as x increases?

Support your answer with tables, graphs, or other representations.

Sec B

Are you ready for more?

1. Jada looks at the function $g(x) = 1.1^x$ and says that this exponential function grows more slowly than the quadratic function $f(x) = x^2$. Do you agree with Jada? Explain your reasoning.

2. Let $f(x) = x^2$. Could you have an exponential function $g(x) = b^x$ so that $g(x) < f(x)$ for all values of x?

Lesson 4 Summary

The graphs of quadratic functions and the graphs of exponential functions with a base that is greater than 1 both curve upward. To compare the two, let's look at the quadratic expression $3n^2$ and the exponential expression 2^n.

A table of values shows that $3n^2$ is initially greater than 2^n, but 2^n eventually becomes greater.

n	$3n^2$	2^n
1	3	2
2	12	4
3	27	8
4	48	16
5	75	32
6	108	64
7	147	128
8	192	256

Here's why exponential growth eventually overtakes quadratic growth.

- When n increases by 1, the exponential expression 2^n always increases by a factor of 2.
- The quadratic expression $3n^2$ increases by different factors, depending on n, but these factors get smaller. For example, when n increases from 2 to 3, the factor is $\frac{27}{12}$ or 2.25. When n increases from 6 to 7, the factor is $\frac{147}{108}$ or about 1.36. As n increases to larger and larger values, $3n^2$ grows by a factor that gets closer and closer to 1.

In general, quadratic functions change with a factor that gets closer and closer to 1 as the input to the function gets larger. Exponential functions always grow with the same factor, so if that growth factor is greater than 1, then the exponential function will eventually grow faster than any quadratic function will.

Lesson 4 Practice Problems

1 The table shows values of the expressions $10x^2$ and 2^x.

a. Describe how the values of each expression change as x increases.

b. Complete the table.

c. Make an observation about how the values of the two expressions change as x becomes greater and greater.

x	$10x^2$	2^x
1	10	2
2	40	4
3	90	8
4	160	16
8		
10		
12		

Sec B

2 Function f is defined by $f(x) = 1.5^x$. Function g is defined by $g(x) = 500x^2 + 345x$.

a. Which function is quadratic? Which one is exponential?

b. The values of which function will eventually be greater for larger and larger values of x?

3 Create a table of values to show that the exponential expression $3(2)^x$ eventually overtakes the quadratic expression $3x^2 + 2x$.

4 The table shows the values of 4^x and $100x^2$ for some values of x.

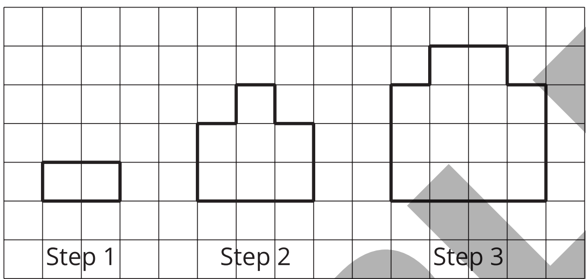
Use the patterns in the table to explain why eventually the values of the exponential expression 4^x will overtake the values of the quadratic expression $100x^2$.

x	4^x	$100x^2$
1	4	100
2	16	400
3	64	900
4	256	1600
5	1024	2500

Sec B

5 from Unit 7, Lesson 2

Here is a pattern of shapes. The area of each small square is 1 sq cm.



a. What is the area of the shape in Step 10?

b. What is the area of the shape in Step n?

c. Explain how you see the pattern growing.

6 from Unit 6, Lesson 5

A bicycle costs \$240, and it loses $\frac{3}{5}$ of its value each year.

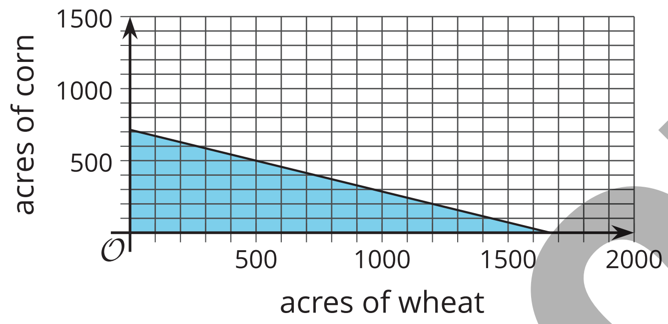
a. Write expressions for the value of the bicycle, in dollars, after 1, 2, and 3 years.

b. When will the bike be worth less than \$1?

c. Will the value of the bike ever be 0? Explain your reasoning.

7 from Unit 4, Lesson 8

A farmer plants wheat and corn. It costs about \$150 per acre to plant wheat and about \$350 per acre to plant corn. The farmer plans to spend no more than \$250,000 planting wheat and corn. The total area of corn and wheat that the farmer plans to plant is less than 1200 acres.



This graph represents the inequality, $150w + 350c \leq 250{,}000$, which describes the cost constraint in this situation. Let w represent the number of acres of wheat and c represent the number of acres of corn.

a. The inequality, $w + c < 1{,}200$ represents the total area constraint in this situation. On the same coordinate plane, graph the solution to this inequality.

b. Use the graphs to find at least two possible combinations of the number of acres of wheat and the number of acres of corn that the farmer could plant.

c. The combination of 400 acres of wheat and 700 acres of corn meets one constraint in the situation but not the other constraint. Which constraint does this meet? Explain your reasoning.

Sec B

Addressing CA CCSSM F-BF.1a, F-IF.2, F-LE.6; building toward F-BF.1; practicing MP7 and MP8

Building Quadratic Functions to Describe Situations (Part 1)

Let's measure falling objects.

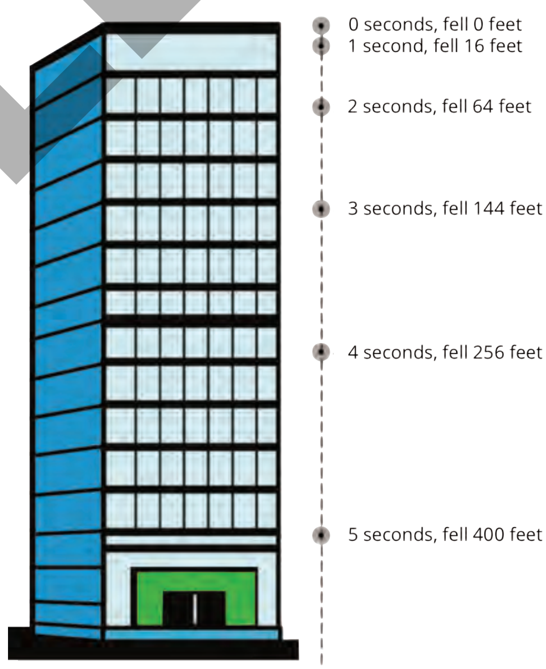
5.1 Notice and Wonder: An Interesting Numerical Pattern

Study the table. What do you notice? What do you wonder?

x	0	1	2	3	4	5
y	0	16	64	144	256	400

5.2 Falling from the Sky

A rock is dropped from the top floor of a 500-foot tall building. A camera captures the distance the rock traveled, in feet, after each second.



1. How far will the rock have fallen after 6 seconds? Show your reasoning.

2. Jada noticed that the distances fallen are all multiples of 16.
She wrote down:

$$\begin{aligned} 16 &= 16 \cdot 1 \\ 64 &= 16 \cdot 4 \\ 144 &= 16 \cdot 9 \\ 256 &= 16 \cdot 16 \\ 400 &= 16 \cdot 25 \end{aligned}$$

Then, she noticed that 1, 4, 9, 16, and 25 are $1^2, 2^2, 3^2, 4^2$ and 5^2.

a. Use Jada's observations to predict the distance that a rock would fall, after 7 seconds, if it were dropped from an even taller building. (Assume that the building is tall enough that an object dropped from the top of it will continue falling for at least 7 seconds without hitting the ground.) Show your reasoning.

b. Write an equation for the function, with d representing the distance dropped after t seconds.

Sec B

5.3 Galileo and Gravity

Galileo Galilei, an Italian scientist, and other medieval scholars studied the motion of free-falling objects. The law they discovered can be expressed by the equation $d = 16 \cdot t^2$, which gives the distance fallen in feet, d, as a function of time, t, in seconds.

An object is dropped from a height of 576 feet.

1. How far does it fall in 0.5 seconds?
2. To find out where the object is after the first few seconds after it was dropped, Elena and Diego created different tables.

Elena's table:

time (seconds)	distance fallen (feet)
0	0
1	16
2	64
3	
4	
t	

Diego's table:

time (seconds)	distance from the ground (feet)
0	576
1	560
2	512
3	
4	
t	

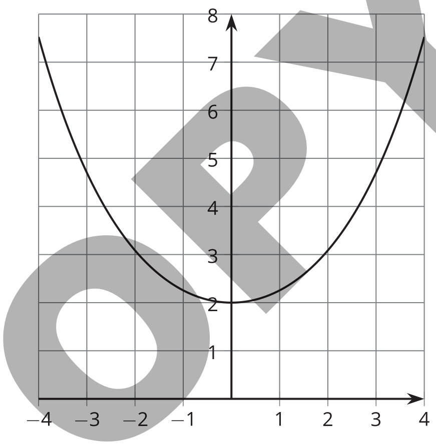
a. How are the two tables alike? How are they different?

b. Complete Elena's and Diego's tables. Be prepared to explain your reasoning.

Are you ready for more?

Galileo correctly observed that gravity causes objects to fall in a way in which the distance fallen is a quadratic function of the time elapsed. He got a little carried away, however, and assumed that a hanging rope or chain could also be modeled by a quadratic function.

Here is a graph of such a shape (called a "catenary"), along with a table of approximate values.



x	-4	-3	-2	-1	0	1	2	3	4
y	7.52	4.70	3.09	2.26	2	2.26	3.09	4.70	7.52

Show that an equation of the form $y = ax^2 + b$ cannot model this data well.

Lesson 5 Summary

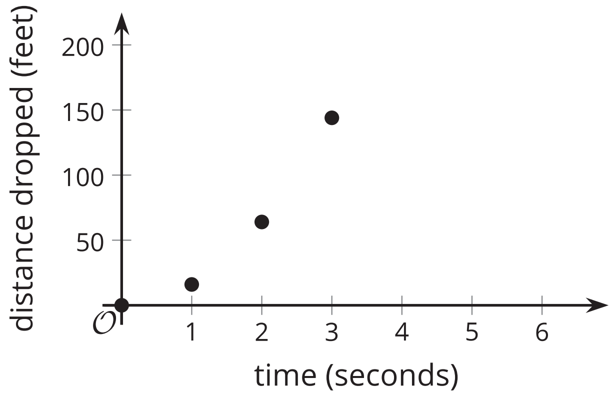
The distance traveled by a falling object in a given amount of time is an example of a quadratic function. Galileo is said to have dropped balls of different mass from the Leaning Tower of Pisa, which is about 190 feet tall, to show that they travel the same distance in the same time. In fact the equation $d = 16t^2$ models the distance d, in feet, that a metal ball falls after t seconds, no matter what its mass.

Because $16 \cdot 4^2 = 256$, and the tower is only 190 feet tall, a metal ball hits the ground before 4 seconds.

Here is a table showing how far a metal ball has fallen over the first few seconds.

time (seconds)	distance fallen (feet)
0	0
1	16
2	64
3	144

Here are the time and distance pairs plotted on a coordinate plane:



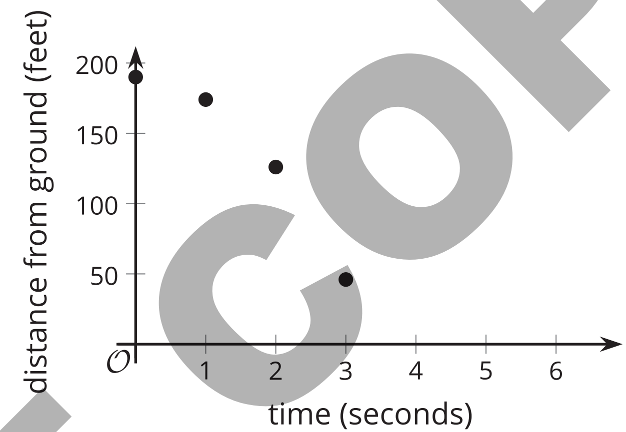
Notice that the distance fallen is increasing each second. The average rate of change is increasing each second, which means that the metal ball is speeding up over time. This comes from the influence of gravity, which is represented by the quadratic expression $16t^2$. It is the exponent 2 in that expression that makes it increase by larger and larger amounts.

Another way to study the change in the position of the metal ball is to look at its distance from the ground as a function of time.

Here is a table showing the distance from the ground in feet at 0, 1, 2, and 3 seconds.

time (seconds)	distance from the ground (feet)
0	190
1	174
2	126
3	46

Here are those time and distance pairs plotted on a coordinate plane:

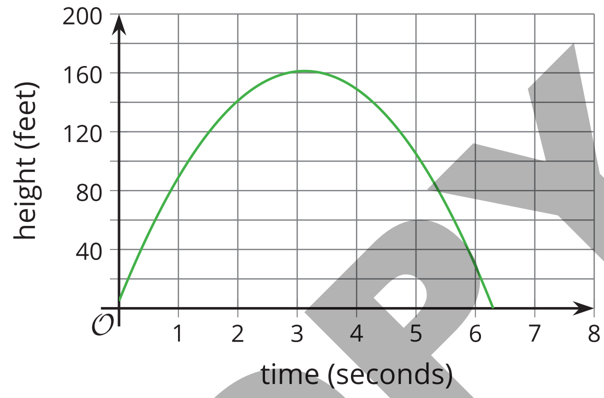


The expression that defines the distance from the ground as a function of time is $190 - 16t^2$. It tells us that the metal ball's distance from the ground is 190 feet before it is dropped and has decreased by $16t^2$ when t seconds have passed.

Lesson 5 Practice Problems

1 A rocket is launched in the air, and its height, in feet, is modeled by a function, h. Here is a graph representing h.

Select **all** true statements about the situation.



A. The rocket is launched from a height of less than 20 feet above the ground.

B. The rocket is launched from about 20 feet above the ground.

C. The rocket reaches its maximum height after about 3 seconds.

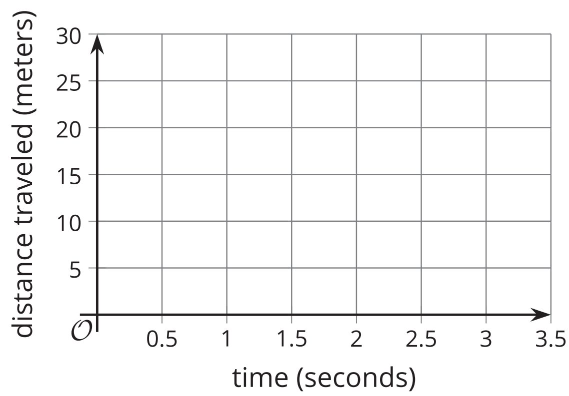
D. The rocket reaches its maximum height after about 160 seconds.

E. The maximum height of the rocket is about 160 feet.

2 A baseball travels d meters t seconds after being dropped from the top of a building. The distance traveled by the baseball can be modeled by the equation $d = 5t^2$.

a. Complete the table and plot the data on the coordinate plane.

t (seconds)	d (meters)
0	
0.5	
1	
1.5	
2	



b. Is the baseball traveling at a constant speed? Explain how you know.

3 A rock is dropped from a bridge over a river. Which table could represent the distance in feet fallen as a function of time in seconds?

Table A

time (seconds)	distance fallen (feet)
0	0
1	48
2	96
3	144

Table B

time (seconds)	distance fallen (feet)
0	0
1	16
2	64
3	144

Table C

time (seconds)	distance fallen (feet)
0	180
1	132
2	84
3	36

Table D

time (seconds)	distance fallen (feet)
0	180
1	164
2	116
3	36

A. Table A

B. Table B

C. Table C

D. Table D

4 from Unit 7, Lesson 4

Determine whether $5n^2$ or 3^n will have the greater value when:

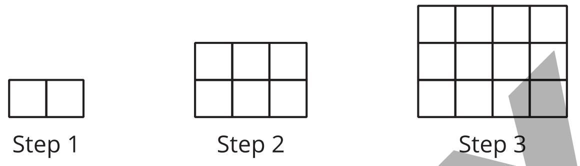
a. $n = 1$

b. $n = 3$

c. $n = 5$

5 from Unit 7, Lesson 3

Select **all** of the expressions that give the number of small squares in Step n.



A. $2n$

B. n^2

C. $n+1$

D. n^2+1

E. $n(n+1)$

F. n^2+n

G. $n+n+1$

6 A small ball is dropped from a 100-foot tall building. Which equation could represent the ball's height, h, in feet, relative to the ground, as a function of time, t, in seconds?

A. $h=100-16t$

B. $h=100-16t^2$

C. $h=100-16^t$

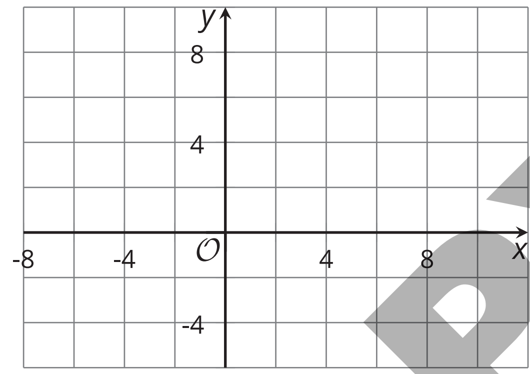
D. $h=100-\frac{16}{t}$

Sec B

7 from Unit 5, Lesson 12

Use the rule for function f to draw its graph.

$$f(x) = \begin{cases} 2, & -5 \le x < -2 \\ 6, & -2 \le x < 4 \\ x, & 4 \le x < 8 \end{cases}$$



8 from Unit 7, Lesson 4

Diego claimed that $10 + x^2$ is always greater than 2^x and used this table as evidence.

Do you agree with Diego? Explain your reasoning.

x	$10 + x^2$	2^x
1	11	2
2	14	4
3	19	8
4	26	16

Addressing CA CCSSM F-BF.1, F-BF.1a, F-IF.5, F.IF.7a, F-LE.6; practicing MP2, MP5, MP8

Building Quadratic Functions to Describe Situations (Part 2)

Let's look at the objects being launched in the air.

6.1 Sky Bound (Part 1)

A person with a t-shirt launcher is standing in the center of the field at a soccer stadium. He is holding the launcher so that the mouth of the launcher, where the t-shirts exit the launcher, is 5 feet above the ground. The launcher sends a t-shirt straight up with a velocity of 90 feet per second.

Imagine that there is no gravity and that the t-shirt continues to travel upward with the same velocity.

1. Complete the table with the heights of the t-shirt at different times.

seconds	0	1	2	3	4	5	t
distance above ground (feet)	5						

2. Write an equation to model the distance in feet, d, of the t-shirt t seconds after it was launched if there was no gravity.

Sec B

6.2 Sky Bound (Part 2)

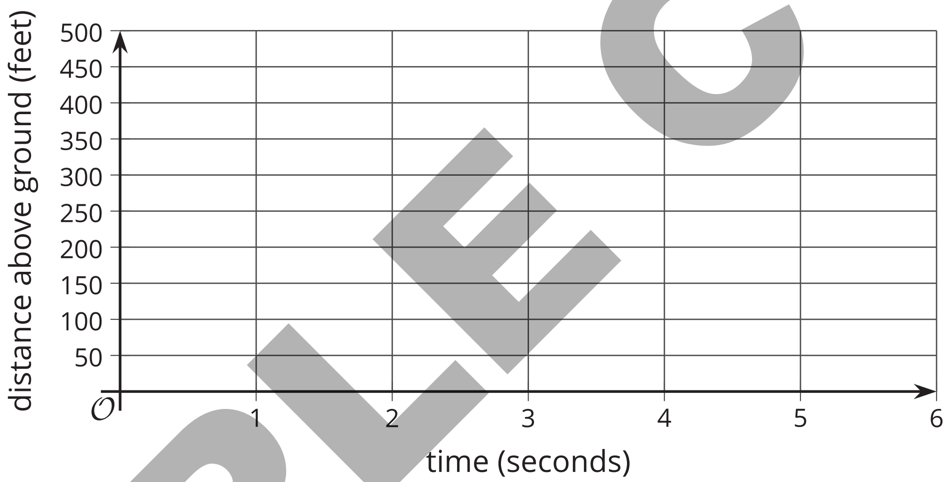
Earlier, you completed a table that represents the height of a t-shirt, in feet, as a function of time, in seconds, if there were no gravity.

1. This table shows the actual heights of the t-shirt at different times.

seconds	0	1	2	3	4	5
distance above ground (feet)	5	79	121	131	109	55

Compare the values in this table with those in the table that you completed earlier. Make at least 2 observations.

2. a. Plot the two sets of data that you have on the same coordinate plane.



b. How are the two graphs alike? How are they different?

3. Write an equation to model the actual distance, d, in feet, of the t-shirt t seconds after it was launched. If you get stuck, consider the differences in distances and the effects of gravity from an earlier lesson.

6.3 Graphing a Cannonball

The function defined by $d = 50 + 312t - 16t^2$ gives the height in feet of a cannonball t seconds after the ball leaves the cannon.

1. What do the terms 50, $312t$, and -$16t^2$ tell us about the cannonball?

2. Use graphing technology to graph the function. Adjust the graphing window to show: $0 < t < 25$ and $0 < y < 2{,}000$.

3. Observe the graph and:

   a. Describe the shape of the graph. What does it tell us about the movement of the cannonball?

   b. Estimate the maximum height that the ball reaches. When does this happen?

   c. Estimate when the ball hits the ground.

4. What domain is appropriate for this function, based on the situation? Explain your reasoning.

Are you ready for more?

The same cannonball is fired upward at 800 feet per second. Does it reach a mile (5,280 feet) in height? Explain your reasoning.

Sec B

Lesson 6 Summary

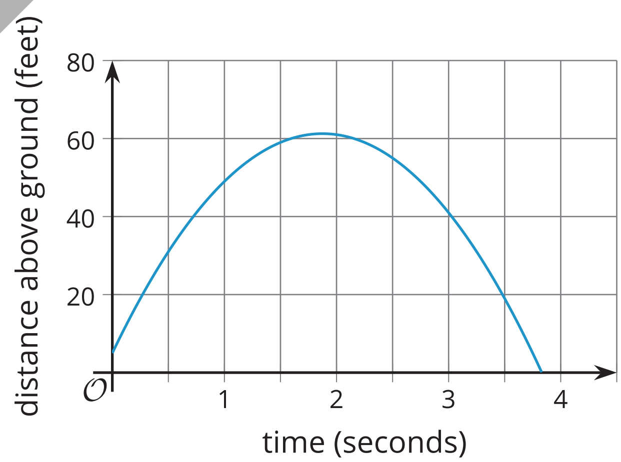
In this lesson, we looked at the height of objects that are launched upward and then come back down because of gravity.

An object is thrown upward from a height of 5 feet with a velocity of 60 feet per second. Its height, $h(t)$, in feet, after t seconds is modeled by the function $h(t) = 5 + 60t - 16t^2$.

- The linear expression $5 + 60t$ represents the height that the object would have at time t if there were no gravity. The object would keep going up at the same speed at which it was thrown. The graph would be a line with a slope of 60, which relates to the constant speed of 60 feet per second.
- The expression $-16t^2$ represents the effect of gravity, which eventually causes the object to slow down, stop, and start falling back again.

Notice the graph intersects the vertical axis at 5, which means that the object was thrown into the air from 5 feet off the ground. The graph indicates that the object reaches its peak height of about 60 feet after a little less than 2 seconds. That peak is the point on the graph where the function reaches a maximum value. At that point, the curve changes direction, and the output of the function changes from increasing to decreasing. We call that point the **vertex** of the graph.

Here is the graph of h.



The graph representing any quadratic function is a special kind of "U" shape called a *parabola*. You will learn more about the geometry of parabolas in a future course. Every parabola has a vertex, because there is a point at which it changes direction—from increasing to decreasing, or the other way around.

The object hits the ground a little before 4 seconds. That time corresponds to the horizontal intercept of the graph. An input value that produces an output of 0 is called a **zero** of the function. A zero of function h is approximately 3.8, because $h(3.8) \approx 0$.

In this situation, input values less than 0 seconds or more than about 3.8 seconds would not be meaningful, so an appropriate domain for this function would include all values of t between 0 and about 3.8.

Sec B

Glossary

- vertex (of a graph)
- zero (of a function)

Lesson 6 Practice Problems

1 The height of a diver above the water, is given by $h(t) = -5t^2 + 5t + 3$, where t is time measured in seconds and $h(t)$ is measured in meters. Select **all** statements that are true about the situation.

A. The diver begins 5 meters above the water.

B. The diver begins 3 meters above the water.

C. The function has 1 zero that makes sense in this situation.

D. The function has 2 zeros that make sense in this situation.

E. The graph that represents h starts at the origin and curves upward.

F. The diver begins at the same height as the water level.

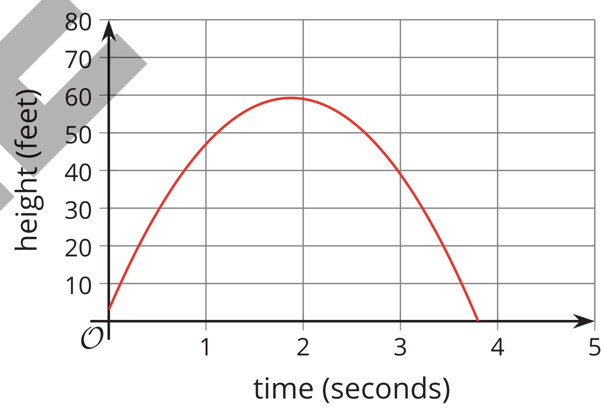
Sec B

2 The height of a baseball, in feet, is modeled by a function, h, given by the equation $h(t) = 3 + 60t - 16t^2$. The graph of the function is shown.

a. About when does the baseball reach its maximum height?

b. About how high is the maximum height of the baseball?

c. About when does the ball hit the ground?



3 *Technology required.* Two rocks are launched straight up into the air. The height of Rock A is given by function f, where $f(t) = 4 + 30t - 16t^2$. The height of Rock B is given by function g, where $g(t) = 5 + 20t - 16t^2$. In both functions, t is time measured in seconds and height is measured in feet.

Use graphing technology to graph both equations. Determine which rock hits the ground first and explain how you know.

4 Each expression represents an object's distance from the ground, in meters, as a function of time, t, in seconds.

Object A: $-5t^2 + 25t + 50$

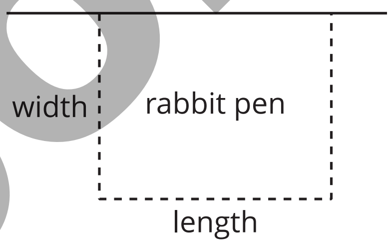
Object B: $-5t^2 + 50t + 25$

a. Which object was launched with the greatest vertical speed?

b. Which object was launched from the greatest height?

5 from Unit 7, Lesson 1

Tyler is building a pen on the side of the garage for his rabbit. He needs to fence in three sides and wants to use 24 ft of fencing.



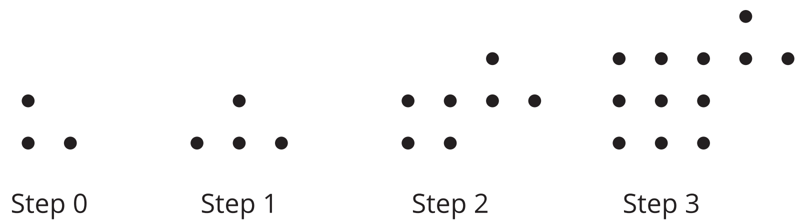
a. The table shows some possible lengths and widths. Complete each area.

b. Which length and width combination should Tyler choose to give his rabbit the most room?

length (ft)	width (ft)	area (sq ft)
8	8	
10	7	
12	6	
14	5	
16	4	

6 from Unit 7, Lesson 2

Here is a pattern of dots.



step	total number of dots
0	
1	
2	
3	

a. Complete the table.

b. How many dots will there be in Step 10?

c. How many dots will there be in Step n?

7 from Unit 7, Lesson 4

A function, f, is defined by $f(x) = 2^x$, and a function, g, is defined by $g(x) = x^2 + 16$.

a. Find the values of f and g when x is 4, 5, and 6.

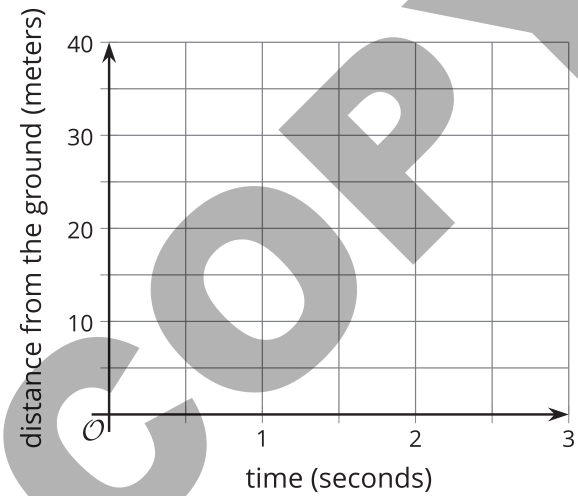
b. Are the values of $f(x)$ always greater than the values of $g(x)$, for all x? Explain how you know.

8 from Unit 7, Lesson 5

Han accidentally drops his water bottle from the balcony of his apartment building. The equation $d = 32 - 5t^2$ gives the distance from the ground, d, in meters, that his water bottle is after t seconds.

a. Complete the table, and then plot the data on the coordinate plane.

t (seconds)	d (meters)
0	
0.5	
1	
1.5	
2	



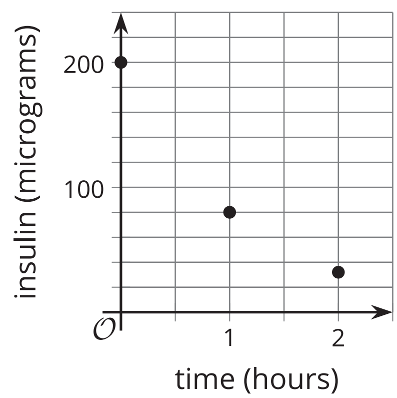
b. Is the water bottle falling at a constant speed? Explain how you know.

9 from Unit 6, Lesson 6

The graph shows how much insulin, in micrograms (mcg), is in a patient's body after receiving an injection. Assume that the amount of insulin continues to decay exponentially.

a. Write an equation giving the number of mcg of insulin, m, in the patient's body h hours after receiving the injection.

b. After 3 hours, will the patient still have at least 10 mcg of insulin in their body? Explain how you know.



Addressing CA CCSSM F-BF.1a, F-IF.5, F-IF.7a; building on F-IF.4; practicing MP2, MP4, MP6, MP8

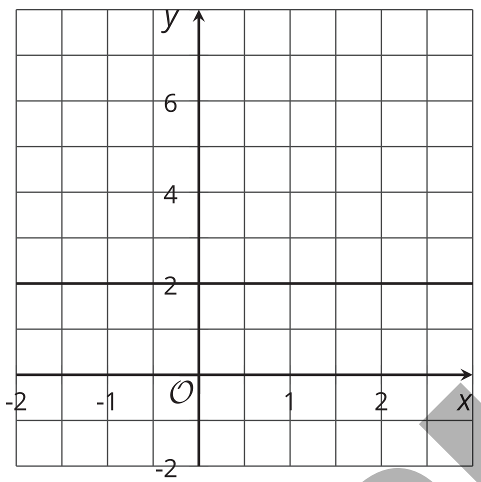
Building Quadratic Functions to Describe Situations (Part 3)

Let's look at how to maximize revenue.

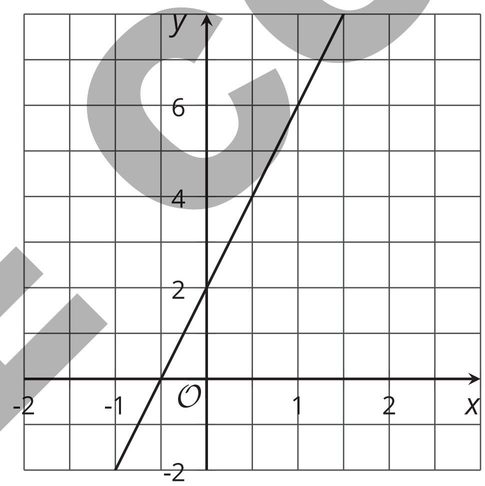
7.1 Which Three Go Together: Graphs of Four Functions

Which three go together? Why do they go together?

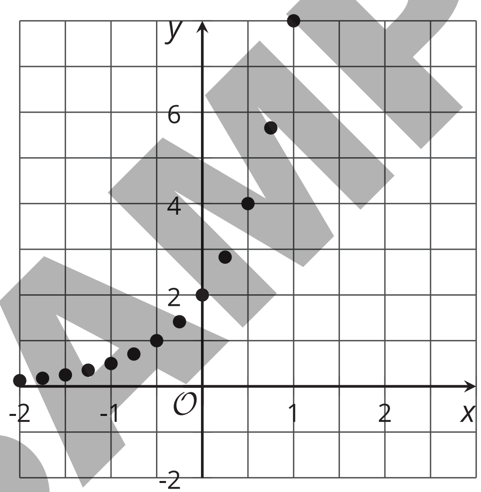
A



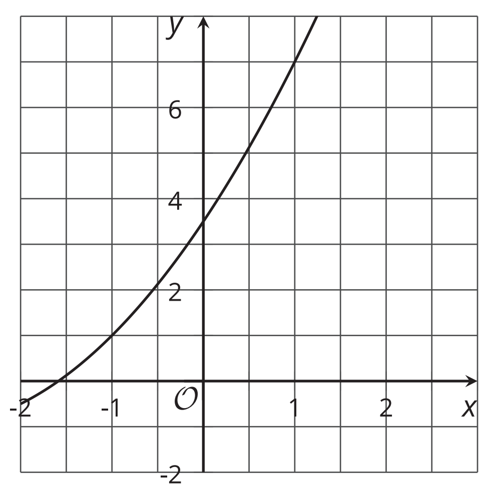
B



C



D



7.2 What Price to Charge?

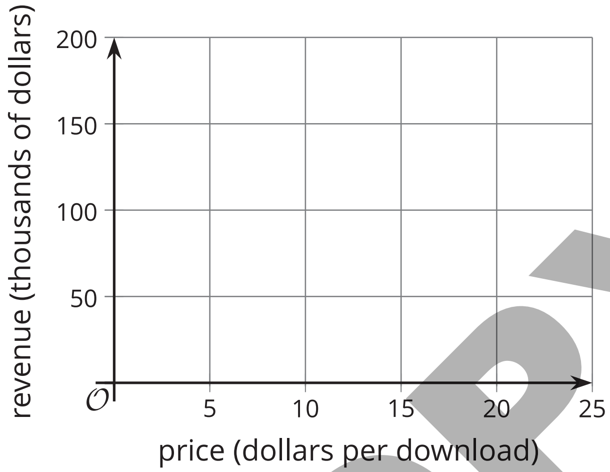
A company that sells movies online is deciding how much to charge customers to download a new movie. Based on data from previous sales, the company predicts that if they charge x dollars for each download, then the number of downloads, in thousands, is $18 - x$.

1. Complete the table to show the predicted number of downloads at each listed price. Then find the revenue at each price. The first row has been completed for you.

price (dollars per download)	number of downloads (thousands)	revenue (thousands of dollars)
3	15	45
5		
10		
12		
15		
18		
x		

2. Is the relationship between the price of the movie and the revenue (in thousands of dollars) quadratic? Explain how you know.

3. Plot the points that represent the revenue, r, as a function of the price of one download in dollars, x.



4. What price would you recommend that the company charge for a new movie? Explain your reasoning.

Are you ready for more?

The function that uses the price (in dollars per download), x, to determine the number of downloads (in thousands), $18 - x$, is an example of a demand function and its graph is known. Economists are interested in factors that can affect the demand function, and therefore the price, that suppliers wish to set.

1. What are some things that could increase the number of downloads predicted for the same given prices?

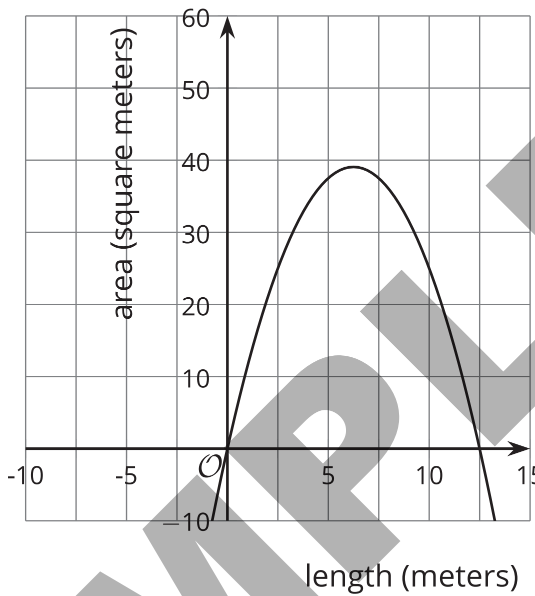
2. If the demand shifted so that we predicted $20 - x$ thousand downloads at a price of x dollars per download, what do you think will happen to the price that gives the maximum revenue? Check what actually happens.

7.3 Domain, Vertex, and Zeros

Here are four sets of descriptions and equations that represent some familiar quadratic functions. The graphs show what graphing technology may produce when the equations are graphed. For each function:

- Describe a domain that is appropriate for the situation. Think about any upper or lower limits for the input, as well as whether all numbers make sense as the input. Then, describe how the graph should be modified to show the domain that makes sense.
- Identify or estimate the vertex on the graph. Describe what it means in the situation.
- Identify or estimate the zeros of the function. Describe what they meant in the situation.

1. The area of a rectangle with a perimeter of 25 meters and a side length of x: $A(x) = x \cdot \frac{(25-2x)}{2}$

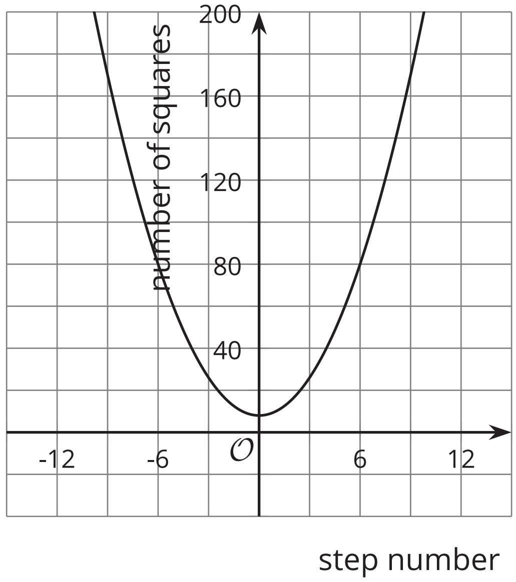


- Domain:
- Vertex:
- Zeros:

2. The number of squares as a function of step number n: $f(n) = n^2 + 4$

- Domain:

Sec B

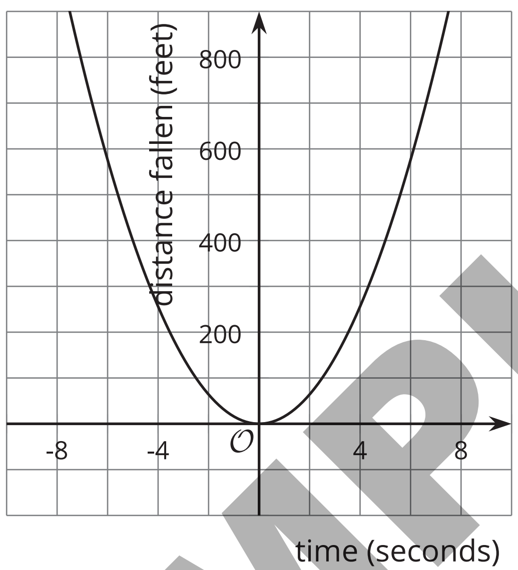


- Vertex:

- Zeros:

3. The distance, in feet, that an object has fallen t seconds after being dropped: $g(t) = 16t^2$

- Domain:

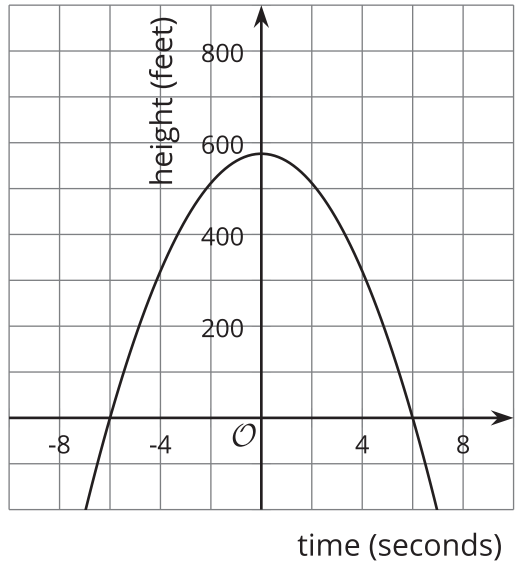


- Vertex:

- Zeros:

4. The height, in feet, of an object t seconds after being dropped: $h(t) = 576 - 16t^2$

- Domain:



- Vertex:

- Zeros:

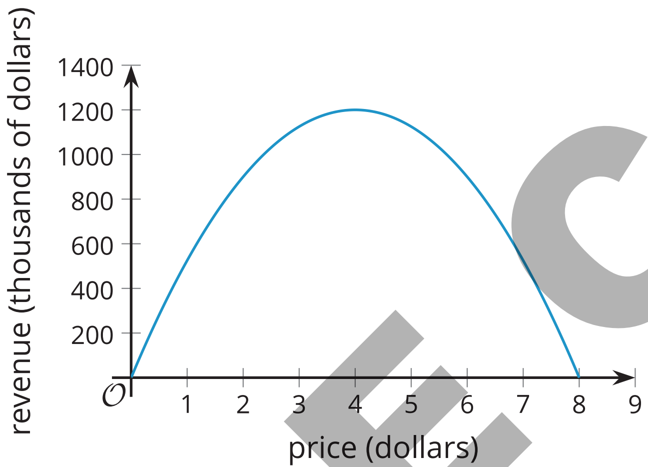
Sec B

Lesson 7 Summary

Quadratic functions often come up when studying revenue, which is the amount of money collected when selling something.

Suppose we are selling raffle tickets and deciding how much to charge for each ticket. When the price of the tickets is higher, typically fewer tickets will be sold.

Let's say that with a price of d dollars, it is possible to sell $600 - 75d$ tickets. We can find the revenue by multiplying the price by the number of tickets expected to be sold. A function that models the revenue, r, collected is $r(d) = d(600 - 75d)$. Here is a graph that represents the function.



When ticket prices are low, a lot of tickets may be sold, but the total revenue is still low because the tickets are cheap. When the ticket prices get close to \$8, not many tickets are sold so the revenue is low again. From the graph, we can tell that the greatest revenue comes when there is a balance between ticket price and number of tickets sold. In this situation, that is \$1,200 of revenue when tickets are sold for \$4 each.

We can also see that for function r, the domain is between 0 and 8. This makes sense because the cost of the tickets can't be negative. If the price is more than \$8, the model doesn't work because the revenue collected can't be negative. A negative revenue (based on a non-negative ticket price) could occur only if the number of tickets sold is negative, which is not possible.

Lesson 7 Practice Problems

1 Based on past musical productions, a theater predicts selling $400 - 8p$ tickets, when each ticket is sold for p dollars.

a. Complete the table to find out how many tickets the theater expects to sell and what revenues it expects to receive at the given ticket prices.

ticket price (dollars)	number of tickets sold	revenue (dollars)
5		
10		
15		
20		
30		
45		
50		
p		

b. For which ticket prices will the theater earn no revenue? Explain how you know.

c. At what ticket prices should the theater sell the tickets if it must earn at least $3,200 in revenue to break even (to not lose money) on the musical production? Explain how you know.

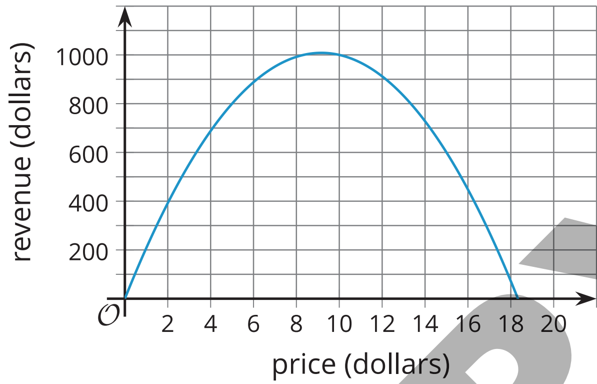
2 A company sells running shoes. If the price of a pair of shoes in dollars is p, the company estimates that it will sell $50,000 - 400p$ pairs of shoes.

Write an expression that represents the revenue in dollars from selling running shoes if a pair of shoes is priced at p dollars.

Sec B

3 A function, f, represents the revenue in dollars that the school can expect to receive if it sells $220 - 12x$ coffee mugs for x dollars each.

Here is the graph of f.



Select **all** the statements that describe this situation.

A. At \$2 per coffee mug, the revenue will be \$196.

B. The school expects to sell 160 mugs if the price is \$5 each.

C. The school will lose money if it sells the mugs for more than \$10 each.

D. The school will earn about \$1,000 if it sells the mugs for \$10 each.

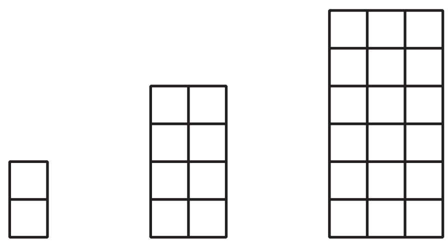
E. The revenue will be more than \$700 if the price is between \$4 and \$14.

F. The expected revenue will increase if the price per mug is greater than \$10.

4 from Unit 7, Lesson 3

a. Write an equation to represent the relationship between the step number, n, and the number of small squares, y.

Briefly describe how each part of the equation relates to the pattern.



Step 1 Step 2 Step 3

b. Is the relationship between the step number and number of small squares quadratic? Explain how you know.

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5 from Unit 7, Lesson 5

Technology required. A small marshmallow is launched straight up in the air with a slingshot. Function h, given by the equation $h(t) = 5 + 20t - 5t^2$, describes the height of the marshmallow, in meters, as a function of time, t, in seconds since it was launched.

a. Use graphing technology to graph function h.

b. About when does the marshmallow reach its maximum height?

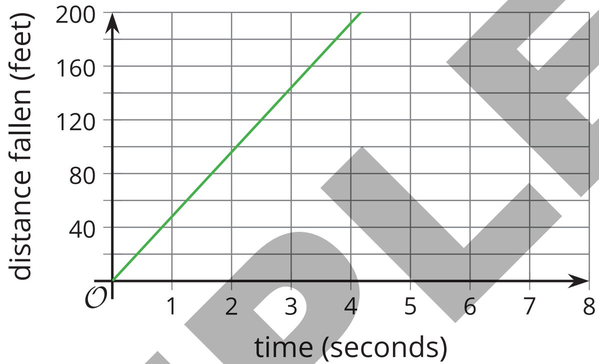
c. About how long does it take before the marshmallow hits the ground?

d. What domain makes sense for function h in this situation?

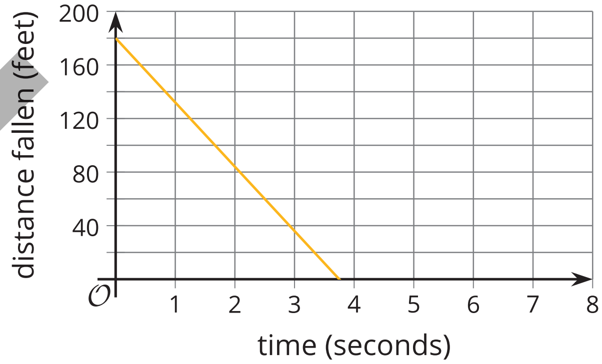
6 from Unit 7, Lesson 5

A rock is dropped from a bridge over a river. Which graph could represent the distance fallen, in feet, as a function of time in seconds?

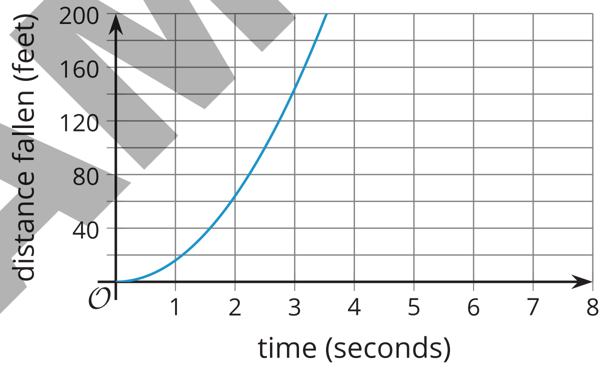
Graph A



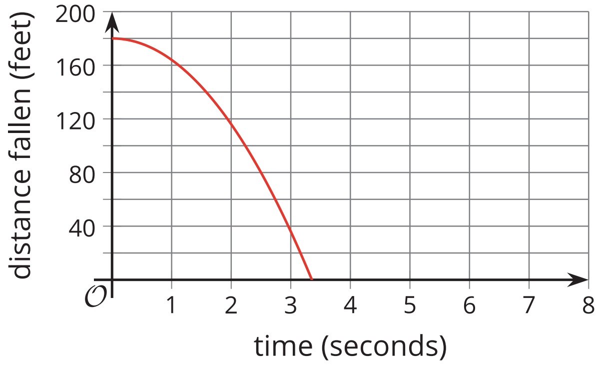
Graph B



Graph C



Graph D



A. Graph A

B. Graph B

C. Graph C

D. Graph D

7 from Unit 6, Lesson 7

Sec B

A bacteria population, p, is modeled by the equation $p = 100{,}000 \cdot 2^d$, where d is the number of days since the population was first measured.

Select **all** statements that are true in this situation.

A. $100{,}000 \cdot 2^{-2}$ represents the bacteria population 2 days before it was first measured.

B. The bacteria population 3 days before it was first measured was 800,000.

C. The population was more than 1,000 one week before it was first measured.

D. The population was more than 1,000,000 one week after it was first measured.

E. The bacteria population 4 days before it was first measured was 6,250.

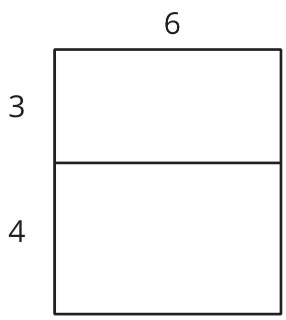
Unit 7, Lesson 8

Addressing CA CCSSM A-APR.1, A-SSE.2, A-SSE.3; building on 6.EE.3, 7.EE.1; building toward A-SSE.3, F-IF.8; practicing MP7

Equivalent Quadratic Expressions

Let's use diagrams to help us rewrite quadratic expressions.

8.1 Diagrams of Products

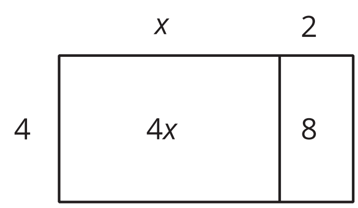


1. Explain why the diagram shows that $6(3+4) = 6 \cdot 3 + 6 \cdot 4$.

2. Draw a diagram to show that $5(x+2) = 5x + 10$.

8.2 Drawing Diagrams to Represent More Products

Applying the distributive property to multiply out the factors of, or expand, $4(x+2)$ gives us $4x+8$, so we know the two expressions are equivalent. We can use a rectangle with side lengths of $(x+2)$ and 4 to illustrate the multiplication.



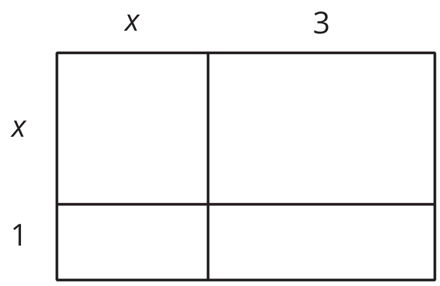
1. Draw a diagram to show that $n(2n+5)$ and $2n^2+5n$ are equivalent expressions.

2. For each expression, use the distributive property to write an equivalent expression. If you get stuck, consider drawing a diagram.

   a. $6\left(\frac{1}{3}n+2\right)$
   b. $p(4p+9)$
   c. $5r\left(r+\frac{3}{5}\right)$
   d. $(0.5w+7)w$

8.3 Using Diagrams to Find Equivalent Quadratic Expressions

1. Here is a diagram of a rectangle with side lengths of $x + 1$ and $x + 3$. Use this diagram to show that $(x + 1)(x + 3)$ and $x^2 + 4x + 3$ are equivalent expressions.

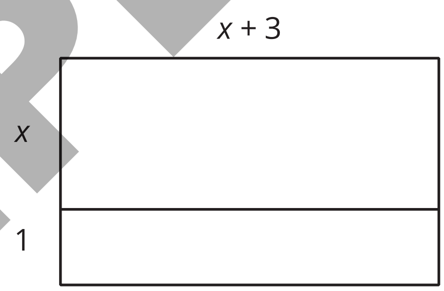


2. Draw diagrams to help you write an equivalent expression for each of the following:

   a. $(2x + 1)(x + 3)$

   b. $(x + 5)^2$

3. Here is a diagram of a rectangle with the same area as in the first question. Use this diagram to show that $(x + 1)(x + 3)$ and $x(x + 3) + 1(x + 3)$ are equivalent expressions. Then explain how you could rewrite that expression as $x^2 + 4x + 3$, without a diagram.

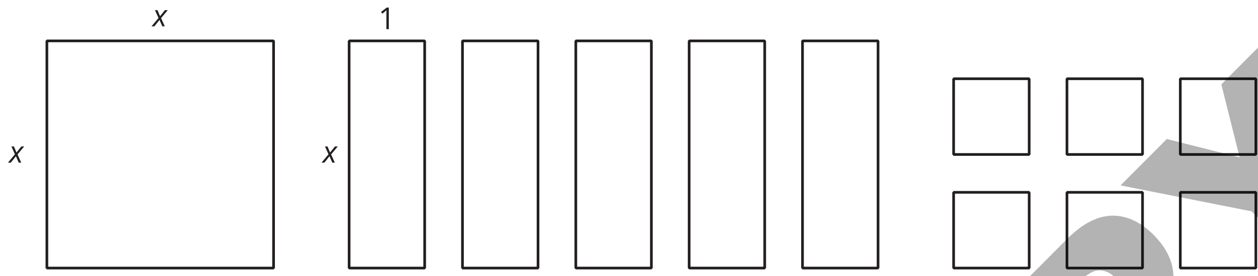


4. Write an equivalent expression for each expression:

   a. $(x + 2)(x + 6)$

   b. $(x + m)(x + n)$

Are you ready for more?



1. Is it possible to arrange an x-by-x square, five x-by-1 rectangles and six 1-by-1 squares into a single large rectangle? Explain or show your reasoning.

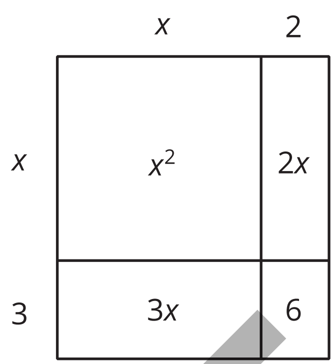
2. What does this tell you about an equivalent expression for $x^2 + 5x + 6$?

3. Keeping the x-by-x square and the five x-by-1 rectangles, can you form a different rectangle by using a different number of 1-by-1 squares than what is shown?

Lesson 8 Summary

A quadratic function can often be defined by many different but equivalent expressions. For example, we saw earlier that the predicted revenue, in thousands of dollars, from selling a downloadable movie at x dollars can be expressed with $x(18 - x)$, which can also be written as $18x - x^2$.

Sometimes a quadratic expression is a product of two factors that are each a linear expression, for example $(x + 2)(x + 3)$. We can write an equivalent expression by thinking about each factor, the $(x + 2)$ and $(x + 3)$, as the side lengths of a rectangle, with each side length being decomposed into a variable expression and a number.



Multiplying $(x + 2)$ and $(x + 3)$ gives the area of the rectangle. Adding the areas of the four sub-rectangles also gives the area of the rectangle. This means that $(x + 2)(x + 3)$ is equivalent to $x^2 + 2x + 3x + 6$, or to $x^2 + 5x + 6$.

Notice that the diagram illustrates the distributive property being applied. Each term of one factor (say, the x and the 2 in $x + 2$) is multiplied by every term in the other factor (the x and the 3 in $x + 3$).

$$\begin{aligned} &(x + 2)(x + 3) \\ =\ &x(x + 3) + 2(x + 3) \\ =\ &x^2 + 3x + 2x + (2)(3) \\ =\ &x^2 + (3 + 2)x + (2)(3) \end{aligned}$$

In general, when a quadratic expression is written in the form of $(x + p)(x + q)$, we can apply the distributive property to rewrite it as $x^2 + px + qx + pq$, or as $x^2 + (p + q)x + pq$.

Sec C

Lesson 8 Practice Problems

1 Draw a diagram to show that $(2x + 5)(x + 3)$ is equivalent to $2x^2 + 11x + 15$.

2 Match each quadratic expression that is written as a product with an equivalent expression that is expanded.

A. $(x + 2)(x + 6)$

B. $(2x + 8)(x + 2)$

C. $(x + 8)(x + 4)$

D. $(x + 2)(2x + 6)$

1. $x^2 + 12x + 32$
2. $2x^2 + 10x + 12$
3. $2x^2 + 12x + 16$
4. $x^2 + 8x + 12$

3 Select **all** expressions that are equivalent to $x^2 + 4x$.

A. $x(x + 4)$

B. $(x + 2)^2$

C. $(x + x)(x + 4)$

D. $(x + 2)^2 - 4$

E. $(x + 4)x$

4 Tyler drew a diagram to expand $(x+5)(2x+3)$.

a. Explain Tyler's mistake.

	$2x$	3
x	$2x^2$	$3x$
5	$7x$	8

b. What is the correct expanded form of $(x+5)(2x+3)$?

5 from Unit 7, Lesson 4

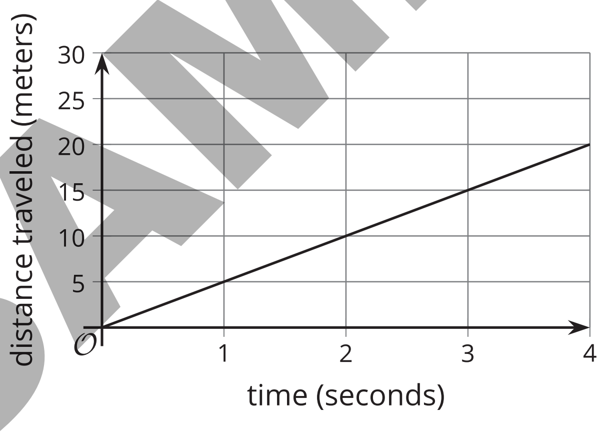
Explain why the values of the exponential expression 3^x will eventually overtake the values of the quadratic expression $10x^2$.

6 from Unit 7, Lesson 5

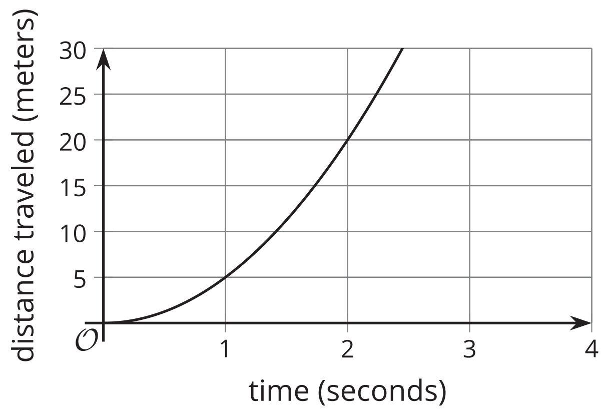
A baseball travels d meters t seconds after being dropped from the top of a building. The distance traveled by the baseball can be modeled by the equation $d = 5t^2$.

Which graph could represent this situation? Explain how you know.

Graph A



Graph B



7 from Unit 5, Lesson 10

Consider a function, q, defined by $q(x) = x^2$. Explain why negative values are not included in the range of q.

8 from Unit 7, Lesson 7

Based on past concerts, a band predicts selling $600 - 10p$ concert tickets when each ticket is sold at p dollars.

a. Complete the table to find out how many concert tickets the band expects to sell and what revenues it expects to receive at the given ticket prices.

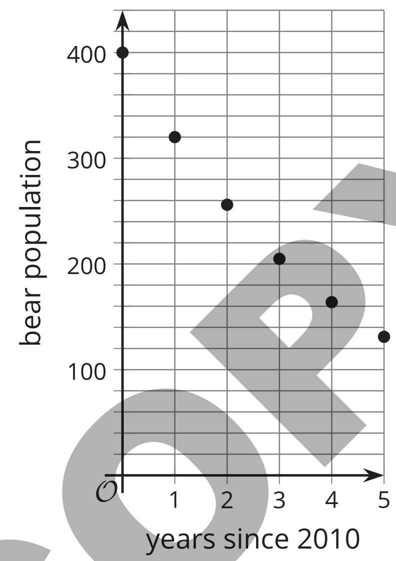
ticket price (dollars)	number of tickets	revenue (dollars)
10		
15		
20		
30		
35		
45		
50		
60		
p		

b. In this model, at what ticket prices will the band earn no revenue at all?

c. At what ticket prices should the band sell the tickets if it must earn at least 8,000 dollars in revenue to break even (to not lose money) on a given concert. Explain how you know.

9 from Unit 6, Lesson 11

A population of bears decreases exponentially. The population was first measured in 2010.



a. What is the annual factor of decrease for the bear population? Explain how you know.

b. Using function notation, represent the relationship between the bear population, b, and the number of years since the population was first measured, t. That is, find a function, f, so that $b = f(t)$.

Sec C

10 from Unit 6, Lesson 11

Equations defining functions a, b, c, d, and f are shown here.

Select **all** the equations that represent exponential functions.

A. $a(x) = 2^3 \cdot x$

B. $b(t) = \left(\frac{2}{3}\right)^t$

C. $c(m) = \frac{1}{5} \cdot 2^m$

D. $d(x) = 3x^2$

E. $f(t) = 3 \cdot 2^t$

Unit 7, Lesson 9

Addressing CA CCSSM A-APR.1, A-SSE.2, A-SSE.3; building toward F-IF.8; practicing MP6, MP7, MP8

Standard Form and Factored Form

A1

Let's write quadratic expressions in different forms.

9.1 Math Talk: Opposites Attract

Solve each equation for n, mentally.

- $40 - 8 = 40 + n$
- $25 + -100 = 25 - n$
- $3 - \frac{1}{2} = 3 + n$
- $72 - n = 72 + 6$

9.2 Finding Products of Differences

1. Show that $(x - 1)(x - 1)$ and $x^2 - 2x + 1$ are equivalent expressions by drawing a diagram or applying the distributive property. Show your reasoning.

2. For each expression, write an equivalent expression. Show your reasoning.
   a. $(x + 1)(x - 1)$

   b. $(x - 2)(x + 3)$

   c. $(x - 2)^2$

Sec C

9.3 What Is the Standard Form? What Is the Factored Form?

The quadratic expression $x^2 + 4x + 3$ is written in **standard form**.

Here are some other quadratic expressions. In one column, the expressions are written in standard form and in the other column the expressions are not.

Written in standard form:	Not written in standard form:
$x^2 - 1$	$(2x + 3)x$
$x^2 + 9x$	$(x + 1)(x - 1)$
$\frac{1}{2}x^2$	$3(x - 2)^2 + 1$
$4x^2 - 2x + 5$	$-4(x^2 + x) + 7$
$-3x^2 - x + 6$	$(x + 8)(-x + 5)$
$1 - x^2$	

1. What are some characteristics of expressions in standard form?

2. $(x + 1)(x - 1)$ and $(2x + 3)x$ in the other column are quadratic expressions written in **factored form**. Why do you think that form is called factored form?

Are you ready for more?

What quadratic expression can be described as being both standard form and factored form? Explain how you know.

Sec C

Lesson 9 Summary

A quadratic function can often be represented by many equivalent expressions. For example, a quadratic function, f, might be defined by $f(x) = x^2 + 3x + 2$. The quadratic expression $x^2 + 3x + 2$ is called the **standard form**, the sum of a multiple of x^2 and a linear expression ($3x + 2$ in this case).

In general, standard form is written as

$$ax^2 + bx + c$$

We refer to a as the coefficient of the squared term x^2, b as the coefficient of the linear term x, and c as the constant term.

Function f can also be defined by the equivalent expression $(x + 2)(x + 1)$. When the quadratic expression is a product of two factors where each one is a linear expression, this is called the **factored form**.

An expression in factored form can be rewritten in standard form by expanding it, which means multiplying out the factors. In a previous lesson we saw how to use a diagram and to apply the distributive property to multiply two linear expressions, such as $(x + 3)(x + 2)$. We can do the same to expand an expression with a sum and a difference, such as $(x + 5)(x - 2)$, or to expand an expression with two differences, for example, $(x - 4)(x - 1)$.

To represent $(x - 4)(x - 1)$ with a diagram, we can think of subtraction as adding the opposite:

	x	-4
x	x^2	$-4x$
-1	$-x$	4

$$(x - 4)(x - 1)$$
$$= (x + -4)(x + -1)$$
$$= x(x + -1) + -4(x + -1)$$
$$= x^2 + -1x + -4x + (-4)(-1)$$
$$= x^2 + -5x + 4$$
$$= x^2 - 5x + 4$$

Glossary

- factored form (of a quadratic expression)
- standard form (of a quadratic expression)

Lesson 9 Practice Problems

1 Write each quadratic expression in standard form. Draw a diagram if needed.

a. $(x + 4)(x - 1)$

b. $(2x - 1)(3x - 1)$

2 Consider the expression $8 - 6x + x^2$.

a. Is the expression in standard form? Explain how you know.

b. Is the expression equivalent to $(x - 4)(x - 2)$? Explain how you know.

3 Which quadratic expression is written in standard form?

A. $(x + 3)x$

B. $(x + 4)^2$

C. $-x^2 - 5x + 7$

D. $x^2 + 2(x + 3)$

4 Explain why $3x^2$ can be said to be in both standard form and factored form.

5 from Unit 7, Lesson 5

Jada dropped her sunglasses from a bridge over a river. Which equation could represent the distance, y, fallen in feet, as a function of time, t, in seconds?

A. $y = 16t^2$

B. $y = 48t$

C. $y = 180 - 16t^2$

D. $y = 180 - 48t$

6 from Unit 7, Lesson 6

A football player throws a football. Function h, given by $h(t) = 6 + 75t - 16t^2$ describes the football's height in feet, t seconds after it is thrown.

Select **all** the statements that are true about this situation.

A. The football is thrown from ground level.

B. The football is thrown from 6 feet off the ground.

C. In the function, $-16t^2$ represents the effect of gravity.

D. The outputs of h decrease then increase in value.

E. The function h has 2 zeros that make sense in this situation.

F. The vertex of the graph of h gives the maximum height of the football.

7 from Unit 7, Lesson 6

Technology required. Two rocks are launched straight up in the air.

- The height of Rock A is given by function f, where $f(t) = 4 + 30t - 16t^2$.
- The height of Rock B is given by function g, where $g(t) = 5 + 20t - 16t^2$.

In both functions, t is time measured in seconds and height is measured in feet. Use graphing technology to graph both equations.

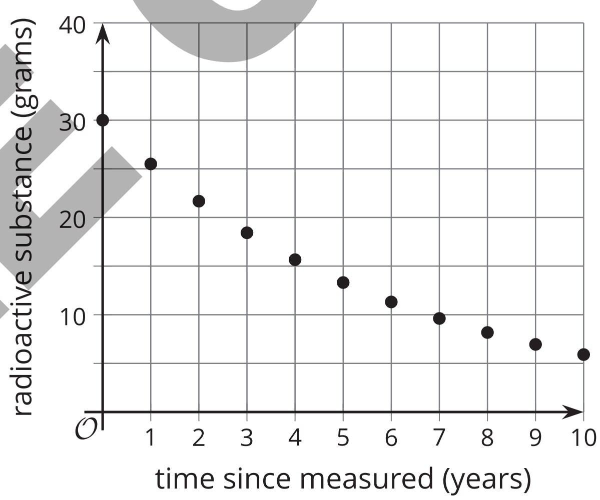
a. What is the maximum height of each rock?

b. Which rock reaches its maximum height first? Explain how you know.

8 from Unit 6, Lesson 13

The graph shows the number of grams of a radioactive substance in a sample at different times after the sample was first analyzed.

a. What is the average rate of change for the substance during the 10 year period?



b. Is the average rate of change a good measure for the change in the radioactive substance during these 10 years? Explain how you know.

9 from Unit 6, Lesson 14

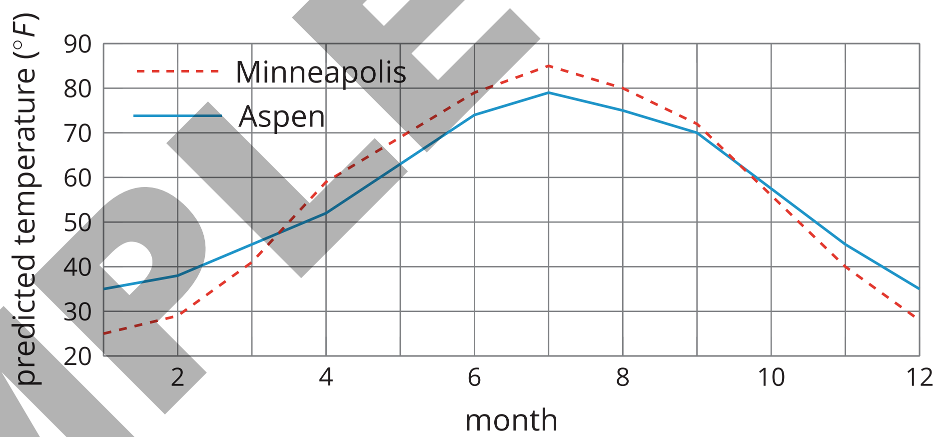
Each day after an outbreak of a new strain of the flu virus, a public health scientist receives a report of the number of new cases of the flu reported by area hospitals.

time since outbreak in days	2	3	4	5	6	7
number of new cases of the flu	20	28	38	54	75	105

Would a linear or exponential model be more appropriate for this data? Explain how you know.

10 from Unit 5, Lesson 9

$A(t)$ is a model for the temperature in Aspen, Colorado, t months after the start of the year. $M(t)$ is a model for the temperature in Minneapolis, Minnesota, t months after the start of the year. Temperature is measured in degrees Fahrenheit.



a. What does $A(8)$ mean in this situation? Estimate $A(8)$.

b. Which city has a higher predicted temperature in February?

c. Are the two cities' predicted temperatures ever the same? If so, when?

Unit 7, Lesson 10

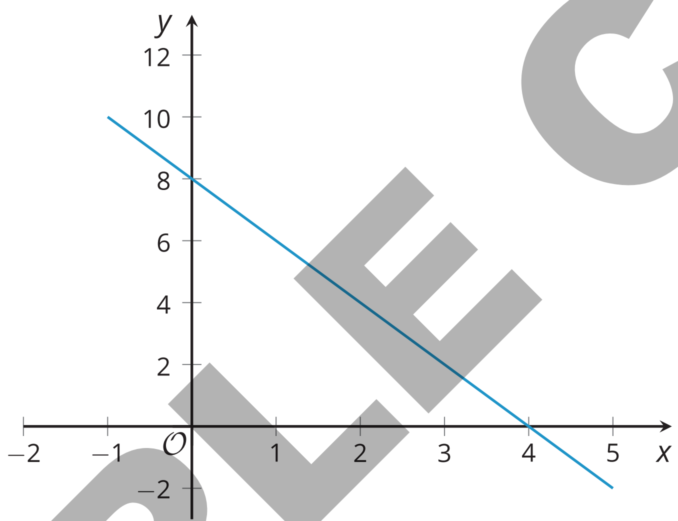
Addressing CA CCSSM A-SSE.3, F-LE.6; building on F-IF.4; building toward A-SSE.3; practicing MP7 and MP8

Graphs of Functions in Standard and Factored Forms

Let's find out what quadratic expressions in standard and factored forms can reveal about the properties of their graphs.

10.1 A Linear Equation and Its Graph

Here is a graph of the equation $y = 8 - 2x$.



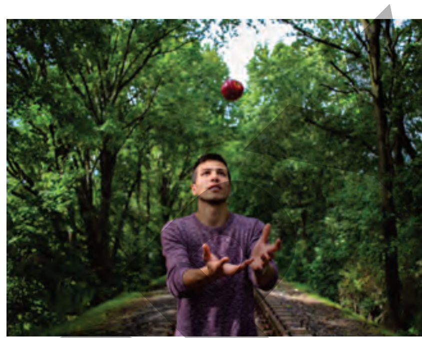
1. Where do you see the 8 from the equation in the graph?

2. Where do you see the -2 from the equation in the graph?

3. What is the x-intercept of the graph? How does this relate to the equation?

10.2 Revisiting Projectile Motion

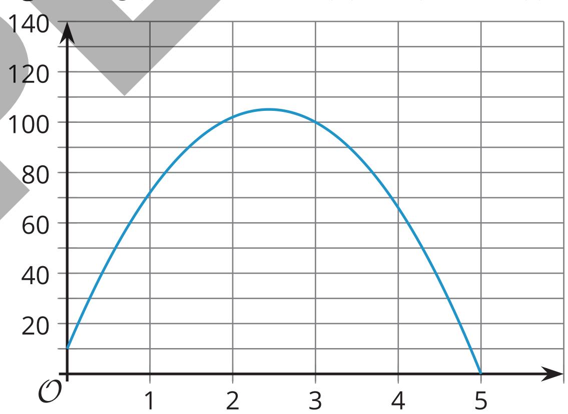
In an earlier lesson, we saw that an equation such as $h(t) = 10 + 78t - 16t^2$ can model the height of an object thrown upward from a height of 10 feet with a vertical velocity of 78 feet per second.



1. Is the expression $10 + 78t - 16t^2$ written in standard form? Explain how you know.

2. Jada said that the equation $g(t) = (\text{-}16t - 2)(t - 5)$ also defines the same function, written in factored form. Show that Jada is correct.

3. Here is a graph representing both $g(t) = (\text{-}16t - 2)(t - 5)$ and $h(t) = 10 + 78t - 16t^2$.



   a. Identify or approximate the vertical and horizontal intercepts.
   b. What do each of these points mean in this situation?

10.3 Relating Expressions and Their Graphs

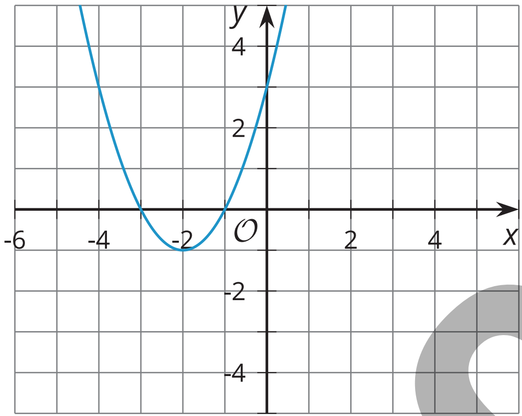
Here are pairs of expressions in standard and factored forms. Each pair of expressions define the same quadratic function, which can be represented with the given graph.

1. Identify the x-intercepts and the y-intercept of each graph.

Function f

$x^2 + 4x + 3$

$(x + 3)(x + 1)$



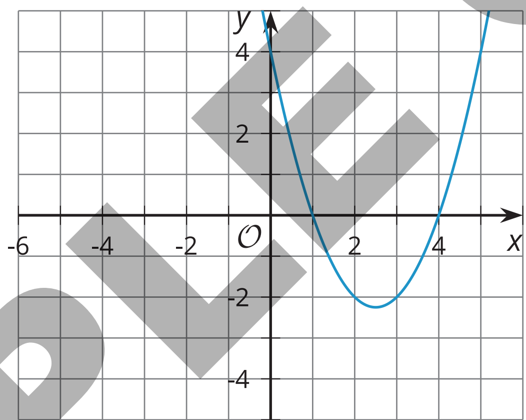
x-intercepts:

y-intercept:

Function g

$x^2 - 5x + 4$

$(x - 4)(x - 1)$



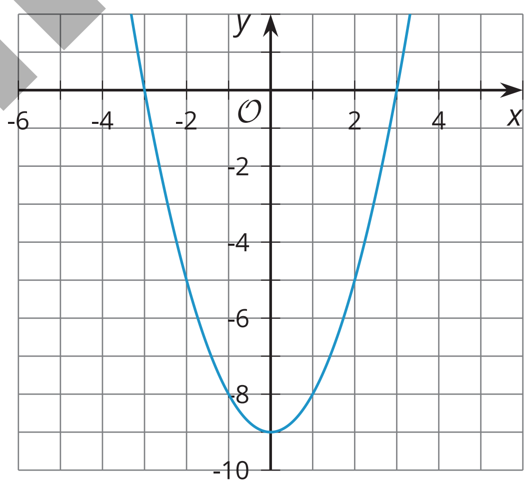
x-intercepts:

y-intercept:

Function h

$x^2 - 9$

$(x - 3)(x + 3)$



x-intercepts:

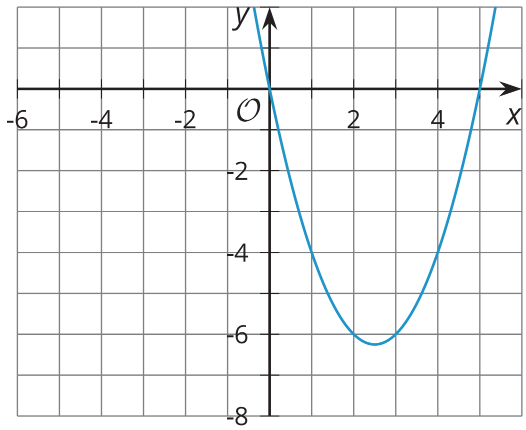
y-intercept:

Sec C

Function i

$x^2 - 5x$

$x(x - 5)$



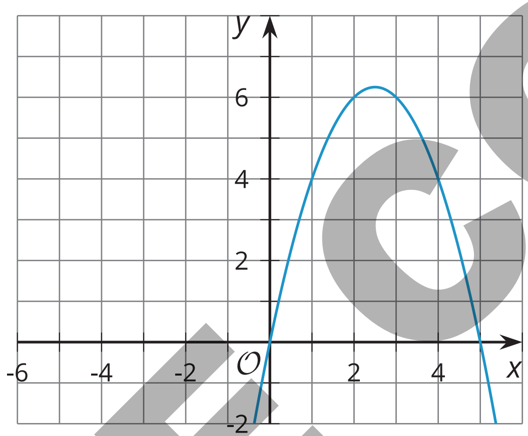
x-intercepts:

y-intercept:

Function j

$5x - x^2$

$x(5 - x)$



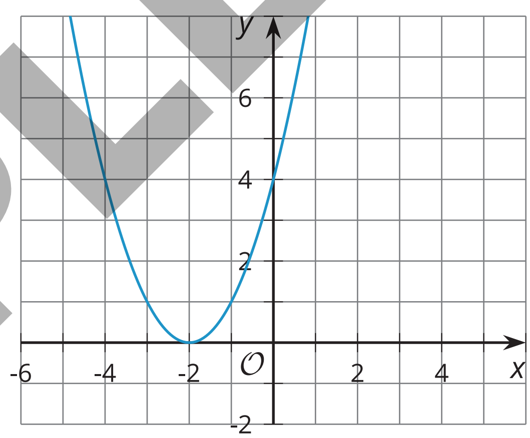
x-intercepts:

y-intercept:

Function k

$x^2 + 4x + 4$

$(x + 2)(x + 2)$



x-intercepts:

y-intercept:

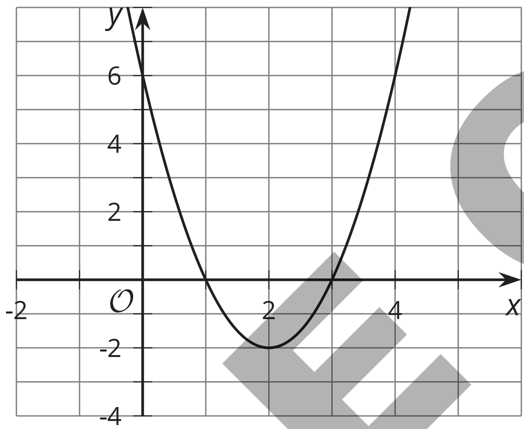
2. What do you notice about the x-intercepts, the y-intercept, and the numbers in the expressions defining each function? Make a couple of observations.

Sec C

3. Here is an expression that models function p, another quadratic function: $(x-9)(x-1)$. Predict the x-intercepts and the y-intercept of the graph that represent this function.

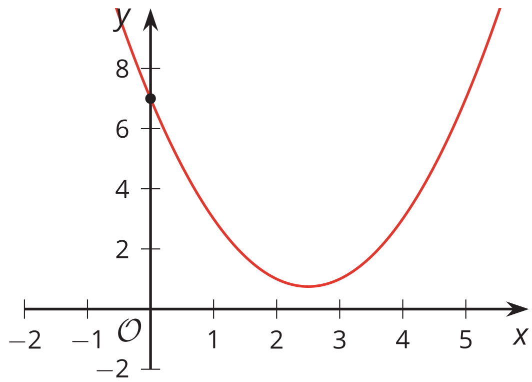
Are you ready for more?

Find the values of a, p, and q that will make $y = a(x-p)(x-q)$ be the equation represented by the graph.



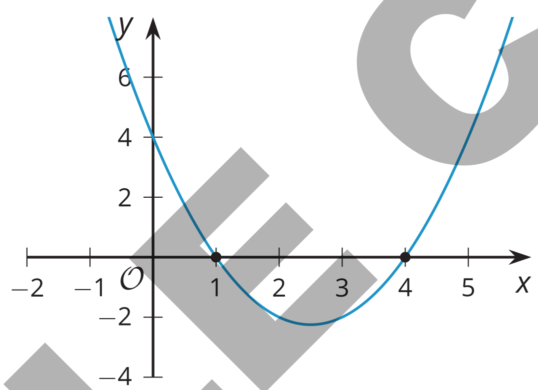
Lesson 10 Summary

Different forms of quadratic functions can tell us interesting information about the function's graph. When a quadratic function is expressed in standard form, it can tell us the y-intercept of the graph that represents the function. For example, the graph representing $y = x^2 - 5x + 7$ has its y-intercept at $(0, 7)$. This makes sense because the y-coordinate is the y-value when x is 0. Evaluating the expression at $x = 0$ gives $y = 0^2 - 5(0) + 7$, which equals 7.

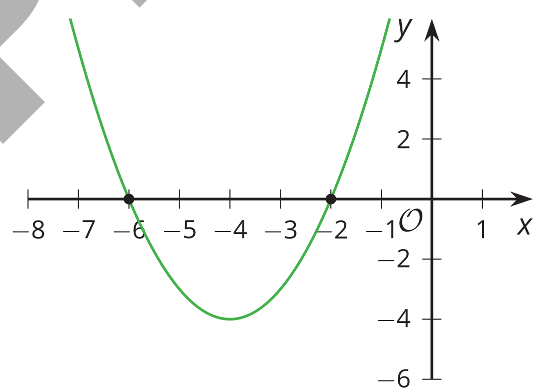


When a function is expressed in factored form, it can help us see the x-intercepts of its graph. Let's look at function f, given by $f(x) = (x - 4)(x - 1)$ and function g, given by $g(x) = (x + 2)(x + 6)$.

If we graph $y = f(x)$, we see that the x-intercepts of the graph are $(1, 0)$ and $(4, 0)$. Notice that 1 and 4 also appear in $f(x) = (x - 4)(x - 1)$, and they are subtracted from x.



If we graph $y = g(x)$, we see that the x-intercepts are at $(-2, 0)$ and $(-6, 0)$. Notice that 2 and 6 are also in the equation $g(x) = (x + 2)(x + 6)$, but they are added to x.



The connection between the factored form and the x-intercepts of the graph tells us about the zeros of the function (the input values that produce an output of 0).

Lesson 10 Practice Problems

1 A quadratic function, f, is defined by $f(x) = (x - 7)(x + 3)$.

a. Without graphing, identify the x-intercepts of the graph of f. Explain how you know.

b. Expand $(x - 7)(x + 3)$ and use the expanded form to identify the y-intercept of the graph of f.

2 What are the x-intercepts of the graph of the function defined by $(x - 2)(2x + 1)$?

A. $(2, 0)$ and $(-1, 0)$

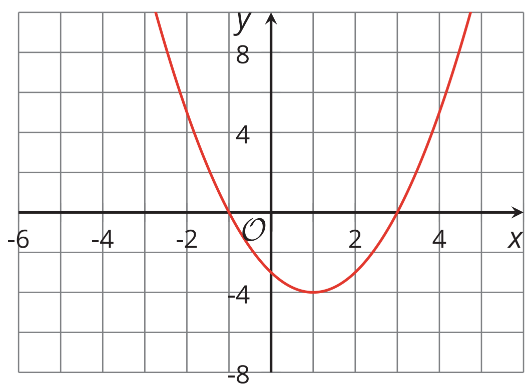
B. $(2, 0)$ and $\left(-\frac{1}{2}, 0\right)$

C. $(-2, 0)$ and $(1, 0)$

D. $(-2, 0)$ and $(\frac{1}{2}, 0)$

3 Here is a graph that represents a quadratic function.

Which expression could define this function?



A. $(x + 3)(x + 1)$

B. $(x + 3)(x - 1)$

C. $(x - 3)(x + 1)$

D. $(x - 3)(x - 1)$

4 a. What is the y-intercept of the graph of the equation $y = x^2 - 5x + 4$?

b. An equivalent way to write this equation is $y = (x - 4)(x - 1)$. What are the x-intercepts of this equation's graph?

5 Noah said that if we graph $y = (x - 1)(x + 6)$, the x-intercepts will be at $(1, 0)$ and $(-6, 0)$. Explain how you can determine, without graphing, whether Noah is correct.

Sec C

6 from Unit 7, Lesson 7

A company sells a video game. If the price of the game in dollars is p, the company estimates that it will sell $20{,}000 - 500p$ games.

Which expression represents the revenue in dollars from selling games if the game is priced at p dollars?

A. $(20{,}000 - 500p) + p$

B. $(20{,}000 - 500p) - p$

C. $\dfrac{20,000 - 500p}{p}$

D. $(20{,}000 - 500p) \cdot p$

7 from Unit 7, Lesson 9

Write each quadratic expression in standard form. Draw a diagram if needed.

a. $(x-3)(x-6)$

b. $(x-4)^2$

c. $(2x+3)(x-4)$

d. $(4x-1)(3x-7)$

8 from Unit 7, Lesson 9

Consider the expression $(5+x)(6-x)$.

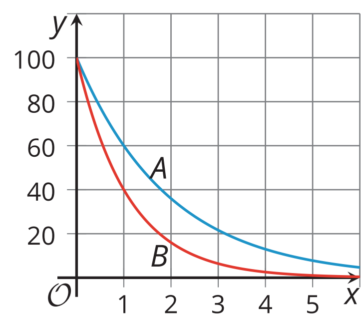
a. Is the expression equivalent to x^2+x+30? Explain how you know.

b. Is the expression $30+x+x^2$ in standard form? Explain how you know.

9 from Unit 6, Lesson 15

Here are graphs of functions f and g, given by $f(x)=100\cdot\left(\frac{3}{5}\right)^x$ and $g(x)=100\cdot\left(\frac{2}{5}\right)^x$.

Which graph corresponds to f and which graph corresponds to g? Explain how you know.



Sec C

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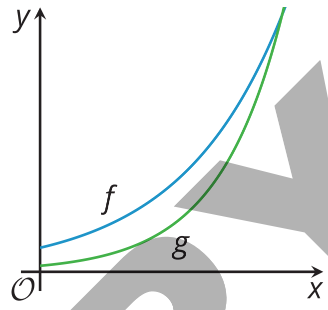
10 from Unit 6, Lesson 16

Here are graphs of two functions, f and g.

An equation defining f is $f(x) = 100 \cdot 2^x$.

Which of these could be an equation defining function g?



A. $g(x) = 25 \cdot 3^x$

B. $g(x) = 50 \cdot (1.5)^x$

C. $g(x) = 100 \cdot 3^x$

D. $g(x) = 200 \cdot (1.5)^x$

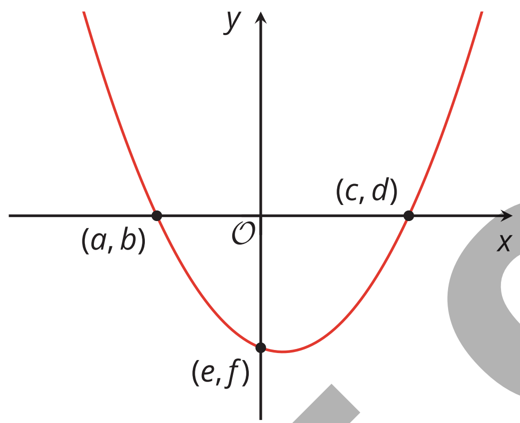
Sec C

Unit 7, Lesson 11
Addressing CA CCSSM A-SSE.1-2, F-IF.7a; practicing MP6, MP7, MP8

Graphing from the Factored Form

Let’s graph some quadratic functions in factored form.

11.1 Finding Coordinates



Sec D

Here is a graph of a function, w, defined by $w(x) = (x + 1.6)(x - 2)$. Three points on the graph are labeled.

Find the values of $a, b, c, d, e,$ and f. Be prepared to explain your reasoning.

11.2 Comparing Two Graphs

Consider two functions defined by $f(x) = x(x + 4)$ and $g(x) = x(x - 4)$.

1. Complete the table of values for each function. Then determine the x-intercepts and vertex of each graph. Be prepared to explain how you know.

x	$f(x)$
-5	5
-4	
-3	
-2	-4
-1	-3
0	
1	
2	
3	
4	32
5	

x-intercepts:

Vertex:

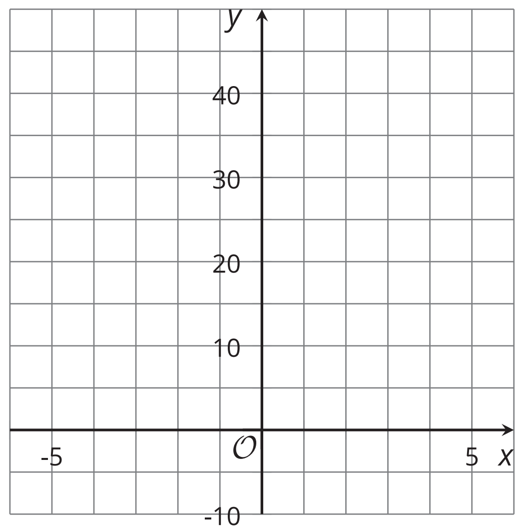
x	$g(x)$
-5	45
-4	
-3	
-2	12
-1	5
0	
1	
2	
3	-3
4	
5	

x-intercepts:

Vertex:

2. Plot the points from the tables on the same coordinate plane. (Consider using different colors or markings for each set of points so you can tell them apart.)

   Then make a couple of observations about how the two graphs compare.

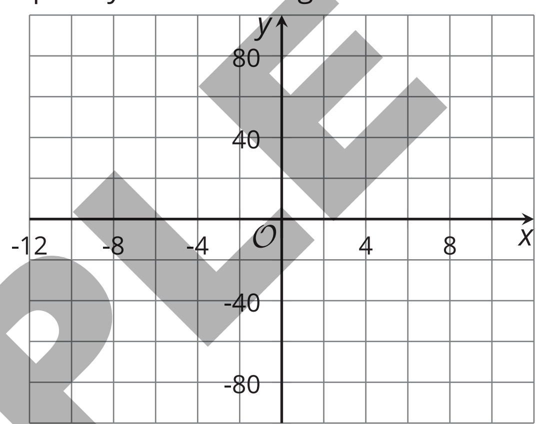


11.3 What Do We Need to Sketch a Graph?

1. Functions f, g, and h are given. Predict the x-intercepts and the x-coordinate of the vertex of each function.

equation	x-intercepts	x-coordinate of the vertex
$f(x) = (x + 3)(x - 5)$		
$g(x) = 2x(x - 3)$		
$h(x) = (x + 4)(4 - x)$		

2. Use graphing technology to graph functions f, g, and h. Use the graphs to check your predictions.
3. Without using technology, sketch a graph that represents the equation $y = (x - 7)(x + 11)$ and that shows the x-intercepts and the vertex. Think about how to find the y-coordinate of the vertex. Be prepared to explain your reasoning.



Are you ready for more?

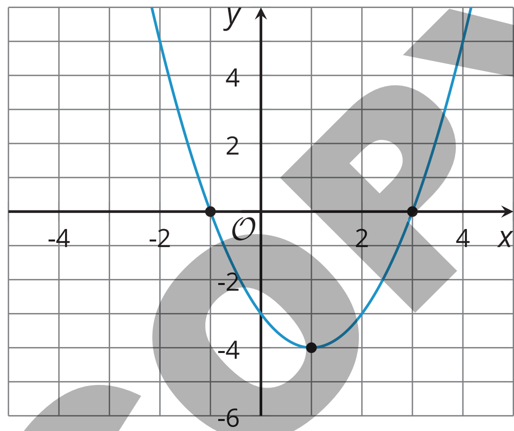
A quadratic function, f, is given by $f(x) = x^2 + 2x + 6$.

1. Find $f(\text{-}2)$ and $f(0)$.
2. What is the x-coordinate of the vertex of the graph of this quadratic function?
3. Does the graph have any x-intercepts? Explain or show how you know.

Lesson 11 Summary

Function f, given by $f(x) = (x+1)(x-3)$, is written in factored form. Recall that this form is helpful for finding the zeros of the function (where the function has the value 0) and for telling us the x-intercepts on the graph that represents the function.

Here is a graph representing f. It shows two x-intercepts: one at $x = -1$ and one at $x = 3$.



If we use -1 and 3 as inputs to f, what are the outputs?

- $f(-1) = (-1+1)(-1-3) = (0)(-4) = 0$
- $f(3) = (3+1)(3-3) = (4)(0) = 0$

Because the inputs -1 and 3 produce an output of 0, they are the zeros of function f. And because both x values have 0 for their y value, they also give us the x-intercepts of the graph (the points where the graph crosses the x-axis, which always have a y-coordinate of 0). So, the zeros of a function have the same values as the x-coordinates of the x-intercepts of the graph of the function.

The factored form can also help us identify the vertex of the graph, which is the point where the function reaches its minimum value. Notice that due to the symmetry of the parabola, the x-coordinate of the vertex is 1, and that 1 is halfway between -1 and 3. Once we know the x-coordinate of the vertex, we can find its y-coordinate by evaluating the function: $f(1) = (1+1)(1-3) = 2(-2) = -4$. So the vertex is at $(1, -4)$.

When a quadratic function is in standard form, the y-intercept is clear: its y-coordinate is the constant term c in $ax^2 + bx + c$. To find the y-intercept from factored form, we can evaluate the function at $x = 0$, because the y-intercept is the point at which the graph has an input value of 0. $f(0) = (0+1)(0-3) = (1)(-3) = -3$.

Sec D

Lesson 11 Practice Problems

1 Select **all** true statements about the graph that represents $y = 2x(x - 11)$.

A. Its x-intercepts are at $(-2, 0)$ and $(11, 0)$.

B. Its x-intercepts are at $(0, 0)$ and $(11, 0)$.

C. Its x-intercepts are at $(2, 0)$ and $(-11, 0)$.

D. It has only one x-intercept.

E. The x-coordinate of its vertex is -4.5.

F. The x-coordinate of its vertex is 11.

G. The x-coordinate of its vertex is 4.5.

H. The x-coordinate of its vertex is 5.5.

2 Select **all** equations whose graphs have a vertex with x-coordinate 2.

A. $y = (x - 2)(x - 4)$

B. $y = (x - 2)(x + 2)$

C. $y = (x - 1)(x - 3)$

D. $y = x(x + 4)$

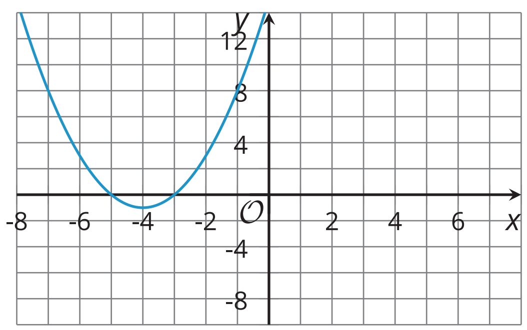
E. $y = x(x - 4)$

3 Determine the x-intercepts and the x-coordinate of the vertex of the graph that represents each equation.

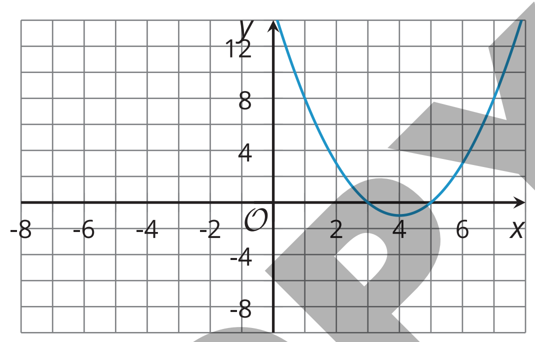
equation	x-intercepts	x-coordinate of the vertex
$y = x(x - 2)$		
$y = (x - 4)(x + 5)$		
$y = -5x(3 - x)$		

4 Which one is the graph of the equation $y = (x - 3)(x + 5)$?

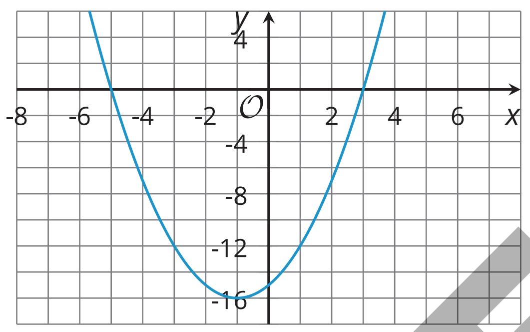
Graph A



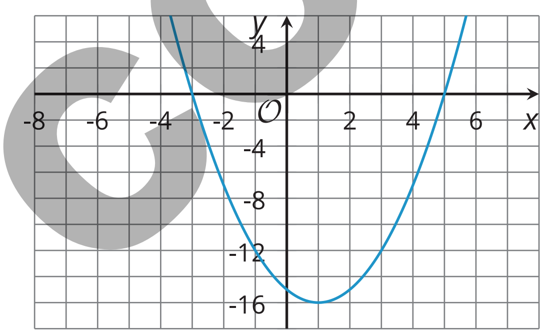
Graph B



Graph C



Graph D



A. Graph A

B. Graph B

C. Graph C

D. Graph D

5 a. What are the x-intercepts of the graph of $y = (x - 2)(x - 4)$?

b. Find the coordinates of another point on the graph. Show your reasoning.

c. Sketch a graph of the equation $y = (x - 2)(x - 4)$.

6 from Unit 7, Lesson 7

A company sells calculators. If the price of the calculator in dollars is p, the company estimates that it will sell $10{,}000 - 120p$ calculators.

Write an expression that represents the revenue in dollars from selling calculators if a calculator is priced at p dollars.

7 from Unit 7, Lesson 8

Is $(s+t)^2$ equivalent to $s^2 + 2st + t^2$? Explain or show your reasoning.

8 from Unit 6, Lesson 17

Tyler is shopping for a truck. He found two trucks that he likes. One truck sells for \$7,200. A slightly older truck sells for 15% less. How much does the older truck cost?

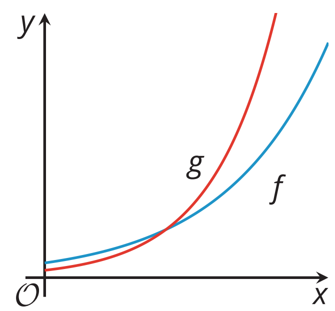
Sec D

9 from Unit 6, Lesson 16

Here are graphs of two exponential functions, f and g.

Function f is given by $f(x) = 100 \cdot 2^x$, and function g is given by $g(x) = a \cdot b^x$.

Based on the graphs of the functions, what can you conclude about a and b?



10 from Unit 5, Lesson 2

Suppose G takes a student's grade and gives a student's name as the output. Explain why G is not a function.

Unit 7, Lesson 12

Addressing CA CCSSM F-BF.3, F-IF.7a, F-IF.7, F-IF.9, F-LE.2; practicing MP2, MP3, MP5, MP7, MP8

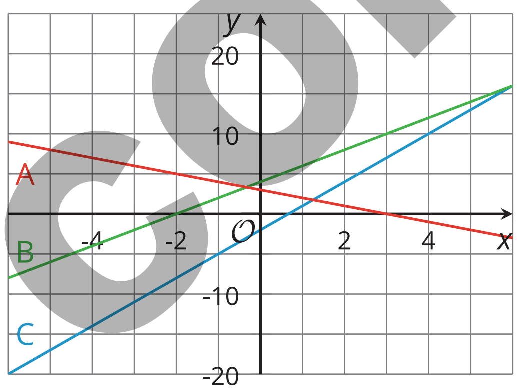
Graphing the Standard Form (Part 1)

Let's see how the numbers in expressions like $-3x^2 + 4$ affect their graph.

12.1 Matching Graphs to Linear Equations

Which graph corresponds to which equation? Explain your reasoning.

1. $y = 2x + 4$
2. $y = 3 - x$
3. $y = 3x - 2$

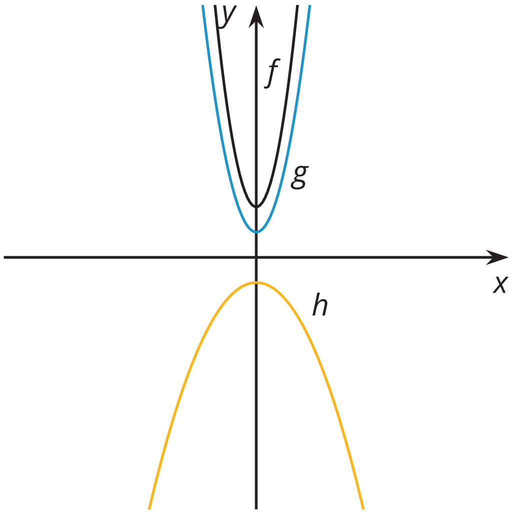


12.2 Quadratic Graphs Galore

Using graphing technology, graph $y = x^2$, and then experiment with each of the following changes to the function. Record your observations (include sketches, if helpful).

1. Add different constant terms to x^2 (for example: $x^2 + 5$, $x^2 + 10$, or $x^2 - 3$)
2. Multiply x^2 by different positive coefficients greater than 1 (for example: $3x^2$ or $7.5x^2$)
3. Multiply x^2 by different negative coefficients less than or equal to -1 (for example: $-x^2$ or $-4x^2$)
4. Multiply x^2 by different coefficients between -1 and 1 (for example: $\frac{1}{2}x^2$ or $-0.25x^2$)

Are you ready for more?



Here are the graphs of three quadratic functions. What can you say about the coefficients of x^2 in the expressions that define f (at the top), g (in the middle), and h (at the bottom)? Can you find the values of the coefficients? How do they compare?

Sec D

12.3 What Do These Tables Reveal?

1. a. Complete the table with values of $x^2 + 10$ and $x^2 - 3$ at different values of x.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 10$							
$x^2 - 3$							

   b. Earlier, you observed the effects on the graph of adding or subtracting a constant term to or from x^2. Study the values in the table. Use them to explain why the graphs changed the way they did when a constant term was added or subtracted.

2. a. Complete the table with values of $2x^2$, $\frac{1}{2}x^2$, and $-2x^2$ at different values of x.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x^2$							
$\frac{1}{2}x^2$							
$-2x^2$							

   b. You also observed the effects on the graph of multiplying x^2 by different coefficients. Study the values in the table. Use them to explain why the graphs changed the way they did when x^2 is multiplied by a number greater than 1, by a negative number less than or equal to -1, and by numbers between -1 and 1.

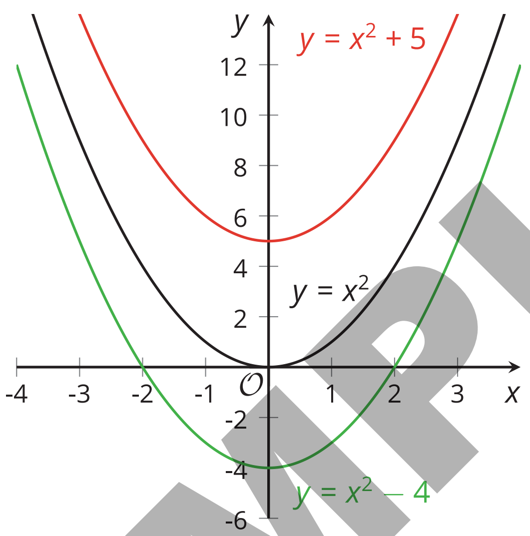
12.4 Card Sort: Representations of Quadratic Functions

Your teacher will give your group a set of cards. Each card contains a graph or an equation. Sort the cards into sets so that each set contains two equations and a graph that all represent the same quadratic function. Record your matches, and be prepared to explain your reasoning.

Lesson 12 Summary

Remember that the graph representing any quadratic function is a shape called a *parabola*. People often say that a parabola "opens upward" when the lowest point on the graph is the vertex (where the graph changes direction), and "opens downward" when the highest point on the graph is the vertex. Each coefficient in a quadratic expression written in standard form $ax^2 + bx + c$ tells us something important about the graph that represents it.

The graph of $y = x^2$ is a parabola opening upward with vertex at $(0, 0)$. Adding a constant term 5 gives $y = x^2 + 5$ and raises the graph by 5 units. Subtracting 4 from x^2 gives $y = x^2 - 4$ and moves the graph 4 units down.



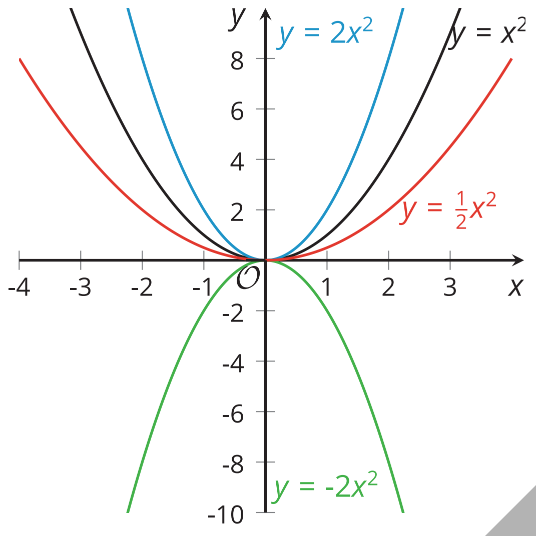
x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 5$	14	9	6	5	6	9	14
$x^2 - 4$	5	0	-3	-4	-3	0	5

A table of values can help us see that adding 5 to x^2 increases all the output values of $y = x^2$ by 5, which explains why the graph moves up 5 units. Subtracting 4 from x^2 decreases all the output values of $y = x^2$ by 4, which explains why the graph shifts down by 4 units.

In general, the constant term of a quadratic expression in standard form influences the vertical position of the graph. An expression with no constant term (such as x^2 or $x^2 + 9x$) means that the constant term is 0, so the y-intercept of the graph is on the x-axis. It's not shifted up or down relative to the x-axis.

The coefficient of the squared term in a quadratic function also tells us something about its graph. The coefficient of the squared term in $y = x^2$ is 1. Its graph is a parabola that opens upward.

- Multiplying x^2 by a number greater than 1 makes the graph steeper, so the parabola is narrower than that representing x^2.
- Multiplying x^2 by a number less than 1 but greater than 0 makes the graph less steep, so the parabola is wider than that representing x^2.
- Multiplying x^2 by a number less than 0 makes the parabola open downward.



x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x^2$	18	8	2	0	2	8	18
$-2x^2$	-18	-8	-2	0	-2	-8	-18

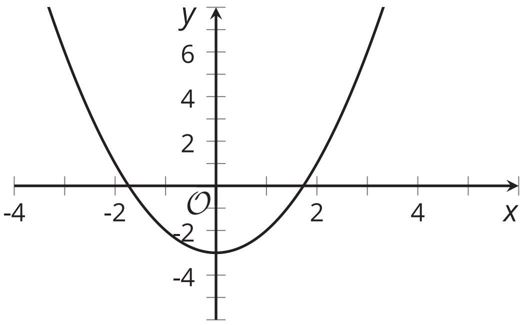
If we compare the output values of $2x^2$ and $-2x^2$, we see that they are opposites, which suggests that one graph would be a reflection of the other across the x-axis.

Sec D

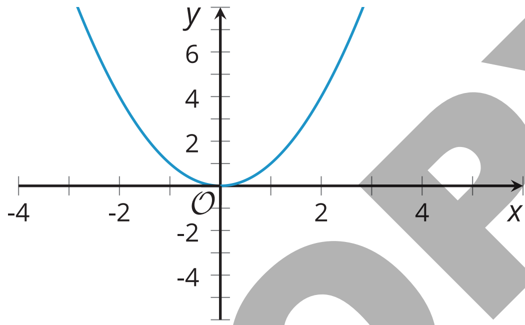
Lesson 12 Practice Problems

1 Here are four graphs. Match each graph with a quadratic equation that it represents.

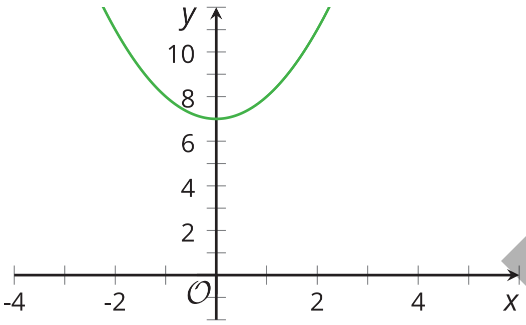
Graph A



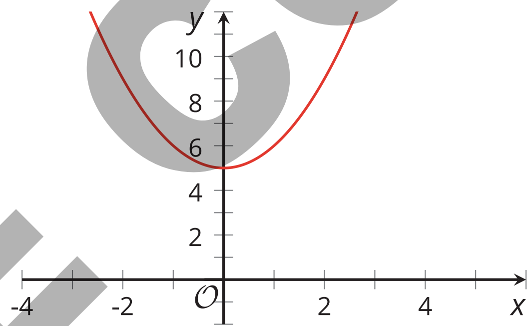
Graph B



Graph C



Graph D



A. Graph A
B. Graph B
C. Graph C
D. Graph D

1. $y = x^2$
2. $y = x^2 + 5$
3. $y = x^2 + 7$
4. $y = x^2 - 3$

2 The two equations $y = (x+2)(x+3)$ and $y = x^2 + 5x + 6$ are equivalent.

a. Which equation helps find the x-intercepts most efficiently?

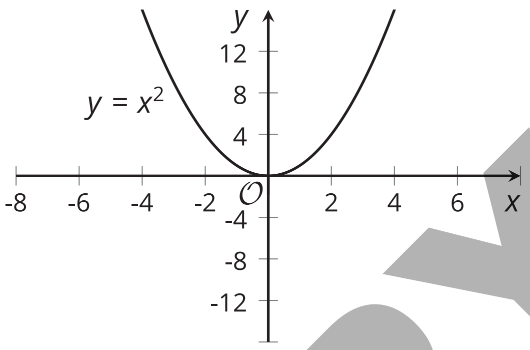
b. Which equation helps find the y-intercept most efficiently?

3 Here is a graph that represents $y = x^2$.

On the same coordinate plane, sketch and label the graph that represents each equation:

a. $y = x^2 - 4$

b. $y = -x^2 + 5$



4 Select **all** equations whose graphs have a y-intercept with a positive y-coordinate.

A. $y = x^2 + 3x - 2$

B. $y = x^2 - 10x$

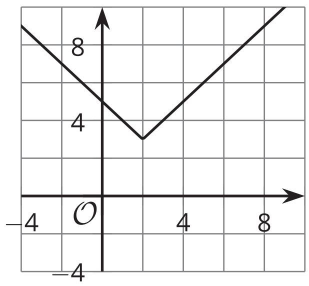
C. $y = (x - 1)^2$

D. $y = 5x^2 - 3x - 5$

E. $y = (x + 1)(x + 2)$

5 from Unit 5, Lesson 14

a. Describe how the graph of $A(x) = |x|$ has to be shifted to match the given graph.

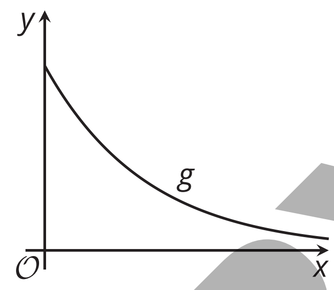


b. Write an equation for the function represented by the graph.

6 from Unit 6, Lesson 16

Here is a graph of a function, g, given by $g(x) = a \cdot b^x$.

What can you say about the value of b? Explain how you know.



7 from Unit 7, Lesson 11

a. What are the x-intercepts of the graph that represents $y = (x+1)(x+5)$? Explain how you know.

b. What is the x-coordinate of the vertex of the graph that represents $y = (x+1)(x+5)$? Explain how you know.

c. Find the y-coordinate of the vertex. Show your reasoning.

d. Sketch a graph of $y = (x+1)(x+5)$.

8 from Unit 7, Lesson 11

Determine the x-intercepts, the vertex, and the y-intercept of the graph of each equation.

equation	x-intercepts	vertex	y-intercept
$y = (x-5)(x-3)$			
$y = 2x(8-x)$			

9 from Unit 6, Lesson 18

Equal amounts of money were invested in stock A and stock B. In the first year, stock A increased in value by 20%, and stock B decreased by 20%. In the second year, stock A decreased in value by 20%, and stock B increased by 20%.

Was one stock a better investment than the other? Explain your reasoning.

Unit 7, Lesson 13

Addressing CA CCSSM A-SSE.3, F-BF.3, F-IF.7, F-IF.7a; practicing MP7 and MP8

Graphing the Standard Form (Part 2)

Let's change some other parts of a quadratic expression and see how they affect its graph.

13.1 Equivalent Expressions

1. Complete each row with an equivalent expression in standard form or factored form.

standard form	factored form
x^2	
	$x(x+9)$
$x^2 - 18x$	
	$x(6-x)$
$-x^2 + 10x$	
	$-x(x+2.75)$

2. What do the quadratic expressions in each column have in common (besides the fact that everything in the left column is in standard form and everything in the other column is in factored form)? Be prepared to share your observations.

13.2 What about the Linear Term?

1. Using graphing technology:

   a. Graph $y = x^2$, and then experiment with adding different linear terms (for example, $x^2 + 4x$, $x^2 + 20x$, $x^2 - 50x$). Record your observations.

   b. Graph $y = -x^2$, and then experiment with adding different linear terms. Record your observations.

2. Use your observations to help you complete the table, without graphing the equations.

equation	x-intercepts	x-coordinate of vertex
$y = x^2 + 6x$		
$y = x^2 - 10x$		
$y = -x^2 + 50x$		
$y = -x^2 - 36x$		

3. Some quadratic expressions have no linear terms. Find the x-intercepts and the x-coordinate of the vertex of the graph representing each equation. (Note that it is possible for the graph to not intersect the x-axis.) If you get stuck, try graphing the equations.

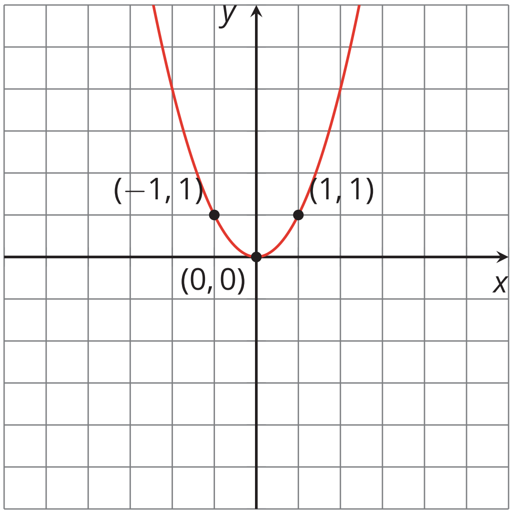
   a. $y = x^2 - 25$

   b. $y = x^2 + 16$

13.3 Writing Equations to Match Graphs

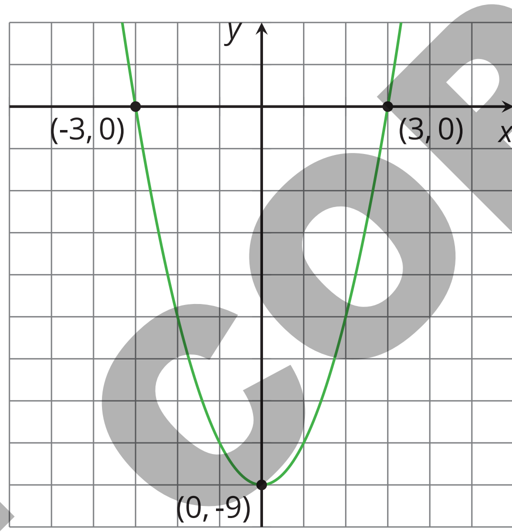
Use graphing technology to graph a function that matches each given graph. Make sure that your graph goes through all 3 points shown!

A



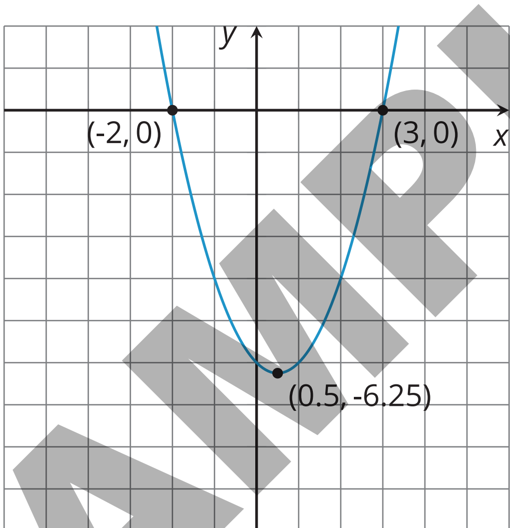
Equation:

B



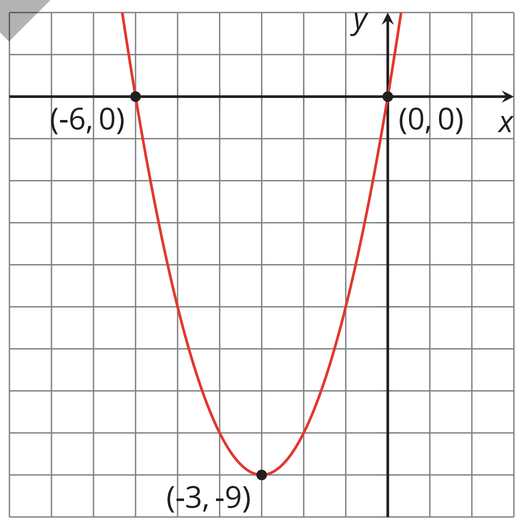
Equation:

C



Equation:

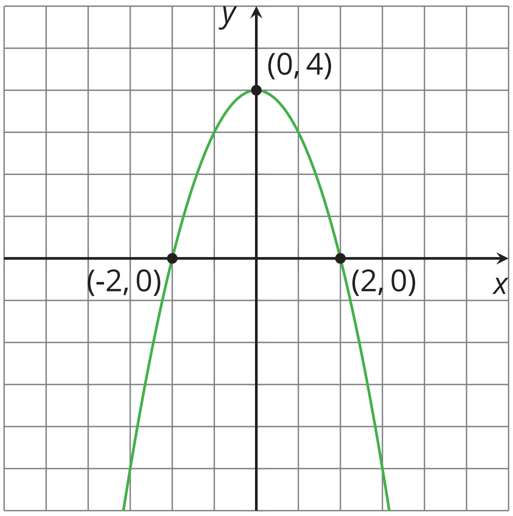
D



Equation:

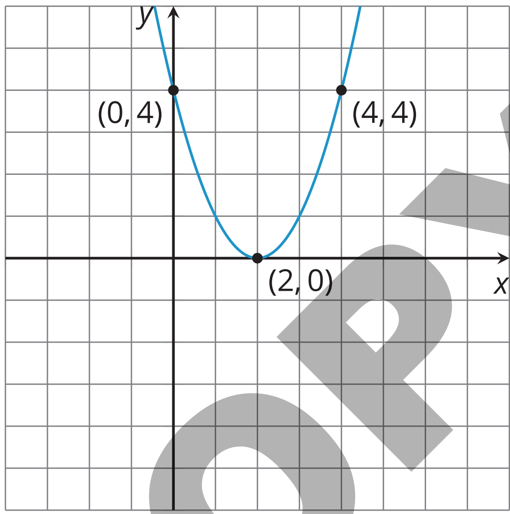
Sec D

E



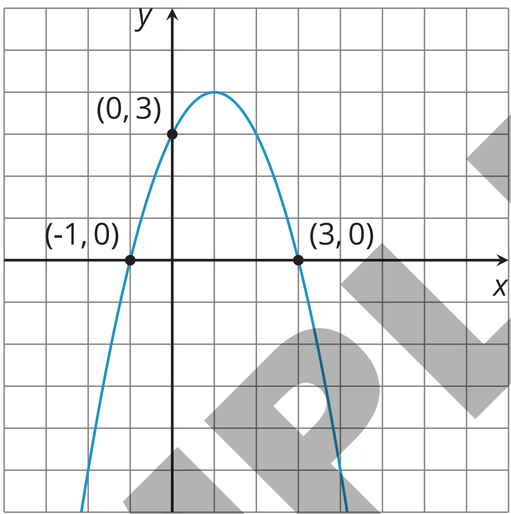
Equation:

F



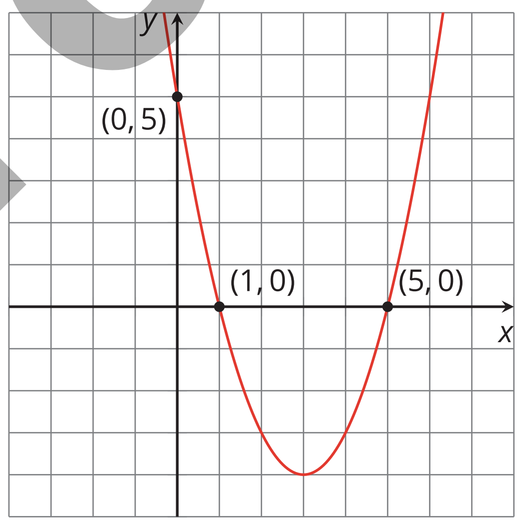
Equation:

G



Equation:

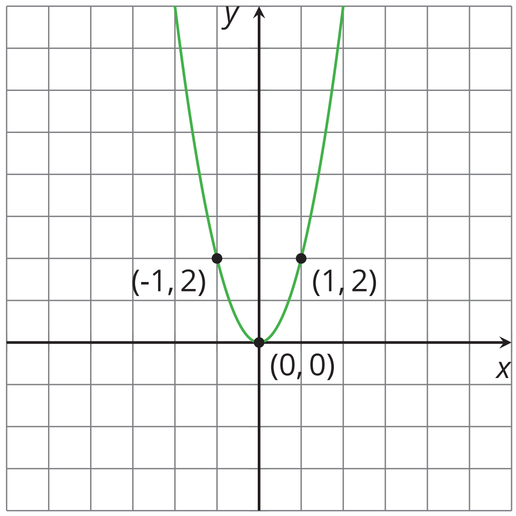
H



Equation:

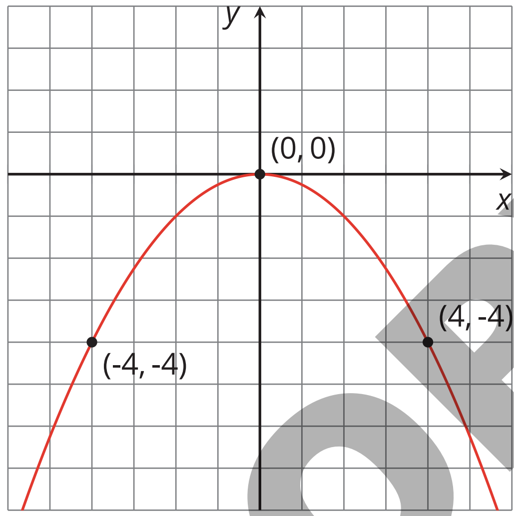
Sec D

I



Equation:

J



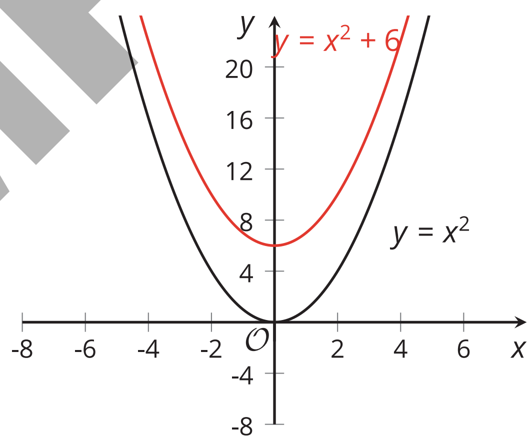
Equation:

Lesson 13 Summary

In an earlier lesson, we saw that a quadratic function written in standard form, $ax^2 + bx + c$, can tell us some things about the graph that represents it. The coefficient a can tell us whether the graph of the function opens upward or downward, and also gives us information about whether it is narrow or wide. The constant term c can tell us about its vertical position.

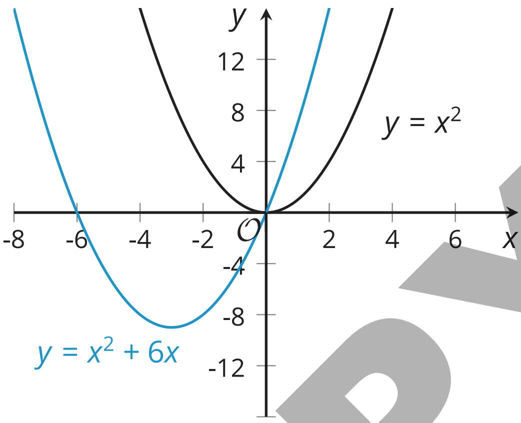
Recall that the graph representing $y = x^2$ is an upward-opening parabola with the vertex at $(0, 0)$. The vertex is also the x-intercept and the y-intercept.

Suppose we add 6 to the squared term: $y = x^2 + 6$. Adding a 6 shifts the graph upward, so the vertex is at $(0, 6)$. The vertex is the y-intercept, and the graph is centered on the y-axis.



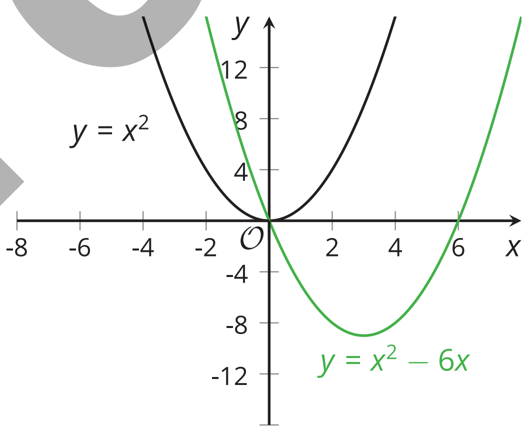
What can the linear term bx tell us about the graph representing a quadratic function?

The linear term has a somewhat mysterious effect on the graph of a quadratic function. The graph seems to shift both horizontally and vertically. When we add bx (where b is not 0) to x^2, the graph of $y = x^2 + bx$ is no longer centered on the y-axis.



Suppose we add $6x$ to the squared term: $y = x^2 + 6x$. Writing the $x^2 + 6x$ in factored form as $x(x + 6)$ gives us the zeros of the function, 0 and -6. Adding the term $6x$ seems to shift the graph to the left and down and the x-intercepts are now $(-6, 0)$ and $(0, 0)$. The vertex is no longer the y-intercept, and the graph is no longer centered on the y-axis.

What if we add $-6x$ to x^2? We know that $x^2 - 6x$ can be rewritten as $x(x - 6)$, which tells us the zeros: 0 and 6. Adding a negative linear term to a squared term seems to shift the graph to the right and down. The x-intercepts are now $(0, 0)$ and $(6, 0)$. The vertex is no longer the y-intercept, and the graph is not centered on the y-axis.

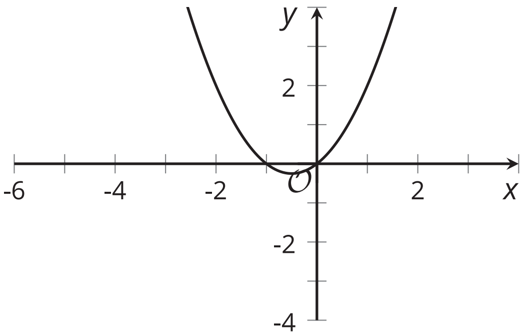


Sec D

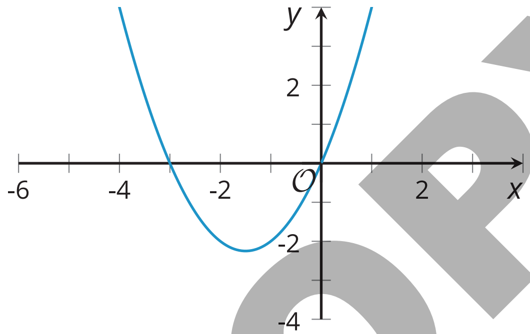
Lesson 13 Practice Problems

1 Here are four graphs. Match each graph with the quadratic equation that it represents.

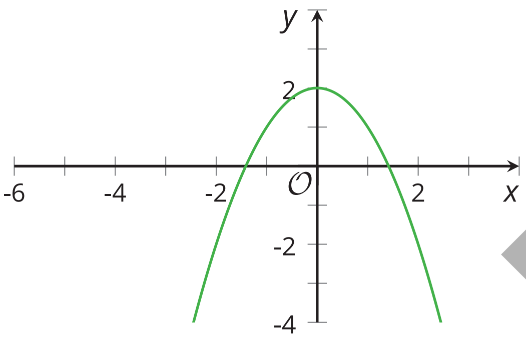
Graph A



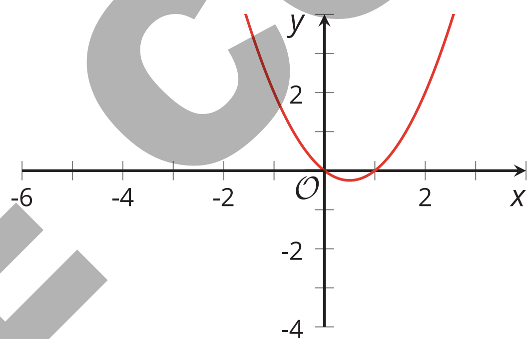
Graph B



Graph C



Graph D



A. Graph A
B. Graph B
C. Graph C
D. Graph D

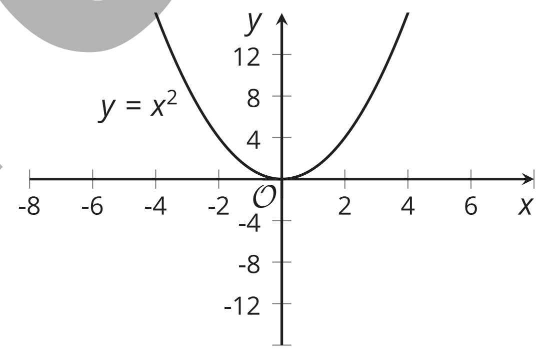
1. $y = x^2 + x$
2. $y = -x^2 + 2$
3. $y = x^2 - x$
4. $y = x^2 + 3x$

2 Complete the table without graphing the equations.

equation	x-intercepts	x-coordinate of the vertex
$y = x^2 + 12x$		
$y = x^2 - 3x$		
$y = -x^2 + 16x$		
$y = -x^2 - 24x$		

3 Here is a graph that represents $y = x^2$.

a. Describe what would happen to the graph if the original equation were changed to $y = x^2 - 6x$. Predict the x- and y-intercepts of the graph and the quadrant where the vertex is located.



b. Sketch the graph of the equation $y = x^2 - 6x$ on the same coordinate plane as $y = x^2$.

4 Select **all** equations whose graph opens upward.

A. $y = -x^2 + 9x$

B. $y = 10x - 5x^2$

C. $y = (2x - 1)^2$

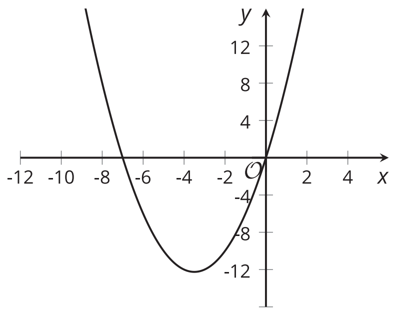
D. $y = (1 - x)(2 + x)$

E. $y = x^2 - 8x - 7$

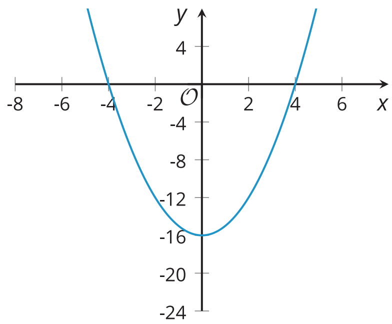
Sec D

5 *Technology required.* Write an equation for a function that can be represented by each given graph. Then, use graphing technology to check each equation you wrote.

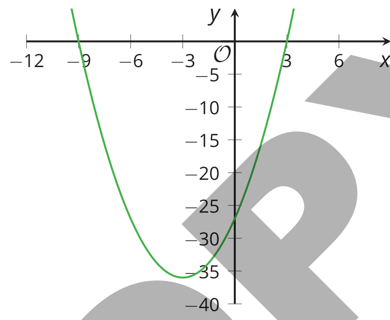
Graph 1



Graph 2



Graph 3



Sec D

6 from Unit 7, Lesson 8

Match each quadratic expression that is written as a product with an equivalent expression that is expanded.

A. $(x+3)(x+4)$

B. $(x+3)(x+7)$

C. $(3x+4)(x+3)$

D. $(x+7)(3x+1)$

1. $x^2+10x+21$
2. $3x^2+13x+12$
3. $3x^2+22x+7$
4. $x^2+7x+12$

7 from Unit 6, Lesson 17

When buying a home, many mortgage companies require a downpayment of 20% of the price of the house. What is the downpayment on a $125,000 home?

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8 from Unit 6, Lesson 18

A bank lends $4,000 to a customer at a $9\frac{1}{2}\%$ annual interest rate.

Write an expression to represent how much the customer will owe, in dollars, after 5 years without payment.

Sec D

Unit 7, Lesson 14

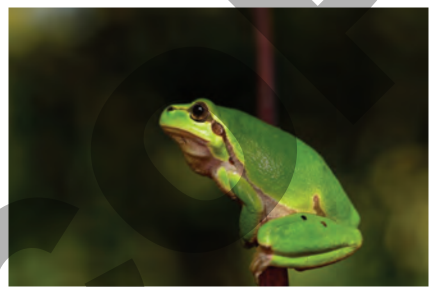
Addressing CA CCSSM F-IF.2, F-IF.4, F-IF.7a, F-IF.8, F-IF.9, F-LE.6; practicing MP1, MP2, MP6

Graphs That Represent Situations

Let's examine graphs that represent the paths of objects being launched into the air.

14.1 A Jumping Frog

The height in inches of a frog's jump is modeled by the equation $h(t) = 60t - 75t^2$, where the time, t, after it jumped is measured in seconds.



1. Find $h(0)$ and $h(0.8)$. What do these values mean in terms of the frog's jump?

2. How much time after it jumped did the frog reach the maximum height? Explain how you know.

Sec D

14.2 A Catapulted Pumpkin

The equation $h = 2 + 23.7t - 4.9t^2$ represents the height of a pumpkin that is catapulted up in the air as a function of time, t, in seconds. The height is measured in meters above ground. The pumpkin is shot upward at a vertical velocity of 23.7 meters per second.

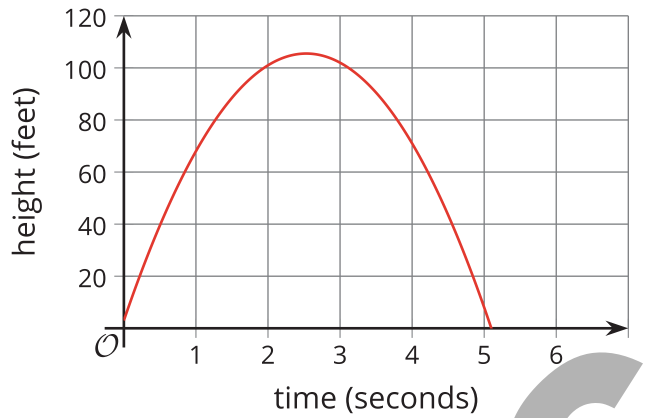
1. Without writing anything down, consider these questions:
   - What do you think the 2 in the equation tells us in this situation? What about the $-4.9t^2$?
   - If we graph the equation, will the graph open upward or downward? Why?
   - Where do you think the vertical intercept would be?
   - What about the horizontal intercepts?
2. Graph the equation using graphing technology.
3. Identify the vertical and horizontal intercepts, and the vertex of the graph. Explain what each point means in this situation.

Are you ready for more?

What approximate initial vertical velocity would this pumpkin need for it to stay in the air for about 10 seconds? (Assume that it is still launched from 2 meters above ground and that the effect of gravity pulling it downward is the same.)

14.3 Flight of Two Baseballs

Here is a graph that represents the height of a baseball, h, in feet, as a function of time, t, in seconds, after it was hit by Player A.



Player B hits a baseball that has its height, in feet, t seconds after it was hit represented by the function $g(t) = (\text{-}16t - 1)(t - 4)$. Without graphing function g, answer the questions, and explain or show how you know.

Sec D

1. Which player's baseball stayed in flight longer?

2. Which player's baseball reached a greater maximum height?

3. How can you find the height at which each baseball was hit?

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14.4 Information Gap: Rocket Math

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the problem card:

1. Silently read your card, and think about what information you need in order to answer the question.
2. Ask your partner for the specific information that you need. "Can you tell me _____?"
3. Explain to your partner how you are using the information to solve the problem. "I need to know _____ because . . . ." Continue to ask questions until you have enough information to solve the problem.
4. Once you have enough information, share the problem card with your partner, and solve the problem independently.
5. Read the data card, and discuss your reasoning.

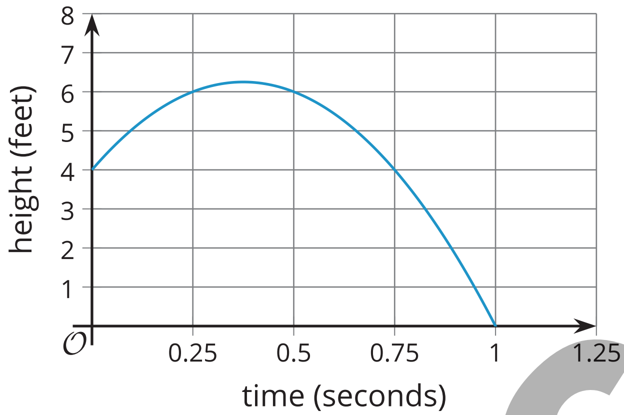
If your teacher gives you the data card:

1. Silently read your card. Wait for your partner to ask for information.
2. Before telling your partner any information, ask, "Why do you need to know _____?"
3. Listen to your partner's reasoning and ask clarifying questions. Give only the information that is on your card. Do not figure out anything for your partner! These steps may be repeated.
4. Once your partner has enough information to solve the problem, read the problem card, and solve the problem independently.
5. Share the data card, and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards, and repeat the activity, trading roles with your partner.

Lesson 14 Summary

Let's say that a tennis ball is hit straight up into the air, and its height, in feet, above the ground is modeled by the equation $f(t) = 4 + 12t - 16t^2$, where t represents the time, in seconds, after the ball is hit. Here is a graph that represents the function, from the time the tennis ball was hit until the time it reached the ground.



In the graph, we can see some information we already know, and some new information:

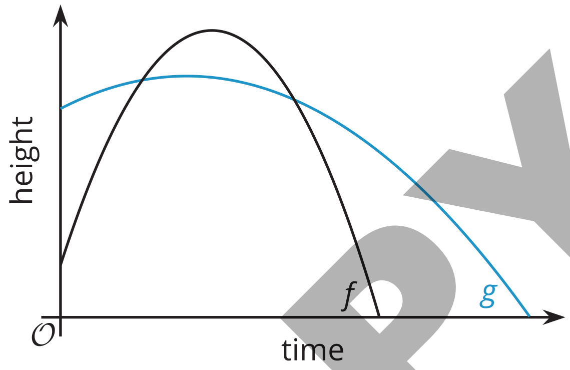
- The 4 in the equation means the graph of the function intersects the vertical axis at 4. It shows that the tennis ball was 4 feet off the ground at $t = 0$, when it was hit.
- The horizontal intercept is $(1, 0)$. It tells us that the tennis ball hit the ground 1 second after it was hit.
- The vertex of the graph is at approximately $(0.4, 6.3)$. This means that about 0.4 second after the ball was hit, it reached the maximum height of about 6.3 feet.

The equation can be written in factored form as $f(t) = (\text{-}16t - 4)(t - 1)$. From this form, we can see that the zeros of the function are $t = 1$ and $t = \text{-}\frac{1}{4}$. The negative zero, $\text{-}\frac{1}{4}$, is not meaningful in this situation, because the time before the ball was hit is irrelevant.

Lesson 14 Practice Problems

1 Two objects are launched into the air at the same time. Functions f and g represent the heights of the objects above the ground, as a function of time, t seconds after being launched.

Here are the graphs of f and g.



a. Which object was launched from a higher point?

b. Which object reached a higher point?

c. Which object was launched with the higher upward velocity?

d. Which object landed last?

2 *Technology required.* A function, h, given by $h(t) = (1 - t)(8 + 16t)$ models the height of a ball, in feet, t seconds after it was thrown.

a. Find the zeros of the function. Show or explain your reasoning.

b. What do the zeros tell us in this situation? Are both zeros meaningful?

c. From what height is the ball thrown? Explain your reasoning.

d. About when does the ball reach its highest point, and about how high does the ball go? Show or explain your reasoning.

3 The height, in feet, of a thrown football is modeled by the equation $f(t) = 6 + 30t - 16t^2$, where time, t, is measured in seconds.

a. What does the constant 6 mean in this situation?

b. What does the $30t$ mean in this situation?

c. How do you think the squared term $-16t^2$ affects the value of the function f? What does this term reveal about the situation?

4 The height, in feet, of an arrow is modeled by the equation $h(t) = (1 + 2t)(18 - 8t)$, where t is seconds after the arrow is shot.

a. When does the arrow hit the ground? Explain or show your reasoning.

b. From what height is the arrow shot? Explain or show your reasoning.

5 Two objects are launched into the air.

- The height, in feet, of Object A is given by the equation $f(t) = 4 + 32t - 16t^2$.
- The height, in feet, of Object B is given by the equation $g(t) = 2.5 + 40t - 16t^2$. In both functions, t is seconds after launch.

a. Which object was launched from a greater height? Explain how you know.

b. Which object was launched with a greater upward velocity? Explain how you know.

6 from Unit 7, Lesson 10

a. Find the x- and y-intercepts of the graph of the quadratic function defined by the equation $y = (x + 6)(x - 6)$, without graphing. Explain or show your reasoning.

b. *Technology required.* Check your predictions by graphing $y = (x + 6)(x - 6)$.

7 from Unit 6, Lesson 18

Technology required. A student needs to get a loan of $12,000 for the first year of college. Bank A has an annual interest rate of 5.75%, Bank B has an annual interest rate of 7.81%, and Bank C has an annual rate of 4.45%.

a. If we graph the amount owed for each loan as a function of years without payment, predict what the three graphs would look like. Describe or sketch your prediction.

b. Use graphing technology to plot the graph of each loan balance.

c. Based on your graph, how much would the student owe for each loan by graduation from college in four years?

d. Based on your graph, if no payments are made, how much would the student owe for each loan after 10 years?

Sec D

8 from Unit 6, Lesson 22

Technology required. Functions f and g are given by $f(x) = 13x + 6$ and $g(x) = 0.1 \cdot (1.4)^x$.

a. Which function eventually grows faster, f or g? Explain how you know.

b. Use graphing technology to decide when the graphs of f and g meet.

Unit 7, Lesson 15
Addressing CA CCSSM F-BF.3, F-IF.7a; building toward F-IF.8a; practicing MP3, MP6, MP7, MP8

Vertex Form

Let's find out about the vertex form.

15.1 Notice and Wonder: Two Sets of Equations

What do you notice? What do you wonder?

Set 1:

$$f(x) = x^2 + 4x$$

$$g(x) = x(x + 4)$$

$$h(x) = (x + 2)^2 - 4$$

Set 2:

$$p(x) = -x^2 + 6x - 5$$

$$q(x) = (5 - x)(x - 1)$$

$$r(x) = -1(x - 3)^2 + 4$$

Sec D

15.2 A Whole New Form

Here are two sets of equations for quadratic functions that you saw earlier. In each set, the expressions that define the output are equivalent.

Set 1:

$$f(x) = x^2 + 4x$$

$$g(x) = x(x+4)$$

$$h(x) = (x+2)^2 - 4$$

Set 2:

$$p(x) = -x^2 + 6x - 5$$

$$q(x) = (5-x)(x-1)$$

$$r(x) = -1(x-3)^2 + 4$$

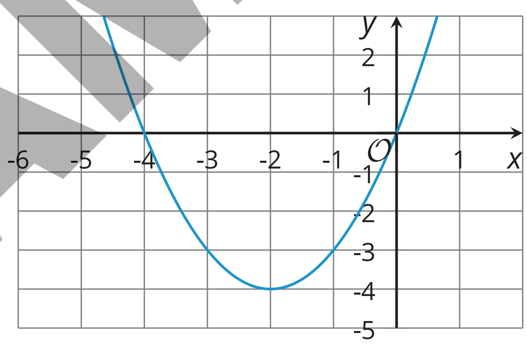
The expressions that define h and r are written in **vertex form**. We can show that h is equivalent to the expression defining f by expanding the expression:

$$\begin{aligned}(x+2)^2 - 4 &= (x+2)(x+2) - 4\\ &= x^2 + 2x + 2x + 4 - 4\\ &= x^2 + 4x\end{aligned}$$

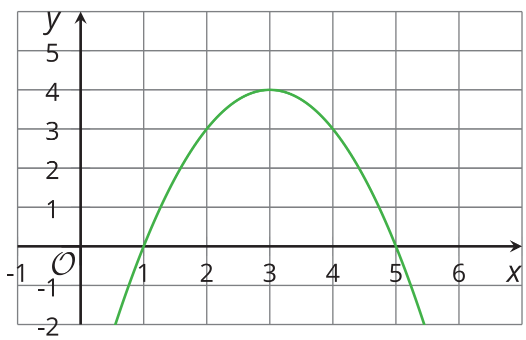
1. Show that the expressions defining r and p are equivalent.

2. Here are graphs representing the quadratic functions. Why do you think expressions such as those defining h and r are said to be written in vertex form?

Graph of h



Graph of r



15.3 Playing with Parameters

1. Using graphing technology, graph $y = x^2$. Then, add different numbers to x before it is squared (for example, $y = (x+4)^2$, $y = (x-3)^2$), and observe how the graph changes. Record your observations.

2. Graph $y = (x-1)^2$. Then, experiment with each of the following changes to the function, and see how they affect the graph and the vertex:

   a. Adding different constant terms to $(x-1)^2$ (for example: $(x-1)^2 + 5$, $(x-1)^2 - 9$).

   b. Multiplying $(x-1)^2$ by different coefficients (for example: $y = 3(x-1)^2$, $y = -2(x-1)^2$).

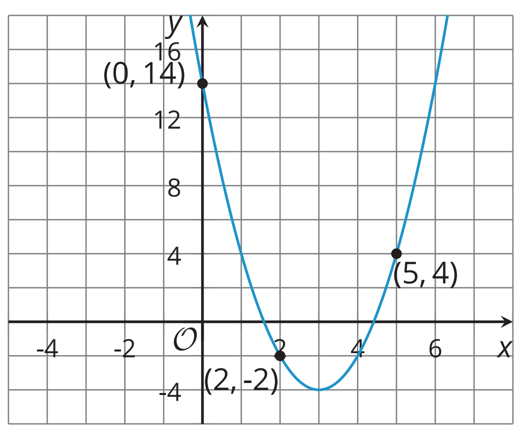
3. Without graphing, predict the coordinates of the vertex of the graphs of these quadratic functions, and predict whether the graph opens upward or opens downward. Ignore the last row until the next question.

equations	coordinates of vertex	graph opens upward or downward?
$y = (x + 10)^2$		
$y = (x - 4)^2 + 8$		
$y = -(x - 4)^2 + 8$		
$y = x^2 - 7$		
$y = \frac{1}{2}(x + 3)^2 - 5$		
$y = -(x + 100)^2 + 50$		
$y = a(x + m)^2 + n$		

4. Use graphing technology to check your predictions. If they are incorrect, revise them. Then complete the last row of the table.

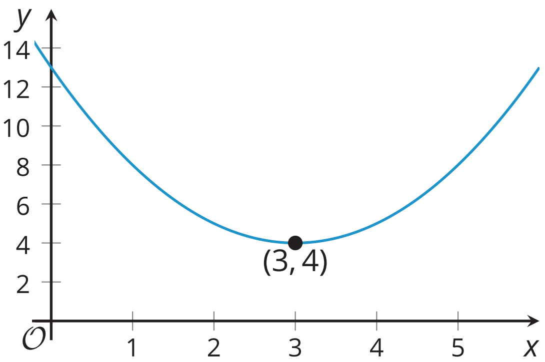
Are you ready for more?

1. What is the vertex of this graph?
2. Find a quadratic equation whose graph has the same vertex, and adjust it (if needed) so that it has the graph shown.



Lesson 15 Summary

Sometimes the expressions that define quadratic functions are written in **vertex form**. The function $f(x) = (x - 3)^2 + 4$ is in vertex form and is shown in this graph.



The vertex form can tell us about the coordinates of the vertex of the graph of a quadratic function. The expression $(x - 3)^2$ reveals that the x-coordinate of the vertex is 3, and the constant term, 4, reveals that the y-coordinate of the vertex is 4. Here the vertex represents the minimum value of function f, and its graph opens upward.

In general, a quadratic function expressed in vertex form is written as

$$y = a(x - h)^2 + k$$

. The vertex of its graph is at (h, k). The graph of the quadratic function opens upward when the coefficient, a, is positive and opens downward when a is negative.

Glossary

- vertex form (of a quadratic expression)

Lesson 15 Practice Problems

1 Select **all** of the quadratic expressions in vertex form.

A. $(x-2)^2+1$

B. x^2-4

C. $x(x+1)$

D. $(x+3)^2$

E. $(x-4)^2+6$

2 Here are two equations. One defines function m and the other defines function p.

$m(x)=x(x+6)$ $\qquad$ $p(x)=(x+3)^2-9$

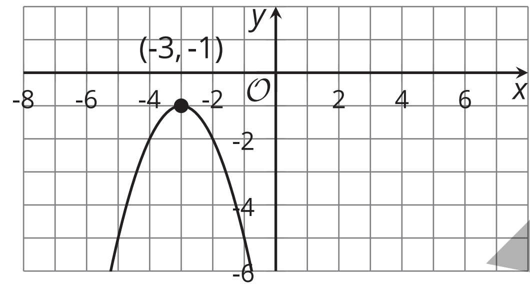
a. Show that the expressions defining m and p are equivalent.

b. What is the vertex of the graph of m? Explain how you know.

c. What are the x-intercepts of the graph of p? Explain how you know.

Sec D

3 Which equation is represented by the graph?



A. $y = (x - 1)^2 + 3$

B. $y = (x - 3)^2 + 1$

C. $y = \text{-}(x + 3)^2 - 1$

D. $y = \text{-}(x - 3)^2 + 1$

4 For each equation, write the coordinates of the vertex of the graph that represents the equation.

a. $y = (x - 3)^2 + 5$

b. $y = (x + 7)^2 + 3$

c. $y = (x - 4)^2$

d. $y = x^2 - 1$

e. $y = 2(x + 1)^2 - 5$

f. $y = \text{-}2(x + 1)^2 - 5$

5 For each function, write the coordinates of the vertex of its graph, and tell whether the graph opens upward or downward.

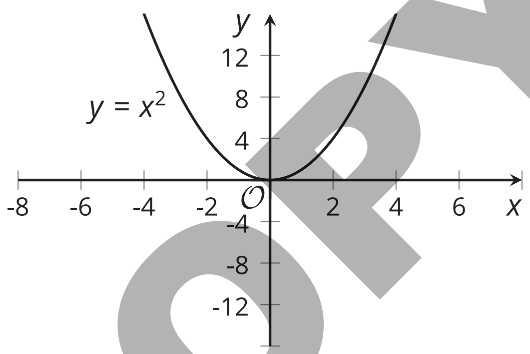
function	coordinates of vertex	graph opens upward or downward?
$f(x) = (x - 4)^2 - 5$		
$g(x) = \text{-}x^2 + 5$		
$h(x) = 2(x + 1)^2 - 4$		

6 from Unit 7, Lesson 12

Here is a graph that represents $y = x^2$.

a. Describe what would happen to the graph if the original equation were modified as follows:

i. $y = -x^2$



ii. $y = 3x^2$

iii. $y = x^2 + 6$

b. Sketch the graph of the equation $y = -3x^2 + 6$ on the same coordinate plane as $y = x^2$.

7 from Unit 6, Lesson 19

Noah is going to put $2,000 in a savings account. He plans on putting the money in an account and leaving it there for 5 years. He can put the money in an account that pays 1% interest monthly, an account that pays 6% interest every six months, or an account that pays 12% interest annually.

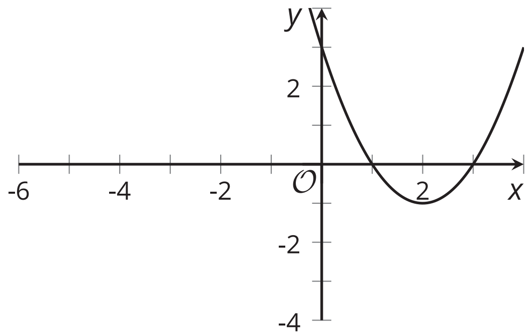
Which account will give him the most money in his account at the end of the 5 years?

Sec D

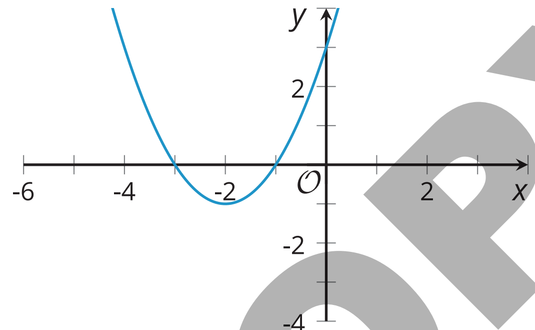
8 from Unit 7, Lesson 12

Here are four graphs. Match each graph with a quadratic equation that it represents.

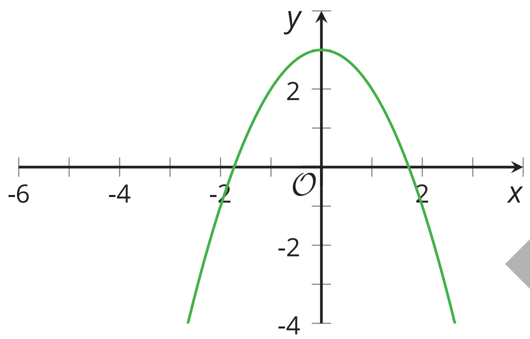
Graph A



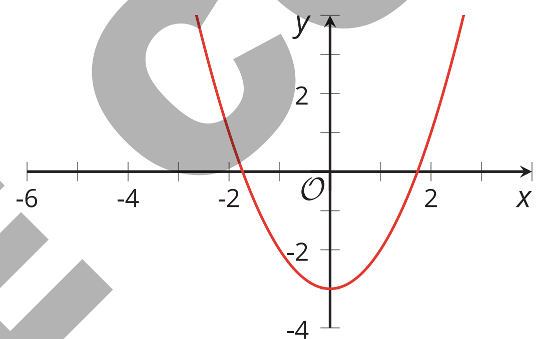
Graph B



Graph C



Graph D



A. Graph A

B. Graph B

C. Graph C

D. Graph D

1. $y = -x^2 + 3$
2. $y = (x + 1)(x + 3)$
3. $y = x^2 - 3$
4. $y = (x - 1)(x - 3)$

Sec D

9 from Unit 5, Lesson 5

The table shows some input and output values of function f. Describe a possible rule for the function by using words or by writing an equation.

x	$f(x)$
-3	-8
0	-2
4	6
10	18

Sec D

Graphing from the Vertex Form

Let's graph equations in vertex form.

16.1 Which Form to Use?

Expressions in different forms can be used to define the same function. Here are three ways to define a function, f.

$f(x) = x^2 - 4x + 3$	standard form
$f(x) = (x - 3)(x - 1)$	factored form
$f(x) = (x - 2)^2 - 1$	vertex form

Which form would you use if you want to find the following features of the graph of f? Be prepared to explain your reasoning.

1. the x-intercepts
2. the vertex
3. the y-intercept

16.2 Sharing a Vertex

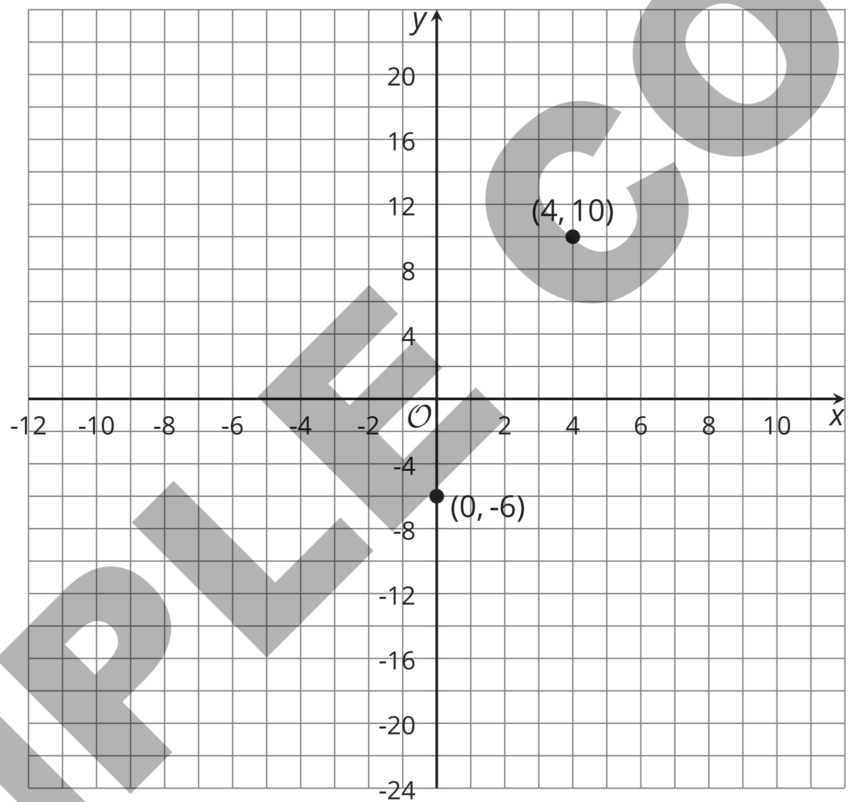
Here are two equations that define quadratic functions.

$p(x) = -(x - 4)^2 + 10$

$q(x) = \frac{1}{2}(x - 4)^2 + 10$

1. The graph of p passes through $(0, -6)$ and $(4, 10)$, as shown on the coordinate plane.

   Find the coordinates of another point on the graph of p. Explain or show your reasoning. Then use the points to sketch and label the graph.



2. On the same coordinate plane, identify the vertex and two other points that are on the graph of q. Explain or show your reasoning. Sketch and label the graph of q.

Sec D

3. Priya says, "Once I know that the vertex is $(4, 10)$, I can find out, without graphing, whether the vertex is the maximum or the minimum of function p. I can just compare the coordinates of the vertex with the coordinates of a point on either side of it."

   Complete the table, and then explain how Priya might have reasoned about whether the vertex is the minimum or maximum.

x	3	4	5
$p(x)$		10	

Are you ready for more?

1. Write a the equation for a quadratic function whose graph has the vertex at $(2, 3)$ and contains the point $(0, -5)$.
2. Sketch a graph of your function.

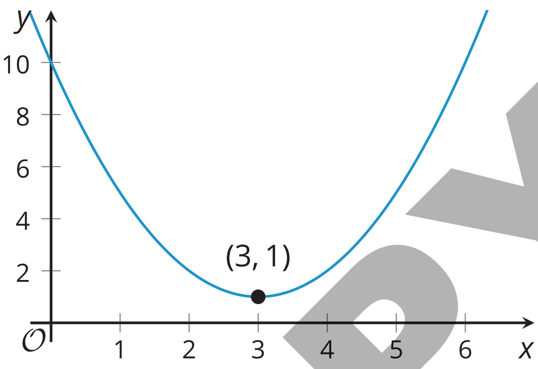
Sec D

16.3 Card Sort: Matching Equations with Graphs

Your teacher will give you a set of cards containing an equation or a graph that represents a quadratic function. Take turns matching each equation to a graph that represents the same function. Record your matches, and be prepared to explain your reasoning.

Lesson 16 Summary

Not surprisingly, vertex form is especially helpful for finding the vertex of a graph of a quadratic function. For example, we can tell that the function, p, given by $p(x) = (x - 3)^2 + 1$ has a vertex at $(3, 1)$.



We also noticed that, when the squared expression $(x - 3)^2$ has a positive coefficient, the graph opens upward. This means that the vertex, $(3, 1)$, represents the minimum function value, $p(x)$.

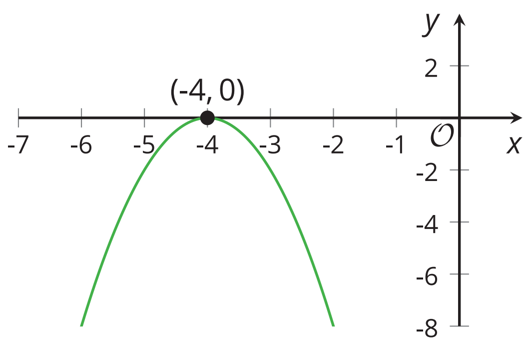
But why does function p take on its minimum value when x is 3?

Here is one way to explain it: When $x = 3$, the squared term $(x - 3)^2$ equals 0, because $(3 - 3)^2 = 0^2 = 0$. When x is any other value besides 3, the squared term $(x - 3)^2$ is a positive number greater than 0. (Squaring any number results in a positive number.) This means that the output when $x \neq 3$ will always be greater than the output when $x = 3$, so function p has a minimum value at $x = 3$.

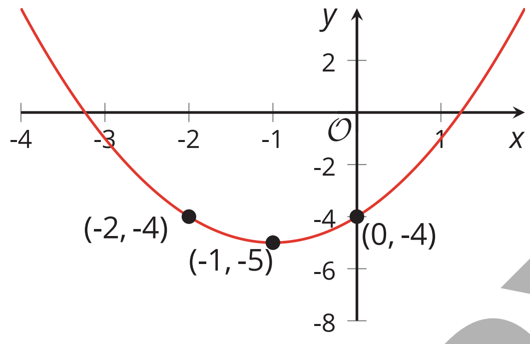
This table shows some values of the function for some values of x. Notice that the output is the least when $x = 3$, and it increases both as x increases and as it decreases.

x	0	1	2	3	4	5	6
$(x - 3)^2 + 1$	10	5	2	1	2	5	10

The squared term sometimes has a negative coefficient, for instance in $h(x) = -2(x + 4)^2$. The x value that makes $(x + 4)^2$ equal 0 is -4, because $(-4 + 4)^2 = 0^2 = 0$. Any other x value makes $(x + 4)^2$ greater than 0. But when $(x + 4)^2$ is multiplied by a negative number like -2, the resulting expression, $-2(x + 4)^2$, ends up being negative. This means that the output when $x \neq -4$ will always be less than the output when $x = -4$, so function h has its maximum value when $x = -4$.



Remember that we can find the y-intercept of the graph representing any function that we have seen. The y-coordinate of the y-intercept is the value of the function when $x = 0$. If g is defined by $g(x) = (x + 1)^2 - 5$, then the y-intercept is $(0, \text{-}4)$ because $g(0) = (0 + 1)^2 - 5 = \text{-}4$. Its vertex is at $(\text{-}1, \text{-}5)$. Another point on the graph with the same y-coordinate is located the same horizontal distance from the vertex but on the other side.



Sec D

Lesson 16 Practice Problems

1 Which equation can be represented by a graph with a vertex at $(1, 3)$?

A. $y = (x - 1)^2 + 3$

B. $y = (x + 1)^2 + 3$

C. $y = (x - 3)^2 + 1$

D. $y = (x + 3)^2 + 1$

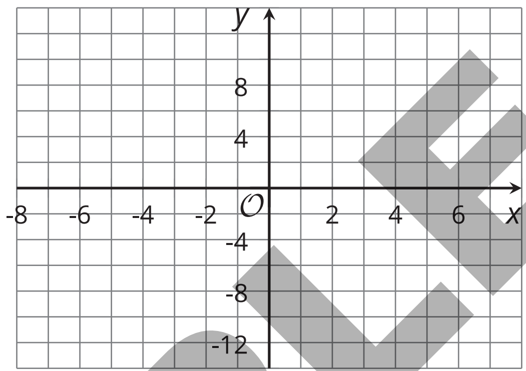
2

a. Where is the vertex of the graph that represents $y = (x - 2)^2 - 8$?

b. Where is the y-intercept? Explain how you know.

c. Identify one other point on the graph of the equation. Explain or show how you know.

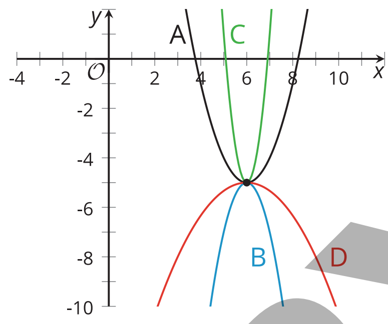
d. Sketch a graph that represents the equation.



3 Function v is defined by $v(x) = \frac{1}{2}(x + 5)^2 - 7$.

Without graphing, determine if the vertex of the graph representing v shows the minimum or maximum value of the function. Explain how you know.

4 Match each graph to an equation that represents it.



A. Graph A

B. Graph B

C. Graph C

D. Graph D

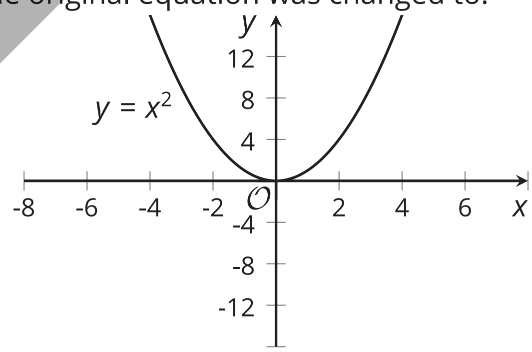
1. $y = -2(x-6)^2 - 5$
2. $y = (x-6)^2 - 5$
3. $y = 6(x-6)^2 - 5$
4. $y = -\frac{1}{3}(x-6)^2 - 5$

5 from Unit 7, Lesson 12

Here is a graph that represents $y = x^2$.

a. Describe what would happen to the graph if the original equation was changed to:

i. $y = \frac{1}{2}x^2$



ii. $y = x^2 - 8$

b. Graph the equation $y = \frac{1}{2}x^2 - 8$ on the same coordinate plane as $y = x^2$.

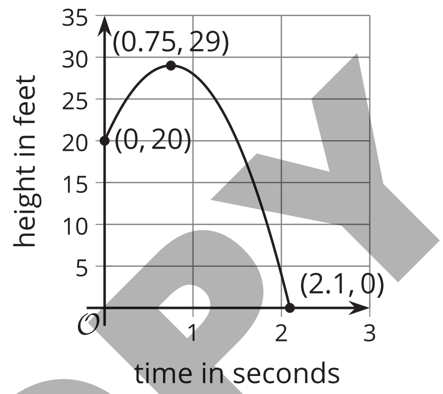
Sec D

6

from Unit 7, Lesson 14

Clare throws a rock into the lake. The graph shows the rock's height above the water, in feet, as a function of time, in seconds.

Select **all** the statements that describe this situation.



A. The vertex of the graph is $(0.75, 29)$.

B. The y-intercept of the graph is $(2.1, 0)$.

C. Clare just dropped the rock into the lake.

D. The maximum height of the rock is about 20 feet.

E. The rock hits the surface of the water after about 2.1 seconds.

F. Clare tossed the rock up into the air from a point 20 feet above the water.

7

from Unit 7, Lesson 14

Technology required. Two objects are launched into the air.

- The height, in feet, of Object A is given by the equation $f(t) = 4 + 32t - 16t^2$.
- The height, in feet, of the Object B is given by the equation $g(t) = 2.5 + 40t - 16t^2$. In both functions, t is seconds after launch.

Use technology to graph each function in the same graphing window.

a. What is the maximum height of each object?

b. Which object hits the ground first? Explain how you know.

Sec D

8 from Unit 7, Lesson 15

Andre thinks the vertex of the graph of the equation $y = (x+2)^2 - 3$ is $(2, \text{-}3)$. Lin thinks the vertex is $(\text{-}2, 3)$. Do you agree with either of them?

9 from Unit 6, Lesson 20

The expression $2{,}000 \cdot \left(1.015^{12}\right)^5$ represents the balance, in dollars, in a savings account.

a. Using the expression, describe the interest rate paid on the account.

b. How many years has the account been accruing interest?

c. How much money was invested?

d. How much money is in the account now?

e. Write an equivalent expression to represent the balance in the savings account.

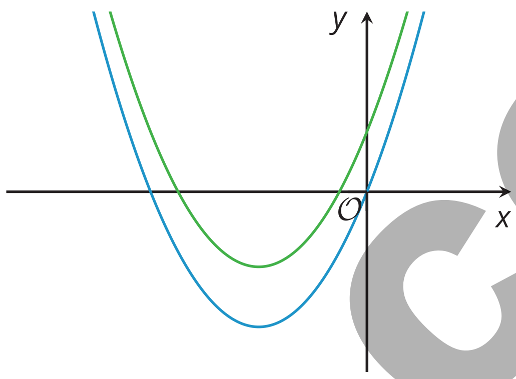
Sec D

Addressing CA CCSSM F-BF.3, F-IF.7a, F-IF.9; practicing MP1, MP3, MP4, MP7

Changing the Vertex

Let’s write new quadratic equations in vertex form to produce certain graphs.

17.1 Graphs of Two Functions



Here are graphs representing two functions, f and g, given by $f(x) = x(x + 6)$ and $g(x) = x(x + 6) + 4$.

1. Which graph represents each function? Explain how you know.

2. Where does the graph of f meet the x-axis? Explain how you know.

17.2 Shifting the Graph

1. How would you change the equation $y = x^2$ so that the vertex of the graph of the new equation is located at the following coordinates and so that the graph opens as described?
   a. $(0, 11)$, opens upward
   b. $(7, 11)$, opens upward
   c. $(7, \text{-}3)$, opens downward
2. Use graphing technology to verify your predictions. Adjust your equations if necessary.
3. Kiran graphed the equation $y = x^2 + 1$ and noticed that the vertex is at $(0, 1)$. He changed the equation to $y = (x - 3)^2 + 1$ and saw that the graph shifted 3 units to the right and the vertex is now at $(3, 1)$.

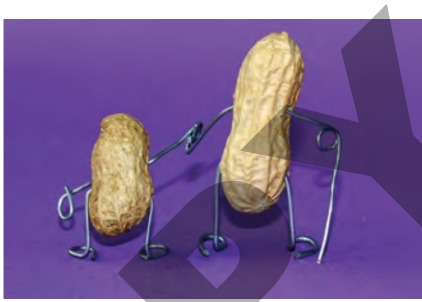
   Next, he graphed the equation $y = x^2 + 2x + 1$ and observed that the vertex is at $(\text{-}1, 0)$. Kiran thought, "If I change the squared term x^2 to $(x - 5)^2$, the graph of $y = (x - 5)^2 + 2x + 1$ will be 5 units to the right and the vertex will be at $(4, 0)$."

   Do you agree with Kiran? Explain or show your reasoning.

Sec D

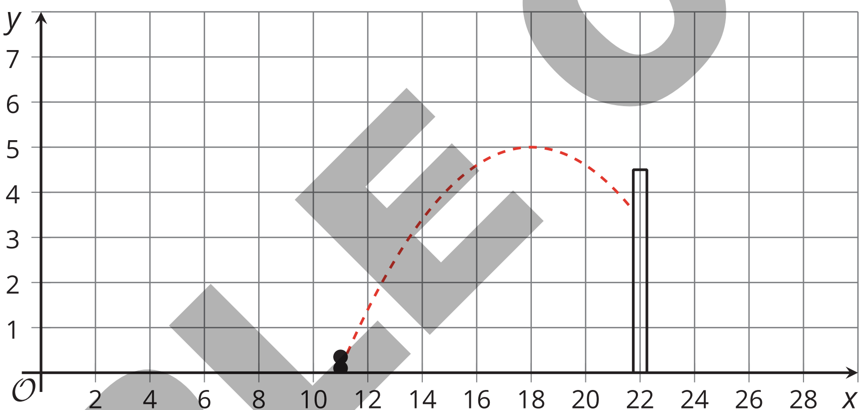
17.3 A Peanut Jumping over a Wall

Mai is learning to create computer animation by programming. In one part of her animation, she uses a quadratic function to model the path of the main character, an animated peanut, jumping over a wall.



Mai uses the equation $y = -0.1(x-h)^2 + k$ to represent the path of the jump. y represents the height of the peanut as a function of the horizontal distance, x, that it travels.

On the screen, the base of the wall is located at $(22, 0)$, with the top of the wall at $(22, 4.5)$. The dashed curve in the picture shows the graph of 1 equation that Mai tried, where the peanut fails to make it over the wall.

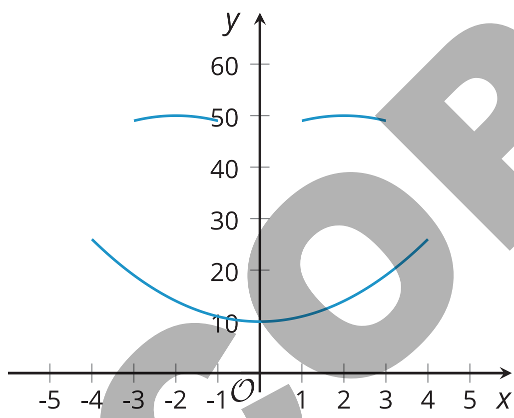


1. What are the values of h and k in this equation?

2. Starting with Mai's equation, choose values for h and k that will guarantee that the peanut stays on the screen but also makes it over the wall. Be prepared to explain your reasoning.

17.4 Smiley Face

Do you see 2 "eyes" and a smiling "mouth" on the graph? The 3 arcs on the graph all represent quadratic functions that were initially defined by $y = x^2$, but whose equations were later modified.



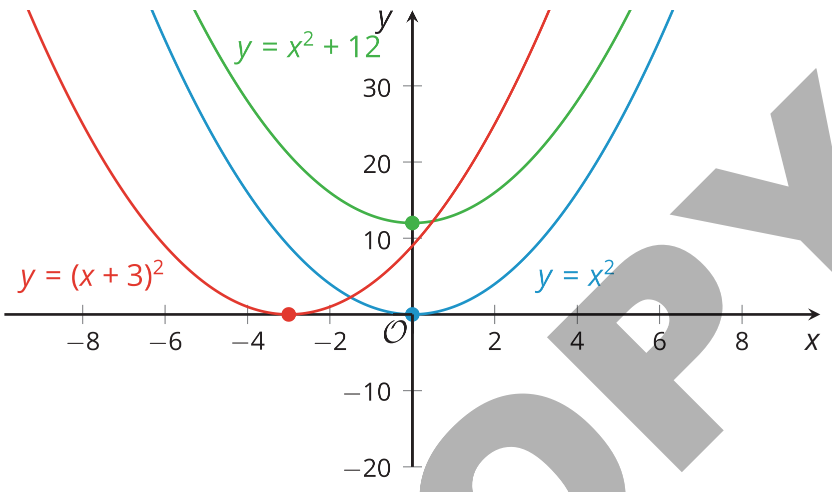
1. Write equations to represent each curve in the smiley face.

2. What domain is used for each function to create this graph?

Sec D

Lesson 17 Summary

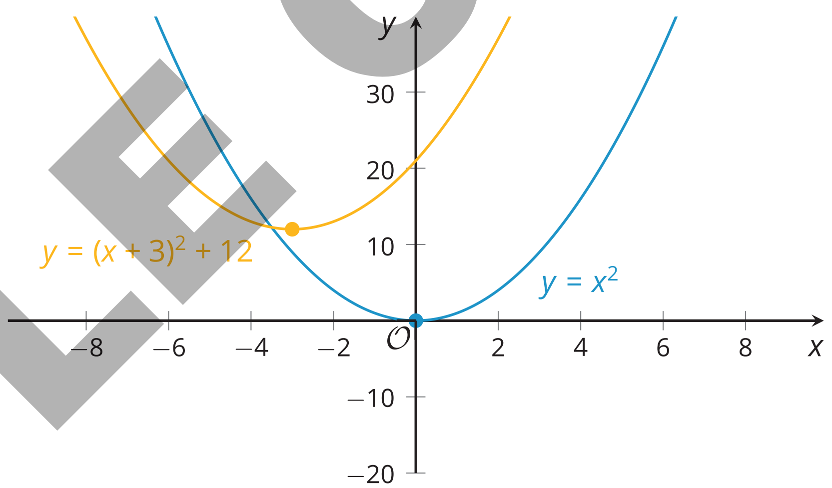
The graphs of $y = x^2$, $y = x^2 + 12$ and $y = (x + 3)^2$ all have the same shape but their locations are different. The graph that represents $y = x^2$ has its vertex at $(0, 0)$.



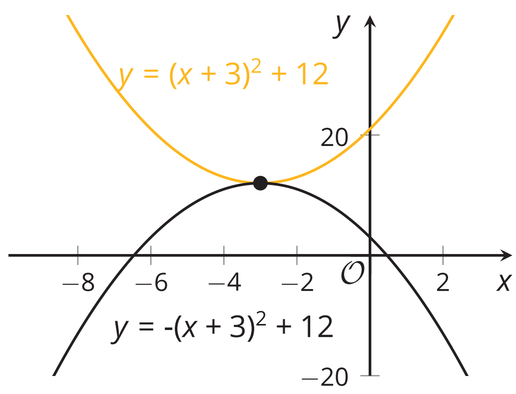
Notice that adding 12 to x^2 raises the graph by 12 units, so the vertex of that graph is at $(0, 12)$. Replacing x^2 with $(x + 3)^2$ shifts the graph 3 units to the left, so the vertex is now at $(-3, 0)$.

We can also shift a graph both horizontally and vertically.

The graph that represents $y = (x + 3)^2 + 12$ will have the same shape as $y = x^2$ but it will be shifted 12 units up and 3 units to the left. Its vertex is at $(-3, 12)$.



The graph representing the equation $y = -(x + 3)^2 + 12$ has the same vertex at $(-3, 12)$, but because the squared term $(x + 3)^2$ is multiplied by a negative number, the graph is flipped over horizontally, so that it opens downward.



Sec D

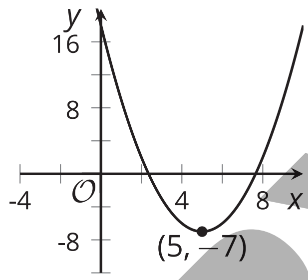
Lesson 17 Practice Problems

1 Here is the graph of quadratic function f.

Andre uses the expression $(x-5)^2+7$ to define f.

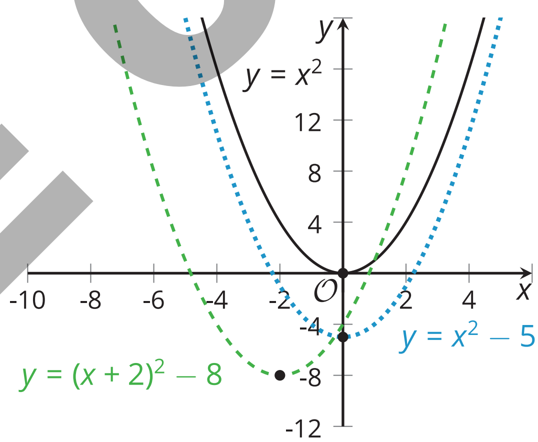
Noah uses the expression $(x+5)^2-7$ to define f.

Do you agree with either of them? Explain your reasoning.



2 Here are the graphs of $y=x^2$, $y=x^2-5$, and $y=(x+2)^2-8$.

a. How do the 3 graphs compare?



b. Compare the graphs of $y=x^2$ and $y=x^2-5$. What role does the -5 play in the comparison?

c. Compare the graphs of $y=x^2$ and $y=(x+2)^2-8$. What role do the 2 and 8 play in the comparison?

3 Select **all** the equations with a graph whose vertex has both a positive x-coordinate and a positive y-coordinate.

A. $y = x^2$

B. $y = (x - 1)^2$

C. $y = (x - 3)^2 + 2$

D. $y = 2(x - 4)^2 - 5$

E. $y = 0.5(x + 2)^2 + 6$

F. $y = -(x - 4)^2 + 3$

G. $y = -2(x - 3)^2 + 1$

4 from Unit 7, Lesson 14

The height, in feet, of a soccer ball is modeled by the equation $g(t) = 2 + 50t - 16t^2$, where time, t, is measured in seconds after it was kicked.

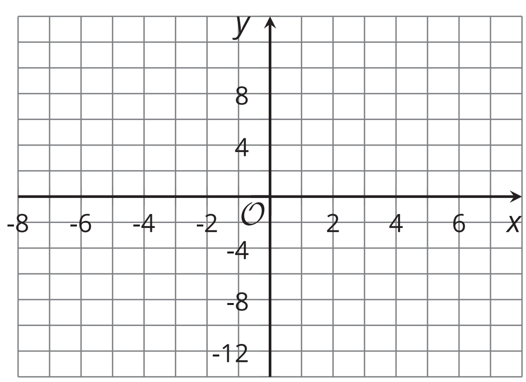
a. How far above the ground was the ball when kicked?

b. What was the initial upward velocity of the ball?

c. Why is the coefficient of the squared term negative?

5 from Unit 7, Lesson 16

a. What is the vertex of the graph of a function, f, defined by $f(x) = -(x - 3)^2 + 6$?



b. Identify the y-intercept and find another point with the same y-coordinate as that point.

c. Sketch the graph of f.

6 from Unit 5, Lesson 7

At 6:00 a.m., Lin began hiking. At noon, she had hiked 12 miles. At 4:00 p.m., Lin finished hiking with a total trip of 26 miles.

During which time interval was Lin hiking faster? Explain how you know.

7 from Unit 5, Lesson 11

Kiran bought a smoothie every day for a week. Smoothies cost $3 each. The amount of money he spends, in dollars, is a function of the number of days of buying smoothies.

a. Sketch a graph of this function. Be sure to label the axes.

b. Describe the domain and range of this function.

8 from Unit 6, Lesson 21

A deposit of $500 has been made in an interest-bearing account. No withdrawals or other deposits (aside from earned interest) are made for 5 years.

Write an expression to represent the account balance for each of the following situations.

a. 6.5% annual interest calculated monthly

b. 6.5% annual interest calculated every two months

c. 6.5% annual interest calculated quarterly

d. 6.5% annual interest calculated semi-annually

Sec D

9 from Unit 6, Lesson 22

Technology required. Function h is defined by $h(x) = 5x + 7$, and function k is defined by $k(x) = (1.005)^x$.

a. Complete the table with values of $h(x)$ and $k(x)$. When necessary, round to 2 decimal places.

x	$h(x)$	$k(x)$
1		
10		
50		
100		

b. Which function do you think eventually grows faster? Explain your reasoning.

c. Use graphing technology to verify your answer to the previous question.

Sec D

Learning Targets

Lesson 1 A Different Kind of Change

- I can create drawings, tables, and graphs that represent the area of a garden.
- I can recognize a situation represented by a graph that increases and then decreases.

Lesson 2 How Does It Change?

- I can describe how a pattern is growing.
- I can tell whether a pattern is growing linearly, exponentially, or quadratically.
- I know a quadratic expression has a squared term.

Lesson 3 Building Quadratic Functions from Geometric Patterns

- I can explain using graphs, tables, or calculations that exponential functions eventually grow faster than do quadratic functions.
- I can recognize quadratic functions written in different ways.
- I can use information from a pattern of shapes to write a quadratic function.

Lesson 4 Comparing Quadratic and Exponential Functions

- I can explain using graphs, tables, or calculations that exponential functions eventually grow faster than do quadratic functions.

Lesson 5 Building Quadratic Functions to Describe Situations (Part 1)

- I can explain the meaning of the terms in a quadratic expression that represents the height of a falling object.
- I can use tables, graphs, and equations to represent the height of a falling object.

Lesson 6 Building Quadratic Functions to Describe Situations (Part 2)

- I can create quadratic functions and graphs that represent a situation.
- I can relate the vertex of a graph and the zeros of a function to a situation.
- I know that the domain of a function can depend on the situation it represents.

Lesson 7 Building Quadratic Functions to Describe Situations (Part 3)

- I can choose a domain that makes sense in a revenue situation.
- I can model revenue with quadratic functions and graphs.
- I can relate the vertex of a graph and the zeros of a function to a revenue situation.

Lesson 8 Equivalent Quadratic Expressions

- I can rewrite quadratic expressions in different forms by using an area diagram or the distributive property.

Lesson 9 Standard Form and Factored Form

- I can rewrite quadratic expressions given in factored form in standard form, using either the distributive property or a diagram.
- I know the difference between "factored form" and "standard form."

Lesson 10 Graphs of Functions in Standard and Factored Forms

- I can explain the meaning of the intercepts on a graph of a quadratic function in terms of the situation it represents.
- I know how the numbers in the factored form of a quadratic expression relate to the intercepts of its graph.

Lesson 11 Graphing from the Factored Form

- I can graph a quadratic function given in factored form.
- I know how to find the vertex and y-intercept of the graph of a quadratic function in factored form without graphing it first.

Lesson 12 Graphing the Standard Form (Part 1)

- I can explain how the a and c in $y = ax^2 + bx + c$ affect the graph of the equation.
- I understand how graphs, tables, and equations that represent the same quadratic function are related.

Lesson 13 Graphing the Standard Form (Part 2)

- I can explain how the b in $y = ax^2 + bx + c$ affects the graph of the equation.
- I can match equations given in standard and factored form with their graph.

Lesson 14 Graphs That Represent Situations

- I can explain how a quadratic equation and its graph relate to a situation.

Lesson 15 Vertex Form

- I can recognize the "vertex form" of a quadratic equation.
- I can relate the numbers in the vertex form of a quadratic equation to its graph.

Lesson 16 Graphing from the Vertex Form

- I can graph a quadratic function given in vertex form, showing a maximum or minimum and the y-intercept.
- I know how to find a maximum or a minimum of a quadratic function given in vertex form

without first graphing it.

Lesson 17 Changing the Vertex

- I can describe how changing a number in the vertex form of a quadratic function affects its graph.

ALGEBRA 1

UNIT

8

Quadratic Equations

Content Connections

In this unit you will interpret, write, and solve equations algebraically. You will make connections by:

- **Reasoning with Data** while noticing patterns that help rewrite quadratic expressions in factored form and recognize that not all are easily factorable.
- **Discovering Shape and Space** while rewriting quadratic expressions from standard form into vertex form to find maximum and minimum values.
- **Exploring Changing Quantities** while analyzing quadratic equations to recognize that there can be 0, 1, or 2 solutions and the solutions can be rational, irrational, or a combination of these.
- **Taking Wholes Apart, Putting Parts Together** while using the factored form of a quadratic expression or a graph of a quadratic function to answer questions about a situation.

Addressing the Standards

As you work your way through **Unit 8 Quadratic Equations,** you will use some mathematical practices that you may have started using in kindergarten and have continued strengthening over your school career. These practices describe types of thinking or behaviors that you might use to solve specific math problems.

Mathematical Practices	Where You Use These MPs
MP1 Make sense of problems and persevere in solving them.	Lessons 1, 2, 3, 9, 10, 21, 22, and 24
MP2 Reason abstractly and quantitatively.	Lessons 1, 2, 17, and 21
MP3 Construct viable arguments and critique the reasoning of others.	Lessons 5, 6, 13, 18, 19, 20, 21, and 23
MP4 Model with mathematics.	Lesson 1
MP5 Use appropriate tools strategically.	Lessons 3, 5, 18, and 21
MP6 Attend to precision.	Lessons 9, 10, 13, 15, 20, and 22
MP7 Look for and make use of structure.	Lessons 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 19, 21, 22, and 23
MP8 Look for and express regularity in repeated reasoning.	Lessons 4, 7, 8, 11, 14, 20, and 21

The California Common Core State Standards for Mathematics (CA CCSSM) describe the topics you will learn in this unit. Many of these topics build upon knowledge you already have and challenge you to expand upon that knowledge. The table below shows what standards are being addressed in this unit.

Big Ideas You Are Studying	California Content Standards	Lessons Where You Learn This
• Model with Functions • Systems of Equations	**A-CED.1** Create equations and inequalities in one variable including ones with absolute value and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*	Lessons 1, 17, and 18
• Systems of Equations	**A-REI.1** Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.	Lessons 3, 5, 13, and 17

Big Ideas You Are Studying	California Content Standards	Lessons Where You Learn This
• Systems of Equations	**A-REI.4** Solve quadratic equations in one variable. a. Use the method of completing the square to transform any quadratic equation in *x* into an equation of the form *(x – p)2 = q* that has the same solutions. Derive the quadratic formula from this form. b. Solve quadratic equations by inspection (e.g., for *x2* = 49), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as *a* ± *bi* for real numbers *a* and *b.*	Lessons 2, 4, and 5
• Systems of Equations	**A-REI.4a** Solve quadratic equations in one variable. a. Use the method of completing the square to transform any quadratic equation in *x* into an equation of the form *(x – p)2 = q* that has the same solutions. Derive the quadratic formula from this form.	Lessons 12, 14, 15, and 19
• Systems of Equations	**A-REI.4b** Solve quadratic equations in one variable. b. Solve quadratic equations by inspection (e.g., for *x2* = 49), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as *a* ± *bi* for real numbers *a* and *b.*	Lessons 3, 5, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, and 24
• Systems of Equations	**A-REI.7** Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.	Lesson 24
• Systems of Equations	**A-REI.10** Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).	Lesson 5
• Systems of Equations	**A-REI.11** Explain why the *x*-coordinates of the points where the graphs of the equations *y = f(x)* and *y = g(x)* intersect are the solutions of the equation *f(x) = g(x);* find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where *f(x)* and/or *g(x)* are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.	Lessons 10 and 15

Big Ideas You Are Studying	California Content Standards	Lessons Where You Learn This
• Features of Functions	**F-LE.6** Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity.	Lessons 2, 4, 10, 17, 18, and 24
• Systems of Equations • Growth and Decay	**A-SSE.1b** Interpret expressions that represent a quantity in terms of its context. b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret P(1 + r)n as the product of P and a factor not depending on P.*	Lesson 10
• Model with Functions • Systems of Equations • Function Investigations • Features of Functions	**A-SSE.2** Use the structure of an expression to identify ways to rewrite it.	Lessons 6, 7, 8, 10, 11, 12, 14, 16, 19, and 22
• Features of Functions	**A-SSE.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. a. Factor a quadratic expression to reveal the zeros of the function it defines. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.	Lesson 22
• Features of Functions	**A-SSE.3a** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. a. Factor a quadratic expression to reveal the zeros of the function it defines.	Lesson 9
• Features of Functions	**A-SSE.3b** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.	Lessons 22 and 23
• Model with Functions • Function Investigations	**F-IF.2** Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.	Lesson 18
• Model with Functions • Function Investigations • Features of Functions • Growth and Decay	**F-IF.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*	Lesson 10

Big Ideas You Are Studying	**California Content Standards**	**Lessons Where You Learn This**
• Model with Functions • Function Investigations • Growth and Decay	**F-IF.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*	Lesson 17
• Model with Functions • Function Investigations	**F-IF.7a** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. a. Graph linear and quadratic functions and show intercepts, maxima, and minima.	Lessons 20 and 23
• Model with Functions • Function Investigations	**F-IF.8a** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.	Lessons 9, 10, 22, and 24
• Model with Functions • Function Investigations • Growth and Decay	**F-IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*	Lesson 23
• Model with Functions • Systems of Equations • Function Investigations	**N-RN.3** Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.	Lessons 20 and 21
• Model with Functions • Function Investigations	**F-BF.1b** 1. Write a function that describes a relationship between two quantities. b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*	Lessons 22, 23, and 24

Note: For a full explanation of the California Common Core State Standards for Mathematics (CA CCSSM) refer to the standards section at the end of this book.

Unit 8, Lesson 1
Addressing CA CCSSM A-CED.1; building on F-IF.4, F-LE.6; building toward A-CED.3; practicing MP1, MP2, MP4

Finding Unknown Inputs

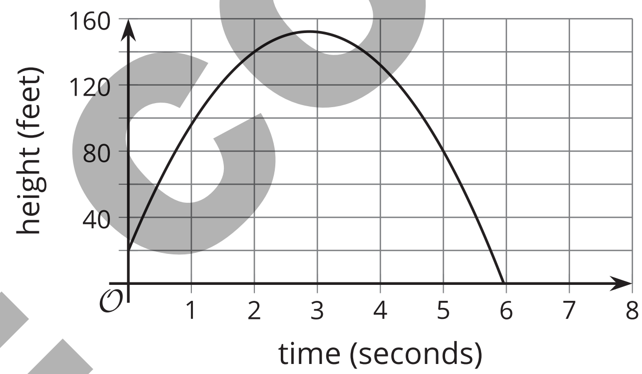
Let's find some new equations to solve.

1.1 What Goes Up Must Come Down

A mechanical device is used to launch a potato vertically into the air. The potato is launched from a platform 20 feet above the ground, with an initial vertical velocity of 92 feet per second.

The function $h(t) = -16t^2 + 92t + 20$ models the height of the potato over the ground, in feet, t seconds after launch.

Here is the graph representing the function.



For each question, be prepared to explain your reasoning.

1. What is the height of the potato 1 second after launch?
2. 8 seconds after launch, will the potato still be in the air?
3. Will the potato reach 120 feet? If so, when will it happen?
4. When will the potato hit the ground?

1.2 A Trip to the Frame Shop

Your teacher will give you a picture that is 7 inches by 4 inches, a piece of framing material measuring 4 inches by 2.5 inches, and a pair of scissors.

Cut the framing material to create a rectangular frame for the picture. The frame should have the same thickness all the way around and have no overlaps. All of the framing material should be used (with no leftover pieces). Framing material is very expensive!

You get 3 copies of the framing material, in case you make mistakes and need to recut.

Sec A

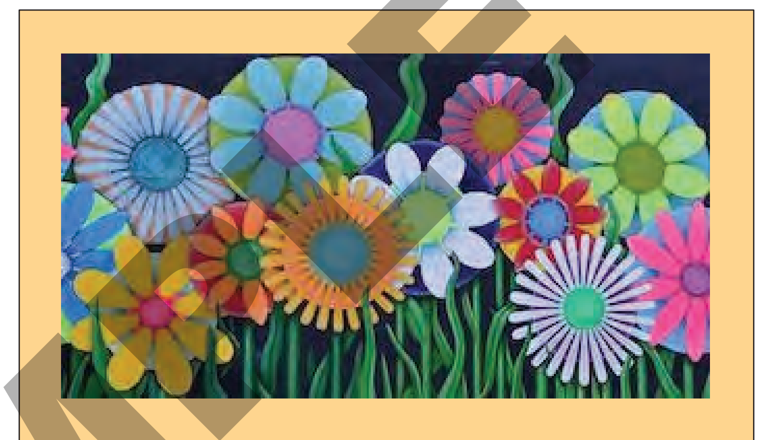
Are you ready for more?

Han says, "The perimeter of the picture is 22 inches. If I cut the framing material into 9 pieces, each one being 2.5 inches by $\frac{4}{9}$ inch, I'll have more than enough material to surround the picture because those pieces would make 22.5 inches for the frame."

Do you agree with Han? Explain your reasoning.

1.3 Representing the Framing Problem

Here is a diagram that shows the picture with a frame that is the same thickness all the way around. The picture is 7 inches by 4 inches. The frame is created from 10 square inches of framing material (in the form of a rectangle measuring 4 inches by 2.5 inches).



1. Write an equation to represent the relationship between the area of the framed picture and the measurements of the picture and of the frame. Be prepared to explain what each part of your equation represents.

2. What would a solution to this equation mean in this situation?

Lesson 1 Summary

The height of a softball, in feet, t seconds after someone throws it straight up, can be defined by $f(t) = \text{-}16t^2 + 32t + 5$. The input of function f is time, and the output is height.

We can find the output of this function at any given input. For instance:

- At the beginning of the softball's journey, when $t = 0$, its height is given by $f(0)$.
- Two seconds later, when $t = 2$, its height is given by $f(2)$.

The values of $f(0)$ and $f(2)$ can be found using a graph or by evaluating the expression $\text{-}16t^2 + 32t + 5$ at those values of t. What if we know the output of the function and want to find the inputs? For example:

- When does the softball hit the ground?

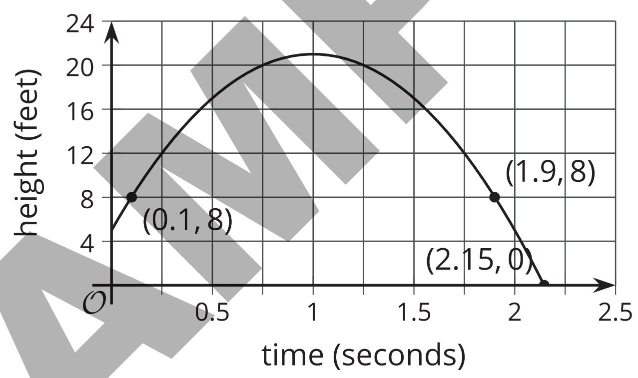
  Answering this question means finding the values of t that make $f(t) = 0$, or solving $\text{-}16t^2 + 32t + 5 = 0$.

- How long will it take the ball to reach 8 feet?

  This means finding one or more values of t that make $f(t) = 8$, or solving the equation $\text{-}16t^2 + 32t + 5 = 8$.

The equations $\text{-}16t^2 + 32t + 5 = 0$ and $\text{-}16t^2 + 32t + 5 = 8$ are *quadratic equations*. One way to solve these equations is by graphing $y = f(t)$.

- To answer the first question, we can look for the horizontal intercepts of the graph, where the vertical coordinate is 0.
- To answer the second question, we can look for the horizontal coordinates that correspond to a vertical coordinate of 8.



We can see that there are two solutions to the equation $\text{-}16t^2 + 32t + 5 = 8$ and one solution to the equation $\text{-}16t^2 + 32t + 5 = 0$.

The softball has a height of 8 feet twice, when going up and when coming down, and these instances occur when t is about 0.1 and 1.9. It has a height of 0 once, when t is about 2.15.

Often, when we are modeling a situation mathematically, an approximate solution is good enough. Sometimes, however, we would like to know exact solutions, and it may not be possible to find them using a graph. In this unit, we will learn more about quadratic equations and how to solve for exact answers using algebraic techniques.

Lesson 1 Practice Problems

Sec A

1 A girl throws a paper airplane from her treehouse. The height of the plane is a function of time and can be modeled by the equation $h(t) = 25 + 2.5t - \frac{1}{2}t^2$. Height is measured in feet, and time is measured in seconds after the airplane is thrown.

a. Evaluate $h(0)$, and explain what this value means in this situation.

b. What would a solution to $h(t) = 0$ mean in this situation?

c. What does the equation $h(9) = 7$ mean?

d. What does the model say about the airplane 2.5 seconds after the girl throws it if each of these statements is true?

$h(2) = 28$ $\quad$ $h(2.5) = 28.125$ $\quad$ $h(3) = 28$

2 A square picture has a frame that is 3 inches thick all the way around. The total side length of the picture and frame is x inches.

Which expression represents the area of the square picture, without the frame? If you get stuck, try sketching a diagram.

A. $(2x + 3)(2x + 3)$

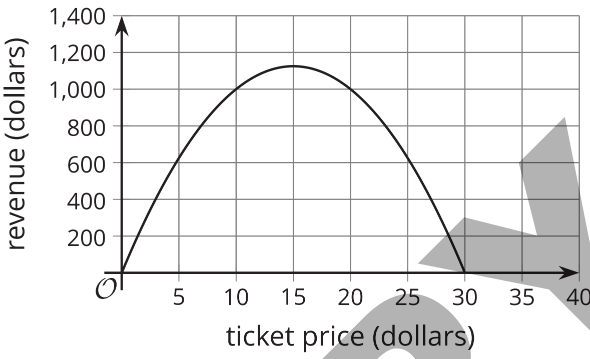
B. $(x + 6)(x + 6)$

C. $(2x - 3)(2x - 3)$

D. $(x - 6)(x - 6)$

3 The revenue from a youth league baseball game depends on the price per ticket, x.

Here is a graph that represents the revenue function, R.



Select **all** the true statements.

A. $R(5)$ is a little more than 600.

B. $R(600)$ is a little less than 5.

C. The maximum possible ticket price is \$15.

D. The maximum possible revenue is about \$1,125.

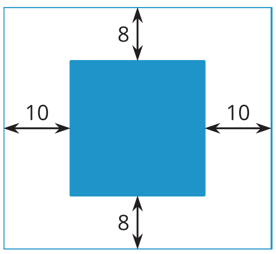
E. If tickets cost \$10, the predicted revenue is \$1,000.

F. If tickets cost \$20, the predicted revenue is \$1,000.

4 A garden designer designed a square decorative pool. The pool is surrounded by a walkway.

On two opposite sides of the pool, the walkway is 8 feet. On the other two opposite sides, the walkway is 10 feet.

Here is a diagram of the design.



The final design for the pool and walkway covers a total area of 1,440 square feet.

a. The side length of the square pool is x. Write an expression that represents:

i. the total length of the rectangle (including the pool and walkway)

ii. the total width of the rectangle (including the pool and walkway)

iii. the total area of the pool and walkway

b. Write an equation of the form: $\text{your expression} = 1{,}440$. What does a solution to the equation mean in this situation?

Sec A

5 from Unit 5, Lesson 17

Suppose m and c each represent the position number of a letter in the alphabet, but m represents the letters in the original message, and c represents the letters in a secret code. The equation $c = m + 2$ is used to encode a message.

a. Write an equation that can be used to decode the secret code into the original message.

b. What does this code say: "OCVJ KU HWP!"?

6 from Unit 5, Lesson 17

An American traveler who is heading to Europe is exchanging some US dollars for euros. At the time of his travel, 1 dollar can be exchanged for 0.91 euros.

a. Find the amount of money in euros that the American traveler would get if he exchanged 100 dollars.

b. What if he exchanged 500 dollars?

c. Write an equation that gives the amount of money in euros, e, as a function of the dollar amount being exchanged, d.

d. Upon returning to America, the traveler has 42 euros to exchange back into US dollars. How many dollars would he get if the exchange rate is still the same?

e. Write an equation that gives the amount of money in dollars, d, as a function of the euro amount being exchanged, e.

7 from Unit 3, Lesson 3

A random sample of people are asked to give a taste score—either "low" or "high"—to two different types of ice cream. The two types of ice cream have identical formulas, except they differ in the percentage of sugar in the ice cream.

What values could be used to complete the table so that it suggests there is an association between taste score and percentage of sugar? Explain your reasoning.

	12% sugar	15% sugar
low taste score	239	
high taste score	126	

Sec A

Unit 8, Lesson 2
Addressing CA CCSSM A-REI.4, F-LE.6; building on A-REI.3; building toward A-CED.1; practicing MP1 and MP2

When and Why Do We Write Quadratic Equations?

Sec A

Let's try to solve some quadratic equations.

2.1 How Many Tickets?

The expression $12t + 2.50$ represents the cost to purchase tickets for a play, where t is the number of tickets. Be prepared to explain your response to each question.

1. A family paid \$62.50 for tickets. How many tickets were bought?

2. A teacher paid \$278.50 for tickets for her students. How many tickets were bought?

2.2 The Flying Potato Again

Here is a function modeling the height of a potato, in feet, t seconds after being fired from a device:

$$f(t) = -16t^2 + 80t + 64$$

1. What equation would we solve to find the time at which the potato hits the ground?

2. Use any method *except graphing* to find a solution to this equation.

2.3 Revenue from Ticket Sales

The expressions $p(200 - 5p)$ and $-5p^2 + 200p$ describe the revenue a school would earn from selling raffle tickets at p dollars each.

1. For each situation, write a **quadratic equation** using these quadratic expressions. Then, find the price, p, of each ticket that would produce the situation. Explain your reasoning.
   a. The school collects \$0 in revenue from raffle sales.

   b. The school collects \$500 in revenue from raffle sales.

Are you ready for more?

Can you find the following prices without graphing?

1. If the school charges \$10, it will collect \$1,500 in revenue. Find another price that would generate \$1,500 in revenue.
2. If the school charges \$28, it will collect \$1,680 in revenue. Find another price that would generate \$1,680 in revenue.
3. Find the price that would produce the maximum possible revenue. Explain your reasoning.

Lesson 2 Summary

The height of a potato that is launched from a mechanical device can be modeled by a function, g, with x representing time in seconds. Here are two expressions that are equivalent and both define function g.

$$-16x^2 + 80x + 96 \qquad\qquad -16(x-6)(x+1)$$

Notice that one expression is in *standard form* and the other is in *factored form*.

Suppose we wish to know, without graphing the function, the time when the potato will hit the ground. We know that the value of the function at that time is 0, so we can write:

$$-16x^2 + 80x + 96 = 0 \qquad\qquad -16(x-6)(x+1) = 0$$

Let's try solving $-16x^2 + 80x + 96 = 0$, using some familiar moves. For example:

- Subtract 96 from each side: $-16x^2 + 80x = -96$
- Apply the distributive property to rewrite the expression on the left: $-16(x^2 - 5x) = -96$
- Divide both sides by -16: $x^2 - 5x = 6$
- Apply the distributive property to rewrite the expression on the left: $x(x-5) = 6$

These steps don't seem to get us any closer to a solution. We need some new moves!

What if we use the other equation? Can we find the solutions to $-16(x-6)(x+1) = 0$?

Earlier, we learned that the *zeros* of a quadratic function can be identified when the expression defining the function is in factored form. The solutions to $-16(x-6)(x+1) = 0$ are the zeros to function g, so this form may be more helpful! We can reason that:

- If x is 6, then the value of $x - 6$ is 0, so the entire expression has a value of 0.
- If x is -1, then the value of $x + 1$ is 0, so the entire expression also has a value of 0.

This tells us that 6 and -1 are solutions to the equation, and that the potato hits the ground after 6 seconds. (A negative value of time is not meaningful, so we can disregard the -1.)

Both equations we see here are quadratic equations. In general, a **quadratic equation** is an equation that can be expressed as $ax^2 + bx + c = 0$, where a, b, and c are constants and $a \neq 0$.

In upcoming lessons, we will learn how to rewrite quadratic equations into forms that make the solutions easy to see.

Glossary

- factored form (of a quadratic expression)
- quadratic equation
- standard form (of a quadratic expression)
- zero (of a function)

Lesson 2 Practice Problems

1 A set of kitchen containers can be stacked to save space. The height of the stack is given by the expression $1.5c + 7.6$, where c is the number of containers.

a. Find the height of a stack made of 8 containers.

b. A tower made of all the containers is 40.6 cm tall. How many containers are in the set?

c. Noah looks at the equation and says, "7.6 must be the height of a single container." Do you agree with Noah? Explain your reasoning.

2 Select **all** values of x that are solutions to the equation $(x - 5)(7x - 21) = 0$.

A. -7

B. -5

C. -3

D. 0

E. 3

F. 5

G. 7

Sec A

3 The expressions $30x^2 - 105x - 60$ and $(5x - 20)(6x + 3)$ define the same function, f.

a. Which expression makes it easier to find $f(0)$? Explain your reasoning.

b. Find $f(0)$.

c. Which expression makes it easier to find the values of x that make the equation $f(x) = 0$ true? Explain or show your reasoning.

d. Find the values of x that make $f(x) = 0$.

4 A band is traveling to a new city to perform a concert. The revenue from their ticket sales is a function of the ticket price, x, and can be modeled with $(x - 6)(250 - 5x)$.

What are the ticket prices at which the band would make no money at all?

5 from Unit 8, Lesson 1

Two students built a small rocket from a kit and attached an altimeter (a device for recording altitude or height) to the rocket. They record in the table the height of the rocket over time since it is launched, using the data from the altimeter.

time (seconds)	0	1	3	4	7	8
height (meters)	0	110.25	236.25	252	110.25	0

Function h gives the height in meters as a function of time in seconds, t.

a. What is the value of $h(3)$?

b. What value of t gives $h(t) = 252$?

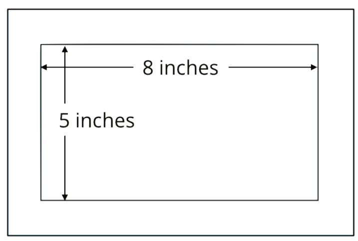
c. Explain why $h(0) = h(8)$.

d. Based on the data, which equation about the function is more likely to be true: $h(2) = 189$ or $h(189) = 2$? Explain your reasoning.

6 from Unit 8, Lesson 1

The screen of a tablet has dimensions 8 inches by 5 inches. The border around the screen has thickness x.

a. Write an expression for the total area of the tablet, including the frame.



b. Write an equation for which your expression is equal to 50.3125. Explain what a solution to this equation means in this situation.

c. Try to find the solution to the equation. If you get stuck, try guessing and checking. It may help to think about tablets that you have seen.

7 from Unit 7, Lesson 1

Here are a few pairs of positive numbers whose sum is 15. The pair of numbers that have a sum of 15 and will produce the largest possible product is *not* shown.

Find this pair of numbers.

first number	second number	product
1	14	14
3	12	36
5	10	50
7	8	56

8 from Unit 5, Lesson 17

A kilometer is a measurement in the metric system, while a mile is a measurement in the customary system. One kilometer equals approximately 0.621 mile.

a. The number of miles, m, is a function of the number of kilometers, k. What equation can be written to represent this function?

b. The number of kilometers, k, is a function of the number of miles, m. What equation can be written to represent this function?

c. How are these two functions related? Explain how you know.

Addressing CA CCSSM A-REI.1, A-REI.4b; practicing MP1, MP5, MP7

Solving Quadratic Equations by Reasoning

Let's find solutions to quadratic equations.

3.1 How Many Solutions?

How many solutions does each equation have? What are the solution(s)? Be prepared to explain how you know.

1. $x^2 = 9$
2. $x^2 = 0$
3. $x^2 - 1 = 3$
4. $2x^2 = 50$
5. $(x + 1)(x + 1) = 0$
6. $x(x - 6) = 0$
7. $(x - 1)(x - 1) = 4$

3.2 Finding Pairs of Solutions

Each of these equations has two solutions. What are they? Explain or show your reasoning.

1. $n^2 + 4 = 404$

2. $432 = 3n^2$

3. $144 = (n+1)^2$

4. $(n-5)^2 - 30 = 70$

Are you ready for more?

1. How many solutions does the equation $(x-3)(x+1)(x+5) = 0$ have? What are the solutions?
2. How many solutions does the equation $(x-2)(x-7)(x-2) = 0$ have? What are the solutions?
3. Write a new equation that has 10 solutions.

Lesson 3 Summary

Some quadratic equations can be solved by performing the same operation on each side of the equal sign and reasoning about which values for the variable would make the equation true.

Suppose we wanted to solve $3(x+1)^2 - 75 = 0$. We can proceed like this:

- Add 75 to each side: $3(x+1)^2 = 75$
- Divide each side by 3: $(x+1)^2 = 25$
- What number can be squared to get 25? $(\boxed{})^2 = 25$
- There are two numbers that work, 5 and -5: $5^2 = 25$ and $(-5)^2 = 25$
- If $x+1=5$, then $x=4$.
- If $x+1=-5$, then $x=-6$.

This means that both $x=4$ and $x=-6$ make the equation true and are solutions to the equation.

Many quadratic equations have 2 solutions, but some have only 1 or no solution.

Sec B

Lesson 3 Practice Problems

1 Consider the equation $x^2 = 9$.

a. Show that 3, -3, $\sqrt{9}$, and -$\sqrt{9}$ are each a solution to the equation.

b. Show that 9 and $\sqrt{3}$ are each *not* a solution to the equation.

2 Solve $(x - 1)^2 = 16$. Explain or show your reasoning.

3 Here is one way to solve the equation $\frac{5}{9}y^2 = 5$. Explain what is done in each step.

$\frac{5}{9}y^2 = 5$	Original equation
$5y^2 = 45$	Step 1
$y^2 = 9$	Step 2
$y = 3$ or $y = -3$	Step 3

4 Diego and Jada are working together to solve the quadratic equation $(x-2)^2 = 100$.

Diego solves the equation by dividing each side of the equation by 2 and then adding 2 to each side. He writes:

$$(x-2) = 50$$
$$x = 52$$

Jada asks Diego why he divides each side by 2 and he says, "I want to find a number that equals 100 when multiplied by itself. That number is half of 100."

a. What mistake is Diego making?

b. If you were Jada, what could you say to Diego to help him realize his mistake?

Sec B

5 from Unit 8, Lesson 1

A billboard installer accidentally drops a tool while working on a billboard. The height of the tool t seconds after it is dropped is given by the function $h(t) = 115 - 16t^2$, where h is in feet.

a. Find $h(2)$. Explain what this value means in this situation.

b. Find $h(0)$. What does this value tell us about the situation?

c. Is the tool still in the air 4 seconds after it is dropped? Explain how you know.

6 from Unit 8, Lesson 2

A zoo offers unlimited drink refills to visitors who purchase its souvenir cup. The cup and the first fill cost \$10, and refills after that are \$2 each. The expression $10 + 2r$ represents the total cost of the cup and r refills.

a. A family visited the zoo several times over a summer. That summer, they paid \$30 for one cup and multiple refills. How many refills did they buy?

b. A visitor has \$18 to spend on drinks at the zoo today and buys a souvenir cup. How many refills can they afford during the visit?

c. Another visitor spent \$10 on this deal. Did they buy any refills? Explain how you know.

Sec B

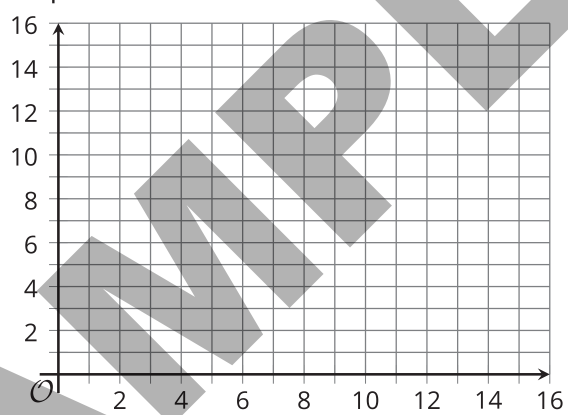
7 from Unit 5, Lesson 17

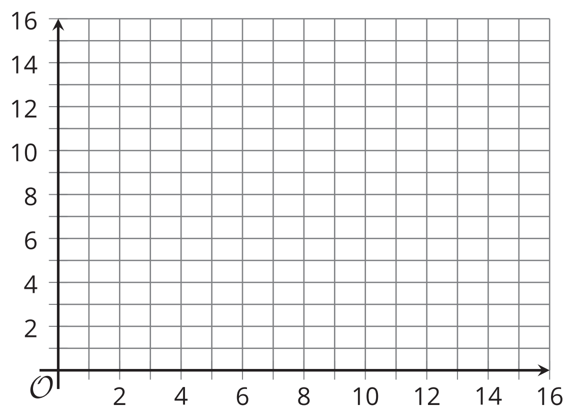
Clare is 5 years older than her sister.

a. Write an equation that defines her sister's age, s, as a function of Clare's age, c.

b. Write an equation that defines Clare's age, c, as a function of her sister's age, s.

c. Graph each function. Be sure to label the axes.



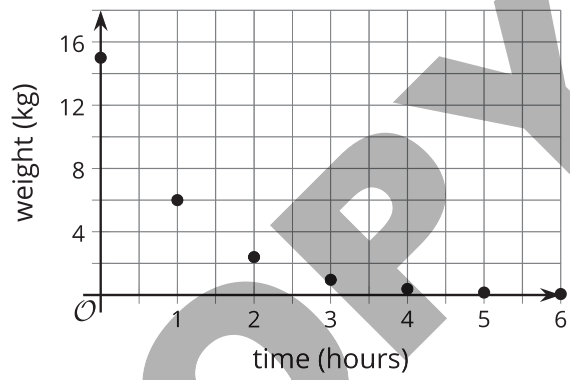


d. Describe how the two graphs compare.

8 from Unit 6, Lesson 4

The graph shows a model for the weight of snow as it melts. The weight decreases exponentially.

a. By what factor does the weight of the snow decrease each hour? Explain how you know.



b. Does the model predict that the weight of the snow will reach 0 kg? Explain your reasoning.

c. Will the weight of the actual snow, represented by the graph, reach 0? Explain how you know.

Unit 8, Lesson 4

Addressing CA CCSSM A-REI.4, F-LE.6; building on A-REI.1, A-REI.3; building toward A-CED.1, A-SSE.3; practicing MP7 and MP8

Solving Quadratic Equations with the Zero Product Property

Let's find solutions to equations that contain products that equal zero.

Sec B

4.1 Math Talk: Solve These Equations

What values of the variables make each equation true?

- $6 + 2a = 0$
- $7b = 0$
- $7(c - 5) = 0$
- $g \cdot h = 0$

4.2 Take the Zero Product Property Out for a Spin

For each equation, find its solution or solutions. Be prepared to explain your reasoning.

1. $x - 3 = 0$
2. $x + 11 = 0$
3. $2x + 11 = 0$
4. $x(2x + 11) = 0$
5. $(x - 3)(x + 11) = 0$
6. $(x - 3)(2x + 11) = 0$
7. $x(x + 3)(3x - 4) = 0$

Are you ready for more?

1. Use factors of 48 to find as many solutions as you can to the equation $(x - 3)(x + 5) = 48$.
2. Once you think you have all the solutions, explain why these must be the only solutions.

4.3 Revisiting a Projectile

We have seen quadratic functions modeling the height of a projectile as a function of time.

Here are two ways to define the same function that approximates the height of a projectile in meters, t seconds after launch:

$$h(t) = -5t^2 + 27t + 18 \qquad h(t) = (-5t - 3)(t - 6)$$

1. Which way of defining the function allows us to use the zero product property to find out when the height of the object is 0 meters?
2. Without graphing, determine at what time the height of the object is 0 meters. Show your reasoning.

Lesson 4 Summary

The **zero product property** says that if the product of two numbers is 0, then one of the numbers must be 0. In other words, if $a \cdot b = 0$, then either $a = 0$ or $b = 0$. This property is handy when an equation we want to solve states that the product of two factors is 0.

Suppose we want to solve $m(m + 9) = 0$. This equation says that the product of m and $(m + 9)$ is 0. For this to be true, either $m = 0$ or $m + 9 = 0$, so both 0 and -9 are solutions.

Here is another equation: $(u - 2.345)(14u + 2) = 0$. The equation says the product of $(u - 2.345)$ and $(14u + 2)$ is 0, so we can use the zero product property to help us find the values of u. For the equation to be true, one of the factors must be 0.

- For $u - 2.345 = 0$ to be true, u would have to be 2.345.
- For $14u + 2 = 0$ or $(14u = \text{-}2)$ to be true, u would have to be $-\frac{2}{14}$, or $-\frac{1}{7}$.

The solutions are 2.345 and $-\frac{1}{7}$.

In general, when a quadratic expression in factored form is on one side of an equation and 0 is on the other side, we can use the zero product property to find its solutions.

This property is unique to 0. Given an equation like $a \cdot b = 6$, the factors could be 2 and 3, 1 and 6, -12 and $-\frac{1}{2}$, π and $\frac{6}{\pi}$, or any other of the infinite number of combinations. This type of equation does not give insight into the value of a or b.

Glossary

- zero product property

Lesson 4 Practice Problems

1 If the equation $(x + 10)x = 0$ is true, which statement is also true according to the zero product property?

A. Only $x = 0$.

B. Either $x = 0$ or $x + 10 = 0$.

C. Either $x^2 = 0$ or $10x = 0$.

D. Only $x + 10 = 0$.

Sec B

2 What are the solutions to the equation $(10 - x)(3x - 9) = 0$?

A. -10 and 3

B. -10 and 9

C. 10 and 3

D. 10 and 9

3 Solve each equation.

a. $(x - 6)(x + 5) = 0$

b. $(x - 3)(\frac{2}{3}x - 6) = 0$

c. $(-3x - 15)(x + 7) = 0$

4 Consider the expressions $(x - 4)(3x - 6)$ and $3x^2 - 18x + 24$.

Show that the two expressions define the same function.

5 Kiran saw that if the equation $(x + 2)(x - 4) = 0$ is true, then by the zero product property, either $x + 2$ is 0 or $x - 4$ is 0. He then reasoned that, if $(x + 2)(x - 4) = 72$ is true, then either $x + 2$ is equal to 72 or $x - 4$ is equal to 72.

Explain why Kiran's conclusion is incorrect.

6 from Unit 8, Lesson 2

Andre wants to solve the equation $5x^2 - 4x - 18 = 20$. He uses a graphing calculator to graph $y = 5x^2 - 4x - 18$ and $y = 20$ and finds that the graphs cross at the points $(-2.39, 20)$ and $(3.19, 20)$.

a. Substitute each x-value that Andre found into the expression $5x^2 - 4x - 18$. Then evaluate the expression.

b. Why did neither solution make $5x^2 - 4x - 18$ equal exactly 20?

7 from Unit 8, Lesson 3

Select **all** the solutions to the equation $7x^2 = 343$.

A. 49

B. $-\sqrt{7}$

C. 7

D. -7

E. $\sqrt{49}$

F. $\sqrt{-49}$

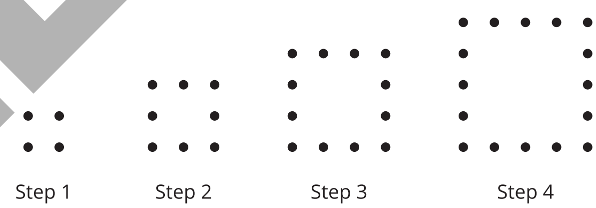
G. $-\sqrt{49}$

Sec B

8 from Unit 7, Lesson 2

Han says this pattern of dots can be represented by a quadratic relationship because the dots are arranged in a square in each step.

Do you agree? Explain your reasoning.



Addressing CA CCSSM A-REI.1, A-REI.4, A-REI.4b, A-REI.10; building on 6.EE.5, F-IF.7a; building toward A-REI.4; practicing MP3, MP5, MP7

How Many Solutions?

Let's use graphs to investigate quadratic equations that have two solutions, one solution, or no solutions.

5.1 Math Talk: Four Equations

Decide whether each statement is true or false.

- 3 is the only solution to $x^2 - 9 = 0$.
- A solution to $x^2 + 25 = 0$ is -5.
- $x(x - 7) = 0$ has two solutions.
- 5 and -7 are the solutions to $(x - 5)(x + 7) = 12$.

5.2 Solving by Graphing

Han is solving three equations by graphing.

$$(x - 5)(x - 3) = 0$$

$$(x - 5)(x - 3) = -1$$

$$(x - 5)(x - 3) = -4$$

1. To solve the first equation, $(x - 5)(x - 3) = 0$, he graphs $y = (x - 5)(x - 3)$, and then looks for the x-intercepts of the graph.

   a. Explain why the x-intercepts can be used to solve $(x - 5)(x - 3) = 0$.

   b. What are the solutions?

2. To solve the second equation, Han rewrites it as $(x-5)(x-3)+1=0$. He then graphs $y=(x-5)(x-3)+1$.

   Use graphing technology to graph $y=(x-5)(x-3)+1$. Then, use the graph to solve the equation. Be prepared to explain how you use the graph for solving.

3. Solve the third equation using Han's strategy.
4. Think about the strategy you used and the solutions you found.
   a. Why might it be helpful to rearrange each equation to equal 0 on one side and then graph the expression on the nonzero side?

   b. How many solutions does each of the three equations have?

Are you ready for more?

The equations $(x-3)(x-5)=\text{-}1$, $(x-3)(x-5)=0$, and $(x-3)(x-5)=3$ all have whole-number solutions.

1. Use graphing technology to graph each of the following pairs of equations on the same coordinate plane. Analyze the graphs, and explain how each pair helps to solve the related equation.
   - $y=(x-3)(x-5)$ and $y=\text{-}1$
   - $y=(x-3)(x-5)$ and $y=0$
   - $y=(x-3)(x-5)$ and $y=3$
2. Use the graphs to help you find a few other equations of the form $(x-3)(x-5)=z$ that have whole-number solutions.

3. Find a pattern in the values of z that give whole-number solutions.

4. Without solving, determine if $(x - 5)(x - 3) = 120$ and $(x - 5)(x - 3) = 399$ have whole-number solutions. Explain your reasoning.

5.3 Finding All the Solutions

Solve each equation. Be prepared to explain or show your reasoning.

1. $x^2 = 121$

2. $x^2 - 31 = 5$

3. $(x - 4)(x - 4) = 0$

4. $(x + 3)(x - 1) = 5$

5. $(x + 1)^2 = -4$

6. $(x - 4)(x - 1) = 990$

Sec B

5.4 Analyzing Errors in Equation Solving

1. Consider $(x-5)(x+1)=7$. Priya reasons that if this is true, then either $x-5=7$ or $x+1=7$. So, the solutions to the original equation are 12 and 6.

   Do you agree? If not, where was the mistake in Priya's reasoning?

2. Consider $x^2-10x=0$. Diego says to solve we can just divide each side by x to get $x-10=0$, so the solution is 10. Mai says, "I wrote the expression on the left in factored form, which gives $x(x-10)=0$, and ended up with two solutions: 0 and 10."

   Do you agree with either strategy? Explain your reasoning.

Lesson 5 Summary

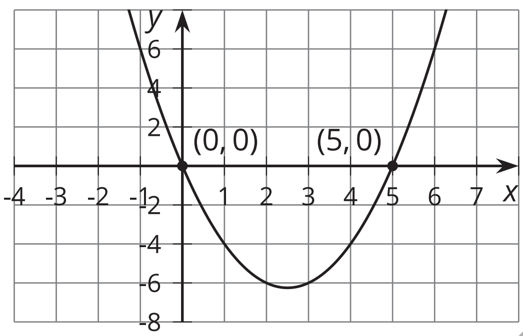
Quadratic equations can have two, one, or no solutions.

We can find out how many solutions a quadratic equation has and what the solutions are by rearranging the equation into the form of an equation with one side equal to 0, graphing the function that the expression defines, and determining its zeros. Here are some examples.

- $x^2 = 5x$

  Let's first subtract $5x$ from each side and rewrite the equation as $x^2 - 5x = 0$. We can think of solving this equation as finding the zeros of a function defined by $x^2 - 5x$.

  If the output of this function is y, we can graph $y = x^2 - 5x$ and identify where the graph intersects the x-axis, or where the y-coordinate is 0.



From the graph, we can see that the x-intercepts are $(0, 0)$ and $(5, 0)$, so $x^2 - 5x$ equals 0 when x is 0 and when x is 5.

The graph readily shows that there are two solutions to the equation.

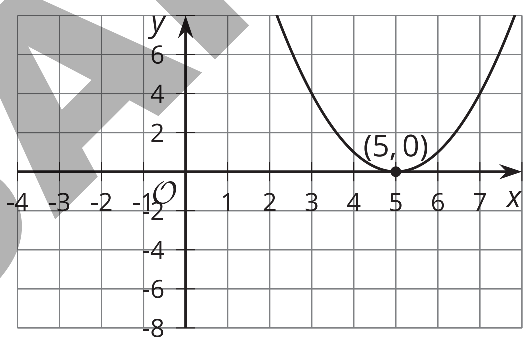
Note that the equation $x^2 = 5x$ can be solved without graphing, but we need to be careful *not* to divide both sides by x. Doing so will give us $x = 5$ but will show no trace of the other solution, $x = 0$!

Even though dividing both sides by the same value is usually acceptable for solving equations, we avoid dividing by the same variable because it may eliminate a solution.

- $(x - 6)(x - 4) = -1$

  Let's rewrite the equation as $(x - 6)(x - 4) + 1 = 0$ and consider it to represent a function defined by $(x - 6)(x - 4) + 1$ and whose output, y, is 0.

  Let's graph $y = (x - 6)(x - 4) + 1$ and identify the x-intercepts.



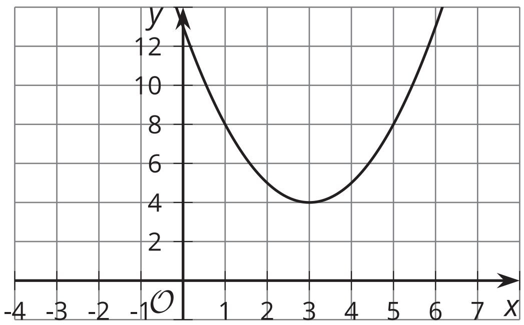
The graph shows one x-intercept at $(5, 0)$. This tells us that the function defined by $(x - 6)(x - 4) + 1$ has only one zero.

It also means that the equation $(x - 6)(x - 4) + 1 = 0$ is true only when $x = 5$. The value 5 is the only solution to the equation.

- $(x-3)(x-3) = \text{-}4$

  Rearranging the equation gives $(x-3)(x-3)+4=0$.

  Let's graph $y=(x-3)(x-3)+4$ and find the x-intercepts.

  The graph does not intersect the x-axis, so there are no x-intercepts.

  This means there are no x-values that can make the expression $(x-3)(x-3)+4$ equal 0, so the function defined by $y=(x-3)(x-3)+4$ has no zeros.

  The equation $(x-3)(x-3) = \text{-}4$ has no solutions.

  We can see that this is the case even without graphing. $(x-3)(x-3) = \text{-}4$ is $(x-3)^2 = \text{-}4$. Because no number can be squared to get a negative value, the equation has no solutions.

Earlier you learned that graphing is not always reliable for showing precise solutions. This is still true here. The x-intercepts of a graph are not always whole-number values. While they can give us an idea of how many solutions there are and what the values may be (at least approximately), for exact solutions we still need to rely on algebraic ways of solving.

Lesson 5 Practice Problems

1 Rewrite each equation so that the expression on one side could be graphed and the x-intercepts of the graph would show the solutions to the equation.

a. $3x^2 = 81$

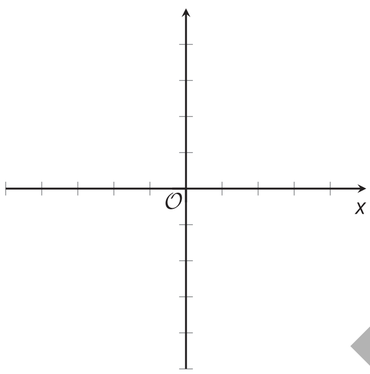
b. $(x - 1)(x + 1) - 9 = 5x$

c. $x^2 - 9x + 10 = 32$

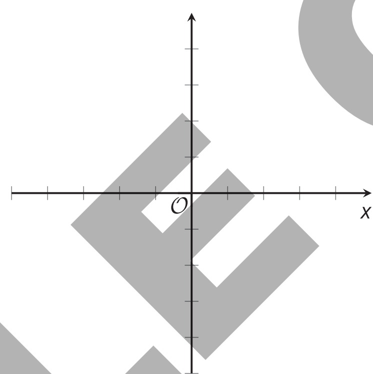
d. $6x(x - 8) = 29$

2 a. Here are equations that define quadratic functions f, g, and h. Sketch a graph, by hand or using technology, that represents each equation.

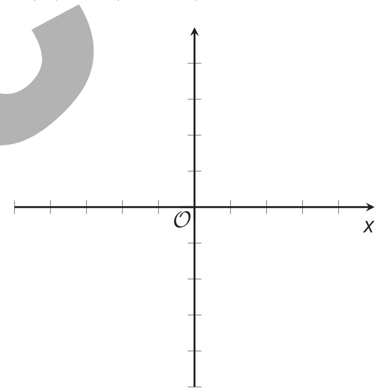
$f(x) = x^2 + 4$



$g(x) = x(x + 3)$



$h(x) = (x - 1)^2$

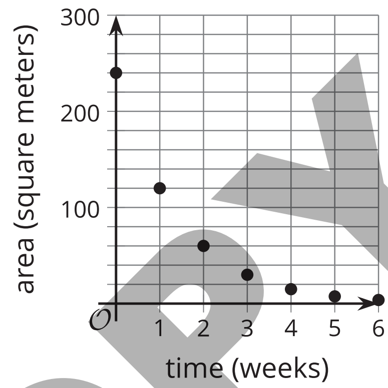


b. Determine how many solutions each of $f(x) = 0$, $g(x) = 0$, and $h(x) = 0$ has. Explain how you know.

3 Mai is solving the equation $(x - 5)^2 = 0$. She writes that the solutions are $x = 5$ and $x =$ -5. Han looks at her work and disagrees. He says that only $x = 5$ is a solution. Who do you agree with? Explain your reasoning.

4 from Unit 6, Lesson 6

The graph shows a model of the number of square meters, A, of a lake that is covered by algae w weeks after it was first measured.



In a second lake, the number of square meters, B, covered by algae is modeled by the equation $B = 975 \cdot \left(\frac{2}{5}\right)^w$, where w is the number of weeks since it was first measured.

For which algae population model is the area decreasing more rapidly? Explain how you know.

5 from Unit 8, Lesson 4

If the equation $(x - 4)(x + 6) = 0$ is true, which is also true according to the zero product property?

A. Only $x - 4 = 0$.

B. Only $x + 6 = 0$.

C. $x - 4 = 0$ or $x + 6 = 0$.

D. $x = -4$ or $x = 6$.

6 from Unit 8, Lesson 3

a. Solve the equation $25 = 4z^2$.

b. Show that your solution or solutions are correct.

Sec B

7 from Unit 8, Lesson 3

To solve the quadratic equation $3(x-4)^2 = 27$, Andre and Clare wrote the following:

Andre

$$\begin{aligned} 3(x-4)^2 &= 27 \\ (x-4)^2 &= 9 \\ x^2 - 4^2 &= 9 \\ x^2 - 16 &= 9 \\ x^2 &= 25 \\ x = 5 \quad &\text{or} \quad x = -5 \end{aligned}$$

Clare

$$\begin{aligned} 3(x-4)^2 &= 27 \\ (x-4)^2 &= 9 \\ x-4 &= 3 \\ x &= 7 \end{aligned}$$

Sec B

a. Identify the mistake each student made.

b. Solve the equation, and show your reasoning.

8 Decide if each equation has 0, 1, or 2 solutions, and explain how you know.

a. $x^2 - 144 = 0$

b. $x^2 + 144 = 0$

c. $x(x-5) = 0$

d. $(x-8)^2 = 0$

e. $(x+3)(x+7) = 0$

Unit 8, Lesson 6

Addressing CA CCSSM A-SSE.2; building on 6.G.1; building toward A-REI.4b, A-SSE.2, A-SSE.3a; practicing MP3 and MP7

Rewriting Quadratic Expressions in Factored Form (Part 1)

Let's write expressions in factored form.

6.1 Puzzles of Rectangles

Here are two puzzles that involve side lengths and areas of rectangles. Can you find the missing area in Figure A and the missing length in Figure B? Be prepared to explain your reasoning.

Figure A

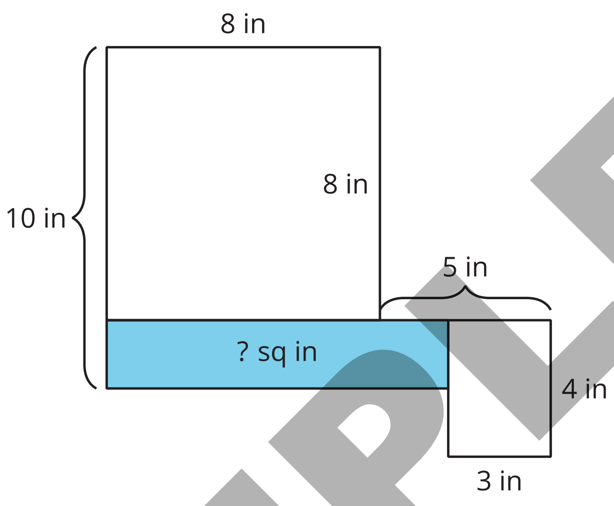
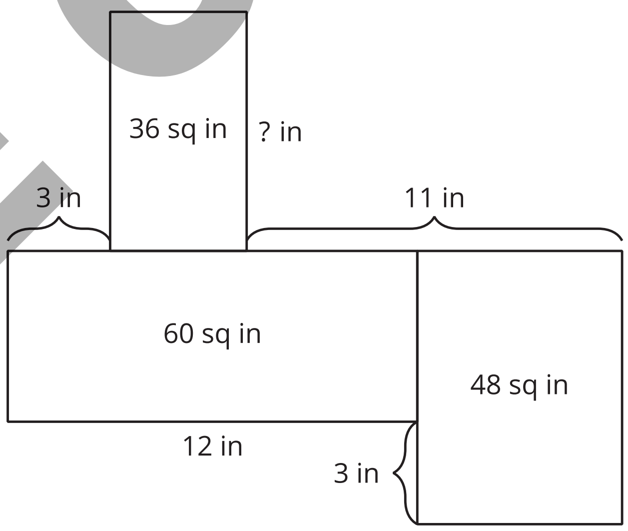


Figure B



6.2 Using Diagrams to Understand Equivalent Expressions

Compare the diagram that shows $(x+m)(x+n) = x^2 + mx + nx + mn$ and the partially completed diagram for $x^2 + 6x + 8$ to determine the values of m and n.

	x	n
x	x^2	nx
m	mx	mn

	x	
x	x^2	$\_\_\_\_\_ x$
	$\_\_\_\_\_ x$	8

Sec B

1. What does the bottom right corner tell you about the connection between the constant term, 8, and the missing parts of the factors, m and n?

2. What is the connection between the linear term, $6x$, and the two sections of the diagram with the underlines? How is that connected to m and n?

3. Find values for m and n, and rewrite $x^2 + 6x + 8$ in factored form.

4. Rewrite these quadratic expressions in factored form. As you work, consider how the values of the constant and linear terms help decide the values of m and n.
   a. $x^2 + 3x$
   b. $x^2 - 6x$
   c. $x^2 + 14x + 40$
   d. $x^2 - 6x + 5$
   e. $x^2 - 8x + 12$

6.3 Let's Rewrite Some Expressions!

Each row in the table contains a pair of equivalent expressions.

Complete the table with the missing expressions. If you get stuck, consider drawing a diagram.

factored form	standard form
$x(x+7)$	
	x^2+9x
	x^2-8x
$(x+6)(x+2)$	
	$x^2+13x+12$
$(x-6)(x-2)$	
	$x^2-7x+12$
	x^2+6x+9
	$x^2+10x+9$
	$x^2-10x+9$
	x^2-6x+9
	$x^2+(m+n)x+mn$

Are you ready for more?

A mathematician threw a party. She told her guests, "I have a riddle for you. I have three daughters. The product of their ages is 72. The sum of their ages is the same as my house number. How old are my daughters?"

The guests went outside to look at the house number. They thought for a few minutes, and then said, "This riddle can't be solved!"

The mathematician said, "Oh yes, I forgot to tell you the last clue. My youngest daughter prefers strawberry ice cream."

With this last clue, the guests could solve the riddle. How old are the mathematician's daughters?

Lesson 6 Summary

Previously, you learned how to expand a quadratic expression in factored form and write it in standard form by applying the distributive property.

For example, to expand $(x+4)(x+5)$, we apply the distributive property to multiply x by $(x+5)$ and 4 by $(x+5)$. Then, we apply the property again to multiply x by x, x by 5, 4 by x, and 4 by 5.

To keep track of all the products, we could make a diagram like this:

	x	4
x		
5		

Next, we could write the products of each pair inside the spaces:

	x	4
x	x^2	$4x$
5	$5x$	$4 \cdot 5$

The diagram helps us see that $(x+4)(x+5)$ is equivalent to $x^2 + 5x + 4x + 4 \cdot 5$, or in standard form, $x^2 + 9x + 20$.

- The **linear term**, or the term with a single factor of x in the standard form of a quadratic expression, is $9x$ and has a *coefficient* of 9, which is the sum of 5 and 4.
- The *constant term*, 20, is the product of 5 and 4.

We can use these observations to reason in the other direction: starting with an expression in standard form and writing it in factored form.

For example, suppose we wish to write $x^2 - 11x + 24$ in factored form.

Let's start by creating a diagram and writing in the terms x^2 and 24.

We need to think of two numbers that multiply to make 24 and add up to -11.

	x	
x	x^2	
		24

After some thinking, we see that -8 and -3 meet these conditions. The product of -8 and -3 is 24. The sum of -8 and -3 is -11.

So, $x^2 - 11x + 24$ written in factored form is $(x - 8)(x - 3)$.

	x	-8
x	x^2	$-8x$
-3	$-3x$	24

Glossary

- linear term

Lesson 6 Practice Problems

1 Find two numbers that satisfy the requirements. If you get stuck, try listing all the factors of the first number.

a. Find two numbers that multiply to 17 and add to 18.

b. Find two numbers that multiply to 20 and add to 9.

c. Find two numbers that multiply to 11 and add to -12.

d. Find two numbers that multiply to 36 and add to -20.

Sec B

2 Use the diagram to show that:

$(x+4)(x+2)$ is equivalent to x^2+6x+8.

	x	2
x		
4		

$(x-10)(x-3)$ is equivalent to $x^2-13x+30$.

	x	-10
x		
-3		

3 Select **all** expressions that are equivalent to $x-5$.

A. $x+(-5)$

B. $x-(-5)$

C. $-5+x$

D. $-5-x$

E. $5-x$

F. $-5-(-x)$

G. $5+x$

4 Here are pairs of equivalent expressions—one in standard form and the other in factored form. Find the missing numbers.

a. $x^2 + \boxed{}x + \boxed{}$ and $(x - 9)(x - 3)$

b. $x^2 + 12x + 32$ and $(x + 4)(x + \boxed{})$

c. $x^2 - 12x + 35$ and $(x - 5)(x + \boxed{})$

d. $x^2 + 5x - 36$ and $(x - 4)(x + \boxed{})$

5 from Unit 8, Lesson 4

Find all the values for the variable that make each equation true.

a. $b(b - 4.5) = 0$

b. $(7x + 14)(7x + 14) = 0$

c. $(2x + 4)(x - 4) = 0$

d. $(-2 + u)(3 - u) = 0$

Sec B

6 from Unit 5, Lesson 10

Consider the function $p(x) = \frac{x-3}{2x-6}$.

a. Evaluate $p(1)$, writing out every step.

b. Evaluate $p(3)$, writing out every step. You will run into some trouble. Describe it.

c. What is a possible domain for p?

7 from Unit 8, Lesson 5

Technology required.

When solving the equation $(2 - x)(x + 1) = 11$, Priya graphs $y = (2 - x)(x + 1) - 11$ and then looks to find where the graph crosses the x-axis.

Tyler looks at her work and says that graphing is unnecessary and Priya can set up the equations $2 - x = 11$ and $x + 1 = 11$, so the solutions are $x =$ -9 and $x = 10$.

a. Do you agree with Tyler? If not, where is the mistake in his reasoning?

b. How many solutions does the equation have? Find out by graphing Priya's equation.

Addressing CA CCSSM A-SSE.2; building on 7.NS.1, 7.NS.2; building toward A-REI.4b, A-SSE.2, A-SSE.3a; practicing MP7 and MP8

Rewriting Quadratic Expressions in Factored Form (Part 2)

Let's write some more expressions in factored form.

7.1 Sums and Products

1. The product of the integers 2 and -6 is -12. List all the other pairs of integers whose product is -12.

2. Of the pairs of factors you found, list all pairs that have a positive sum. Explain why they all have a positive sum.

3. Of the pairs of factors you found, list all pairs that have a negative sum. Explain why they all have a negative sum.

7.2 Negative Constant Terms

1. Each row of this table should have a pair of equivalent expressions. Complete the table. If you get stuck, consider drawing a diagram.

factored form	standard form
$(x+5)(x+6)$	
	$x^2+13x+30$
$(x-3)(x-6)$	
	$x^2-11x+18$

2. Each row in this table should have a pair of equivalent expressions. Complete the table. If you get stuck, consider drawing a diagram.

factored form	standard form
$(x+12)(x-3)$	
	$x^2-9x-36$
	$x^2-35x-36$
	$x^2+35x-36$

3. How are the two tables similar and different?

7.3 Factors of 100 and -100

1. Consider the expression $x^2 + bx + 100$.

   Complete the first table with all factor pairs of 100 that would give positive values of b, and the second table with factors that would give negative values of b.

   For each pair, state the b value they produce. (Use as many rows as needed.)

   positive value of b

factor 1	factor 2	b (positive)

   negative value of b

factor 1	factor 2	b (negative)

2. Consider the expression $x^2 + bx - 100$.

   Complete the first table with all factor pairs of -100 that would result in positive values of b, the second table with factors that would result in negative values of b, and the third table with factors that would result in a zero value of b.

   For each pair of factors, state the b value they produce. (Use as many rows as there are pairs of factors. You may not need all the rows.)

   positive value of b

factor 1	factor 2	b (positive)

   negative value of b

factor 1	factor 2	b (negative)

Sec B

zero value of b

factor 1	factor 2	b (zero)

3. Write each expression in factored form:

   a. $x^2 - 25x + 100$

   b. $x^2 + 15x - 100$

   c. $x^2 - 15x - 100$

   d. $x^2 + 99x - 100$

Sec B

Are you ready for more?

How many different integers b can you find so that the expression $x^2 + 10x + b$ can be written in factored form?

Lesson 7 Summary

When we rewrite expressions in factored form, it is helpful to remember that:

- Multiplying two positive numbers or two negative numbers results in a positive product.
- Multiplying a positive number and a negative number results in a negative product.

This means that if we want to find two factors whose product is 10, the factors must both be positive or both be negative. If we want to find two factors whose product is -10, one of the factors must be positive and the other negative.

Suppose we wanted to rewrite $x^2 - 8x + 7$ in factored form. Recall that subtracting a number can be thought of as adding the opposite of that number, so that expression can also be written as $x^2 + \text{-}8x + 7$. We are looking for two numbers that:

- Have a product of 7. The candidates are 7 and 1, and -7 and -1.
- Have a sum of -8. Only -7 and -1 from the list of candidates meet this condition.

The factored form of $x^2 - 8x + 7$ is therefore $(x + \text{-}7)(x + \text{-}1)$ or, written another way, $(x - 7)(x - 1)$.

To write $x^2 + 6x - 7$ in factored form, we would need two numbers that:

- Multiply to make -7. The candidates are 7 and -1, and -7 and 1.
- Add up to 6. Only 7 and -1 from the list of candidates add up to 6.

The factored form of $x^2 + 6x - 7$ is $(x + 7)(x - 1)$.

Sec B

Lesson 7 Practice Problems

1 Find two numbers that . . .

a. multiply to -40 and add to -6.

b. multiply to -40 and add to 6.

c. multiply to -36 and add to 9.

d. multiply to -36 and add to -5.

If you get stuck, try listing all the factors of the first number.

Sec B

2 Create a diagram to show that $(x-5)(x+8)$ is equivalent to $x^2+3x-40$.

3 Write a + or a – sign in each box so the expressions on each side of the equal sign are equivalent.

a. $(x \boxed{}\ 18)(x \boxed{}\ 3) = x^2 - 15x - 54$

b. $(x \boxed{}\ 18)(x \boxed{}\ 3) = x^2 + 21x + 54$

c. $(x \boxed{}\ 18)(x \boxed{}\ 3) = x^2 + 15x - 54$

d. $(x \boxed{}\ 18)(x \boxed{}\ 3) = x^2 - 21x + 54$

4 Match each quadratic expression in standard form with its equivalent expression in factored form.

A. $x^2 - 2x - 35$

B. $x^2 + 12x + 35$

C. $x^2 + 2x - 35$

D. $x^2 - 12x + 35$

1. $(x+5)(x+7)$
2. $(x-5)(x-7)$
3. $(x+5)(x-7)$
4. $(x-5)(x+7)$

5 Rewrite each expression in factored form. If you get stuck, try drawing a diagram.

a. $x^2 - 3x - 28$

b. $x^2 + 3x - 28$

c. $x^2 + 12x - 28$

d. $x^2 - 28x - 60$

6 from Unit 8, Lesson 5

Which equation has exactly one solution?

A. $x^2 = -4$

B. $(x+5)^2 = 0$

C. $(x+5)(x-5) = 0$

D. $(x+5)^2 = 36$

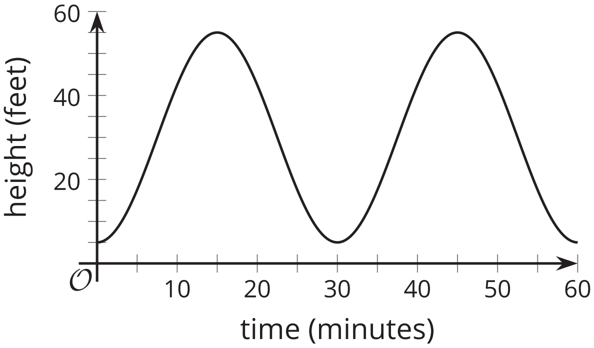
7 from Unit 5, Lesson 11

The graph represents the height of a passenger car on a ferris wheel, in feet, as a function of time, in seconds since the ride starts.

Use the graph to answer these questions:

a. What is $H(0)$?

b. Does $H(t) = 0$ have a solution? Explain how you know.



c. Describe the domain of the function.

d. Describe the range of the function.

Sec B

8 from Unit 8, Lesson 5

Elena solves the equation $x^2 = 7x$ by dividing both sides by x to get $x = 7$. She says the solution is 7.

Lin solves the equation $x^2 = 7x$ by rewriting the equation to get $x^2 - 7x = 0$. When she graphs the equation $y = x^2 - 7x$, the x-intercepts are $(0, 0)$ and $(7, 0)$. She says the solutions are 0 and 7.

Do you agree with either of them? Explain or show how you know.

9 from Unit 6, Lesson 7

A bacteria population, p, can be represented by the equation $p = 100{,}000 \cdot \left(\frac{1}{4}\right)^d$, where d is the number of days since it was measured.

a. What was the population 3 days before it was measured? Explain how you know.

b. What is the last day when the population was more than 1,000,000? Explain how you know.

Addressing CA CCSSM A-SSE.2; building on 6.EE.3; building toward A-APR.1, A-REI.4b, A-SSE.2, A-SSE.3a; practicing MP7 and MP8

A1

Sec B

Rewriting Quadratic Expressions in Factored Form (Part 3)

Let's look closely at some special kinds of factors.

8.1 Math Talk: Products of Large-ish Numbers

Evaluate mentally.

- $9 \cdot 11$
- $19 \cdot 21$
- $99 \cdot 101$
- $119 \cdot 121$

8.2 Can Products Be Written as Differences?

1. Clare claims that $(10 + 3)(10 - 3)$ is equivalent to $10^2 - 3^2$ and $(20 + 1)(20 - 1)$ is equivalent to $20^2 - 1^2$. Do you agree? Show your reasoning.

2. a. Use your observations from the first question to evaluate $(100 + 5)(100 - 5)$. Show your reasoning.

   b. Check your answer by computing $105 \cdot 95$.

3. Is $(x+4)(x-4)$ equivalent to $x^2 - 4^2$? Support your answer:

With a diagram:

	x	4
x		
-4		

Without a diagram:

4. Is $(x+4)^2$ equivalent to $x^2 + 4^2$? Support your answer, either with or without a diagram.

Sec B

Are you ready for more?

1. Explain how your work in the previous questions can help you mentally evaluate $22 \cdot 18$ and $45 \cdot 35$.

2. Here is a shortcut that can be used to mentally square any two-digit number. Let's take 83^2, for example.

   - 83 is $80 + 3$.
   - Compute 80^2 and 3^2, which give 6,400 and 9. Add these values to get 6,409.
   - Compute $80 \cdot 3$, which is 240. Double it to get 480.
   - Add 6,409 and 480 to get 6,889.

   Try using this method to find the squares of some other two-digit numbers. (With some practice, it is possible to get really fast at this!) Then, explain why this method works.

8.3 What If There Is No Linear Term?

Each row has a pair of equivalent expressions.

Complete the table.

If you get stuck, consider drawing a diagram. (Heads up: One of them is impossible.)

factored form	standard form
$(x - 10)(x + 10)$	
$(2x + 1)(2x - 1)$	
$(4 - x)(4 + x)$	
	$x^2 - 81$
	$49 - y^2$
	$9z^2 - 16$
	$25t^2 - 81$
$(c + \frac{2}{5})(c - \frac{2}{5})$	
	$\frac{49}{16} - d^2$
$(x + \sqrt{5})(x - \sqrt{5})$	
	$x^2 - 6$
	$x^2 + 100$

Sec B

Lesson 8 Summary

Sometimes expressions in standard form don't have a linear term. Can they still be written in factored form?

Let's take $x^2 - 9$ as an example. To help us write it in factored form, we can think of it as having a linear term with a coefficient of 0: $x^2 + 0x - 9$.

We know that we need to find two numbers that multiply to make -9 and add up to 0. The numbers 3 and -3 meet both requirements, so the factored form is $(x + 3)(x - 3)$.

To check that this expression is indeed equivalent to $x^2 - 9$, we can expand the factored expression by applying the distributive property: $(x + 3)(x - 3) = x^2 - 3x + 3x + (\text{-}9)$. Adding $\text{-}3x$ and $3x$ gives 0, so the expanded expression is $x^2 - 9$.

In general, a quadratic expression that is a difference of two squares and has the form $a^2 - b^2$ can be rewritten as $(a + b)(a - b)$.

Here is a more complicated example: $49 - 16y^2$. This expression can be written as $7^2 - (4y)^2$, so an equivalent expression in factored form is $(7 + 4y)(7 - 4y)$.

What about $x^2 + 9$? Can it be written in factored form?

Let's think about this expression as $x^2 + 0x + 9$. Can we find two numbers that multiply to make 9 and add up to 0? Here are factors of 9 and their sums:

- 9 and 1, sum: 10
- -9 and -1, sum: -10
- 3 and 3, sum: 6
- -3 and -3, sum: -6

For two numbers to add up to 0, they need to be opposites (a negative and a positive), but a pair of opposites cannot multiply to make positive 9, because multiplying a negative number and a positive number always gives a negative product.

Because there are no numbers that multiply to make 9 and also add up to 0, it is not possible to write $x^2 + 9$ in factored form using the kinds of numbers that we know about.

Lesson 8 Practice Problems

1 Match each quadratic expression given in factored form with an equivalent expression in standard form. One expression in standard form has no match.

A. $(y+x)(y-x)$

B. $(11+x)(11-x)$

C. $(x-11)(x+11)$

D. $(x-y)(x-y)$

1. $121-x^2$
2. $x^2+2xy-y^2$
3. y^2-x^2
4. $x^2-2xy+y^2$
5. x^2-121

2 Both $(x-3)(x+3)$ and $(3-x)(3+x)$ contain a sum and a difference and have only 3 and x in each factor.

If each expression is rewritten in standard form, will the two expressions be the same? Explain or show your reasoning.

3

a. Show that the expressions $(5+1)(5-1)$ and 5^2-1^2 are equivalent.

b. The expressions $(30-2)(30+2)$ and 30^2-2^2 are equivalent and can help us find the product of two numbers. Which two numbers are they?

c. Write $94 \cdot 106$ as a product of a sum and a difference, and then as a difference of two squares. What is the value of $94 \cdot 106$?

4 Write each expression in factored form. If not possible, write "not possible."

a. $x^2 - 144$

b. $x^2 + 16$

c. $25 - x^2$

d. $b^2 - a^2$

e. $100 + y^2$

Sec B

5 from Unit 8, Lesson 4

What are the solutions to the equation $(x - a)(x + b) = 0$?

A. a and b

B. -a and -b

C. a and -b

D. -a and b

6 from Unit 8, Lesson 6

Create a diagram to show that $(x - 3)(x - 7)$ is equivalent to $x^2 - 10x + 21$.

7 from Unit 8, Lesson 6

Select **all** the expressions that are equivalent to $8 - x$.

A. $x - 8$

B. $8 + (\text{-}x)$

C. $\text{-}x - (\text{-}8)$

D. $\text{-}8 + x$

E. $x - (\text{-}8)$

F. $x + (\text{-}8)$

G. $\text{-}x + 8$

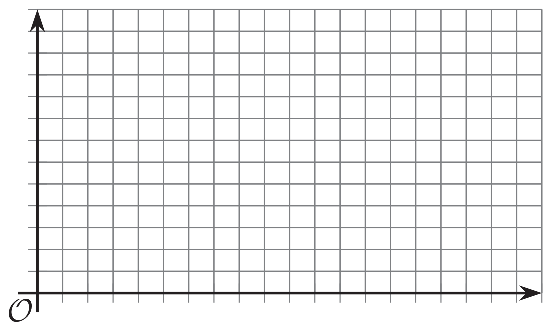
8 from Unit 5, Lesson 11

Mai fills a tall cup with hot cocoa, 12 centimeters in height. She waits 5 minutes for it to cool. Then, she starts drinking in sips, at an average rate of 2 centimeters of height every 2 minutes, until the cup is empty.

The function C gives the height of hot cocoa in Mai's cup, in centimeters, as a function of time, in minutes.

a. Sketch a possible graph of C. Be sure to include a label and a scale for each axis.

b. What quantities do the domain and range represent in this situation?



c. Describe the domain and range of C.

9 from Unit 6, Lesson 7

Two bacteria populations are measured at the same time. One bacteria population, p, is modeled by the equation $p = 250{,}000 \cdot \left(\frac{1}{2}\right)^d$, where d is the number of days since it was first measured. The second bacteria population, q, is modeled by the equation $q = 500{,}000 \cdot \left(\frac{1}{3}\right)^d$.

Which statement is true about the two populations?

A. The second population will always be larger than the first.

B. Both populations are increasing.

C. The second bacteria population decreases more rapidly than the first.

D. When initially measured, the first population is larger than the second.

Addressing CA CCSSM A-REI.4b, A-SSE.3a, F-IF.8a; building on A-REI.4b; building toward A-REI.4; practicing MP1, MP6, MP7

Solving Quadratic Equations by Using Factored Form

Let's solve some quadratic equations that, before now, we could only solve by graphing.

Sec B

9.1 Why Would You Do That?

Let's try to find at least one solution to $x^2 - 2x - 35 = 0$.

1. Choose a whole number between 0 and 10.
2. Evaluate the expression $x^2 - 2x - 35$, using your number for x.
3. If your number doesn't give a value of 0, look for someone in your class who may have chosen a number that does make the expression equal 0. Which number is it?
4. There is another number that would make the expression $x^2 - 2x - 35$ equal 0. Can you find it?

9.2 Let's Solve Some Equations!

1. To solve the equation $n^2 - 2n = 99$, Tyler wrote out the following steps. Analyze Tyler's work. Write down what Tyler did in each step.

$n^2 - 2n = 99$	Original equation
$n^2 - 2n - 99 = 0$	Step 1
$(n - 11)(n + 9) = 0$	Step 2
$n - 11 = 0 \quad \text{or} \quad n + 9 = 0$	Step 3
$n = 11 \quad \text{or} \quad n = -9$	Step 4

2. Solve each equation by rewriting it in factored form and using the zero product property. Show your reasoning.

   a. $x^2 + 8x + 15 = 0$

   b. $x^2 - 8x + 12 = 5$

   c. $x^2 - 10x - 11 = 0$

   d. $49 - x^2 = 0$

   e. $(x + 4)(x + 5) - 30 = 0$

Are you ready for more?

Solve this equation, and explain or show your reasoning.

$$(x^2 - x - 20)(x^2 + 2x - 3) = (x^2 + 2x - 8)(x^2 - 8x + 15)$$

9.3 Revisiting Quadratic Equations with Only One Solution

1. The other day, we saw that a quadratic equation can have 0, 1, or 2 solutions. Sketch graphs that represent three quadratic functions: one that has no zeros, one with 1 zero, and one with 2 zeros.

2. Use graphing technology to graph the function defined by $f(x) = x^2 - 2x + 1$. What do you notice about the x-intercepts of the graph? What do the x-intercepts reveal about the function?

3. Solve $x^2 - 2x + 1 = 0$ by using the factored form and zero product property. Show your reasoning. What solutions do you get?

4. Write an equation to represent another quadratic function that you think will have only one zero. Graph it to check your prediction.

Sec B

Lesson 9 Summary

Recently, we learned strategies for transforming expressions from standard form to factored form. In earlier lessons, we have also seen that when a quadratic expression is in factored form, we can find values of the variable that make the expression equal zero. Suppose we are solving the equation $x(x+4)=0$, which says that the product of x and $x+4$ is 0. By the zero product property, we know this means that either $x=0$ or $x+4=0$, which then tells us that 0 and -4 are solutions.

Together, these two skills—writing quadratic expressions in factored form and using the zero product property when a factored expression equals 0—allow us to solve quadratic equations given in other forms. Here is an example:

$n^2-4n=140$	Original equation
$n^2-4n-140=0$	Subtract 140 from each side so the right side is 0.
$(n-14)(n+10)=0$	Rewrite in factored form.
$n-14=0 \quad \text{or} \quad n+10=0$	Apply the zero product property.
$n=14 \quad \text{or} \quad n=\text{-}10$	Solve each equation.

When a quadratic equation is written as as an expression in factored form equal to 0, we can also see the number of solutions the equation has.

In the previous example, it is not obvious how many solutions there are when the equation is in the form $n^2-4n-140=0$. When the equation is rewritten as $(n-14)(n+10)=0$, we can see that there are two numbers that could make the expression equal 0: 14 and -10.

How many solutions does the equation $x^2-20x+100=0$ have?

Let's rewrite it in factored form: $(x-10)(x-10)=0$. The two factors are identical, which means that there is only one value of x that makes the expression $(x-10)(x-10)$ equal 0. The equation has only one solution: 10.

Lesson 9 Practice Problems

1 Find **all** the solutions to each equation.

a. $x(x-1)=0$

b. $(5-x)(5+x)=0$

c. $(2x+1)(x+8)=0$

d. $(3x-3)(3x-3)=0$

e. $(7-x)(x+4)=0$

2 Rewrite each equation in factored form and solve using the zero product property.

a. $d^2-7d+6=0$

b. $x^2+18x+81=0$

c. $u^2+7u-60=0$

d. $x^2+0.2x+0.01=0$

3 Here is how Elena solves the quadratic equation $x^2-3x-18=0$.

$$\begin{gathered} x^2-3x-18=0 \\ (x-3)(x+6)=0 \\ x-3=0 \quad \text{or} \quad x+6=0 \\ x=3 \quad \text{or} \quad x=-6 \end{gathered}$$

Is her work correct? If you think there is an error, explain the error and correct it.

Otherwise, check her solutions by substituting them into the original equation and showing that the equation remains true.

4 Jada is working on solving a quadratic equation, as shown here.

$$p^2 - 5p = 0$$
$$p(p-5) = 0$$
$$p - 5 = 0$$
$$p = 5$$

She thinks that her solution is correct because substituting 5 for p in the original expression, $p^2 - 5p$, gives $5^2 - 5(5)$, which is $25 - 25$, or 0.

Explain the mistake that Jada made, and show the correct solutions.

5 from Unit 8, Lesson 4

Choose a statement to correctly describe the zero product property.

If a and b are numbers, and $a \cdot b = 0$, then:

A. Both a and b must equal 0.

B. Neither a nor b can equal 0.

C. Either $a = 0$ or $b = 0$.

D. $a + b$ must equal 0.

6 from Unit 8, Lesson 6

Which expression is equivalent to $x^2 - 7x + 12$?

A. $(x+3)(x+4)$

B. $(x-3)(x-4)$

C. $(x+2)(x+6)$

D. $(x-2)(x-6)$

7 from Unit 8, Lesson 6

These quadratic expressions are given in standard form. Rewrite each expression in factored form. If you get stuck, try drawing a diagram.

a. $x^2 + 7x + 6$

b. $x^2 - 7x + 6$

c. $x^2 - 5x + 6$

d. $x^2 + 5x + 6$

8 from Unit 7, Lesson 4

Select **all** the functions whose output values will eventually overtake the output values of function f, defined by $f(x) = 25x^2$.

A. $g(x) = 5(2)^x$

B. $h(x) = 5^x$

C. $j(x) = x^2 + 5$

D. $k(x) = (\frac{5}{2})^x$

E. $m(x) = 5 + 2^x$

F. $n(x) = 2x^2 + 5$

Sec B

9 from Unit 5, Lesson 12

A piecewise function, p, is defined by this rule: $p(x) = \begin{cases} x - 1, & x \leq -2 \\ 2x - 1, & x > -2 \end{cases}$

Find the value of p at each given input.

a. $p(-20)$

b. $p(-2)$

c. $p(4)$

d. $p(5.7)$

Sec B

Addressing CA CCSSM A-REI.4b, A-REI.11, A-SSE.1b, A-SSE.2, F-IF.4, F-IF.8a, F-LE.6; building on A-REI.4b; practicing MP1, MP6, MP7

Rewriting Quadratic Expressions in Factored Form (Part 4)

Let's transform more-complicated quadratic expressions into the factored form.

10.1 Which Three Go Together: Quadratic Expressions

Which three go together? Why do they go together?

A. $(x+4)(x-3)$

B. $3x^2-8x+5$

C. x^2-25

D. x^2+2x+3

10.2 A Little More Advanced

Each row in each table has a pair of equivalent expressions. Complete the tables. If you get stuck, try drawing a diagram.

1.

factored form	standard form
$(3x+1)(x+4)$	
$(3x+2)(x+2)$	
$(3x+4)(x+1)$	

2.

factored form	standard form
	$5x^2 + 21x + 4$
	$3x^2 + 15x + 12$
	$6x^2 + 19x + 10$

Are you ready for more?

Sec B

Here are three quadratic equations, each with two solutions. Find both solutions to each equation, using the zero product property somewhere along the way. Show each step in your reasoning.

$x^2 = 6x$ $\quad$ $x(x+4) = x+4$ $\quad$ $2x(x-1) + 3x - 3 = 0$

10.3 Timing a Blob of Water

An engineer is designing a fountain that shoots out drops of water. The nozzle from which the water is launched is 3 meters above the ground. It shoots out a drop of water at a vertical velocity of 9 meters per second.

Function h models the height in meters, h, of a drop of water t seconds after it is shot out from the nozzle. The function is defined by the equation $h(t) = -5t^2 + 9t + 3$.

How many seconds until the drop of water hits the ground?

1. Write an equation that we could solve to answer the question.

2. Try to solve the equation by writing the expression in factored form and using the zero product property.

3. Solve the equation by graphing the function using graphing technology. Explain how you found the solution.

10.4 Making It Simpler

Here is a clever way to think about quadratic expressions that would make it easier to rewrite them in factored form.

$9x^2 + 21x + 10$
$(3x)^2 + 7(3x) + 10$
$N^2 + 7N + 10$
$(N + 2)(N + 5)$
$(3x + 2)(3x + 5)$

1. Use the distributive property to expand $(3x + 2)(3x + 5)$. Show your reasoning, and write the resulting expression in standard form. Is it equivalent to $9x^2 + 21x + 10$?

2. Study the method and make sense of what was done in each step. Make a note of your thinking and be prepared to explain it.

3. Try the method to write each of these expressions in factored form.

$4x^2 + 28x + 45$ $\qquad$ $25x^2 - 35x + 6$

4. You have probably noticed that the coefficient of the squared term in all of the previous examples is a perfect square. What if that coefficient is not a perfect square?

Here is an example of an expression whose squared term has a coefficient that is not a perfect square.

$$5x^2 + 17x + 6$$
$$\frac{1}{5} \cdot 5 \cdot (5x^2 + 17x + 6)$$
$$\frac{1}{5}(25x^2 + 85x + 30)$$
$$\frac{1}{5}((5x)^2 + 17(5x) + 30)$$
$$\frac{1}{5}(N^2 + 17N + 30)$$
$$\frac{1}{5}(N + 15)(N + 2)$$
$$\frac{1}{5}(5x + 15)(5x + 2)$$
$$(x + 3)(5x + 2)$$

Use the distributive property to expand $(x + 3)(5x + 2)$. Show your reasoning and write the resulting expression in standard form. Is it equivalent to $5x^2 + 17x + 6$?

5. Study the method and make sense of what was done in each step and why. Make a note of your thinking and be prepared to explain it.

6. Try the method to write each of these expressions in factored form.

$3x^2 + 16x + 5$

$10x^2 - 41x + 4$

Lesson 10 Summary

Only some quadratic equations in the form of $ax^2 + bx + c = 0$ can be solved by rewriting the quadratic expression into factored form and using the zero product property. In some cases, finding the right factors of the quadratic expression is quite difficult.

For example, what is the factored form of $6x^2 + 11x - 35$?

We could try $(3x + \boxed{})(2x + \boxed{})$, or $(6x + \boxed{})(x + \boxed{})$, but will the second number in each factor be -5 and 7, 5 and -7, 35 and -1, or -35 and 1? And in which order?

We have to do some guessing and checking before finding the equivalent expression that would allow us to solve the equation $6x^2 + 11x - 35 = 0$.

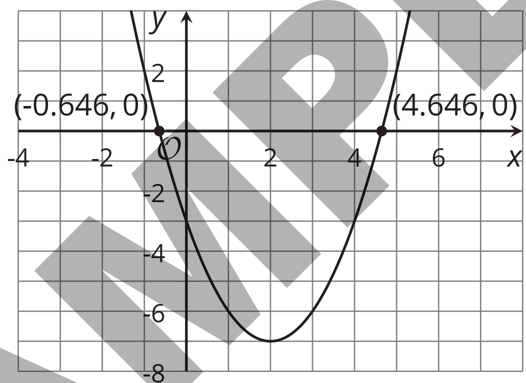
Once we find the right factors, we can proceed to solving using the zero product property, as shown here:

$$6x^2 + 11x - 35 = 0$$
$$(3x - 5)(2x + 7) = 0$$
$$3x - 5 = 0 \quad \text{or} \quad 2x + 7 = 0$$
$$x = \frac{5}{3} \quad \text{or} \quad x = -\frac{7}{2}$$

What is even trickier is that most quadratic expressions can't be written in factored form!

Let's take $x^2 - 4x - 3$ for example. Can you find two numbers that multiply to make -3 and add up to -4? Nope! At least not easy-to-find rational numbers.

We can use technology to graph the function defined by $x^2 - 4x - 3$, which reveals two x-intercepts at around $(-0.646, 0)$ and $(4.646, 0)$. These give the approximate zeros of the function, -0.646 and 4.646, so they are also approximate solutions to $x^2 - 4x - 3 = 0$.



The fact that the zeros of this function don't seem to be simple rational numbers is a clue that it may not be possible to easily rewrite the expression in factored form.

It turns out that rewriting quadratic expressions in factored form and using the zero product property is a very limited tool for solving quadratic equations.

In the next several lessons, we will learn some ways to solve quadratic equations that work for any equation.

Sec B

Lesson 10 Practice Problems

1 To write $11x^2 + 17x - 10$ in factored form, Diego first listed factor pairs of -10.

$(\_\_\_\_ + 5)(\_\_\_\_ + -2)$

$(\_\_\_\_ + 2)(\_\_\_\_ + -5)$

$(\_\_\_\_ + 10)(\_\_\_\_ + -1)$

$(\_\_\_\_ + 1)(\_\_\_\_ + -10)$

a. Use what Diego started to write $11x^2 + 17x - 10$ in factored form.

b. How do you know you've found the right pair of expressions? What did you look for when trying out different possibilities?

2 To rewrite $4x^2 - 12x - 7$ in factored form, Jada listed some factor pairs of $4x^2$:

$(2x + \_\_\_\_)(2x + \_\_\_\_)$

$(4x + \_\_\_\_)(1x + \_\_\_\_)$

Use what Jada started to rewrite $4x^2 - 12x - 7$ in factored form.

3 Rewrite each quadratic expression in factored form. Then, use the zero product property to solve the equation.

a. $7x^2 - 22x + 3 = 0$

b. $4x^2 + x - 5 = 0$

c. $9x^2 - 25 = 0$

4 Han is solving the equation $5x^2 + 13x - 6 = 0$.

Here is his work:

$$5x^2 + 13x - 6 = 0$$
$$(5x - 2)(x + 3) = 0$$
$$x = 2 \quad \text{or} \quad x = -3$$

Describe Han's mistake. Then, find the correct solutions to the equation.

5 from Unit 8, Lesson 1

A picture is 10 inches wide by 15 inches long. The total area of the picture and a frame that is x inches thick can be modeled by the function $A(x) = (2x + 10)(2x + 15)$.

a. Use function notation to write a statement for this description: the total area of the picture and a frame that is 2 inches thick is 266 square inches.

b. What is the total area if the picture has a frame that is 4 inches thick?

Sec B

6 from Unit 8, Lesson 2

To solve the equation $0 = 4x^2 - 28x + 39$, Elena uses technology to graph the function $f(x) = 4x^2 - 28x + 39$. She finds that the graph crosses the x-axis at $(1.919, 0)$ and $(5.081, 0)$.

a. What is the name for the points where the graph of a function crosses the x-axis?

b. Use a calculator to compute $f(1.919)$ and $f(5.081)$.

c. Explain why 1.919 and 5.081 are approximate solutions to the equation $0 = 4x^2 - 28x + 39$ and are not exact solutions.

7 from Unit 8, Lesson 3

Which equation shows a next step in solving $9(x - 1)^2 = 36$ that will lead to the correct solutions?

A. $9(x - 1) = 6$ or $9(x - 1) = -6$

B. $3(x - 1) = 6$

C. $(x - 1)^2 = 4$

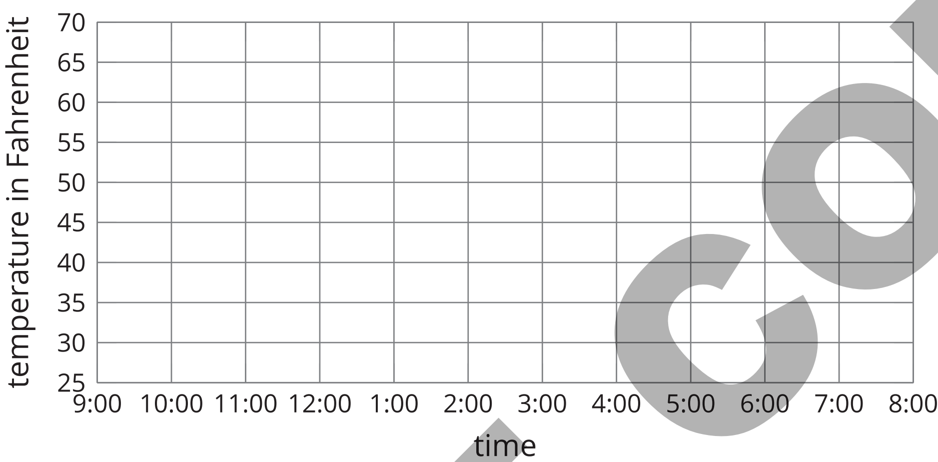
D. $(9x - 9)^2 = 36$

8 from Unit 5, Lesson 8

Here is a description of the temperature at a certain location yesterday.

"It started out cool in the morning, but then the temperature increased until noon. It stayed the same for a while, until it suddenly dropped quickly! It got colder than it was in the morning, and after that, it was cold for the rest of the day."

Sketch a graph of the temperature as a function of time.



9 from Unit 6, Lesson 12

Technology required.

The number of people, p, who watch a weekly TV show is modeled by the equation $p = 100{,}000 \cdot (1.1)^w$, where w is the number of weeks since the show first aired.

a. How many people watched the show the first time it aired? Explain how you know.

b. Use technology to graph the equation.

c. On which week does the show first get an audience of more than 500,000 people?

Addressing CA CCSSM A-REI.4b, A-SSE.2; building toward A-REI.4a, A-REI.4b; practicing MP7 and MP8

What Are Perfect Squares?

Let's see how perfect squares make some equations easier to solve.

11.1 The Thing We Are Squaring

Each of these expressions is a **perfect square**, which means that each can be written as something multiplied by itself.

Rewrite each expression as something multiplied by itself in the form $(\_\_\_\_\_\_)^2$. For example, $16x^2$ can be rewritten as $(4x)^2$.

1. 100
2. $9x^2$
3. $\frac{1}{4}x^2$
4. $x \cdot x$
5. $7x \cdot 7x$
6. $(2x - 9)(2x - 9)$

11.2 Perfect Squares in Different Forms

1. Each expression is written as the product of linear factors. Write an equivalent expression in standard form.
   a. $(3x)^2$
   b. $7x \cdot 7x$
   c. $(x + 4)(x + 4)$
   d. $(x + 1)^2$
   e. $(x - 7)^2$
   f. $(x + n)^2$
2. Why do you think the following expressions can be described as perfect squares?

$$x^2 + 6x + 9 \qquad x^2 - 16x + 64 \qquad x^2 + \frac{1}{3}x + \frac{1}{36}$$

Are you ready for more?

Expand each expression to have 3 terms, similar to standard form.

1. $(x^2 - 15)^2$
2. $(\sqrt{x} + 7)^2$
3. $(5^x + 3)^2$

11.3 Two Methods

Han and Jada solved the same equation with different methods. Here they are:

Han's method:

$$\begin{aligned}(x-6)^2 &= 25\\ (x-6)(x-6) &= 25\\ x^2 - 12x + 36 &= 25\\ x^2 - 12x + 11 &= 0\\ (x-11)(x-1) &= 0\end{aligned}$$

$$x = 11 \quad \text{or} \quad x = 1$$

Jada's method:

$$(x-6)^2 = 25$$

$$\begin{aligned}x - 6 = 5 \quad &\text{or} \quad x - 6 = -5\\ x = 11 \quad &\text{or} \quad x = 1\end{aligned}$$

Work with a partner to solve these equations. For each equation, one partner solves with Han's method, and the other partner solves with Jada's method. Make sure both partners get the same solutions to the same equation. If not, work together to find your mistakes.

$(y - 5)^2 = 49$

$(x + 4)^2 = 9$

$(z + \frac{1}{3})^2 = \frac{4}{9}$

$(v - 0.1)^2 = 0.36$

Sec C

Lesson 11 Summary

These are some examples of perfect squares:

- 49, because 49 is $7 \cdot 7$ or 7^2.
- $81a^2$, because it is equivalent to $(9a) \cdot (9a)$ or $(9a)^2$.
- $(x+5)^2$, because it is equivalent to $(x+5)(x+5)$.
- $x^2 - 12x + 36$, because it is equivalent to $(x-6)^2$ or $(x-6)(x-6)$.

A **perfect square** is an expression that is something times itself. Usually we are interested in situations in which the something is a *rational* number or an expression with rational coefficients.

When expressions that are perfect squares are written in factored form and standard form, there is a predictable pattern.

- $(x+5)(x+5)$ is equivalent to $x^2 + 10x + 25$.
- $(x-6)^2$ is equivalent to $x^2 - 12x + 36$.
- $(x-9)^2$ is equivalent to $x^2 - 18x + 81$.

In general, $(x+n)^2$ is equivalent to $x^2 + (2n)x + n^2$.

Quadratic equations that are in the form $\text{a perfect square} = \text{a perfect square}$ can be solved in a straightforward manner. Here is an example:

$$\begin{aligned} x^2 - 18x + 81 &= 25 \\ (x-9)(x-9) &= 25 \\ (x-9)^2 &= 25 \end{aligned}$$

The equation now expresses that squaring $(x-9)$ gives 25 as a result. This means $(x-9)$ must be 5 or -5.

$$\begin{aligned} x - 9 &= 5 \quad \text{or} \quad x - 9 = \text{-}5 \\ x &= 14 \quad \text{or} \quad x = 4 \end{aligned}$$

Glossary

- perfect square

Sec C

Lesson 11 Practice Problems

1 Select **all** the expressions that are perfect squares.

A. $(x + 5)(x + 5)$

B. $(-9 + c)(c - 9)$

C. $(y - 10)(10 - y)$

D. $(a + 3)(3 + a)$

E. $(2x - 1)(2x + 1)$

F. $(4 - 3x)(3 - 4x)$

G. $(a + b)(b + a)$

2 Each diagram represents the square of an expression or a perfect square.

$(n + 7)^2$

	n	7
n	n^2	$7n$
7	$7n$	7^2

$(5 - m)^2$

	5	$-m$
5	5^2	$5(-m)$
$-m$	$5(-m)$	$(-m)^2$

$(h + \frac{1}{3})^2$

	h	$\frac{1}{3}$
h		
$\frac{1}{3}$		

a. Complete the cells in the last table.

b. How are the contents of the three diagrams alike? This diagram represents $(\text{term\_1} + \text{term\_2})^2$. Describe your observations about cells 1, 2, 3, and 4.

	term_1	term_2
term_1	cell 1	cell 2
term_2	cell 3	cell 4

c. Rewrite the perfect-square expressions $(n+7)^2$, $(5-m)^2$, and $(h+\frac{1}{3})^2$ in standard form: ax^2+bx+c.

d. How are the ax^2, bx, and c of a perfect square in standard form related to the two terms in $(\text{term\_1}+\text{term\_2})^2$?

3 Solve each equation.

a. $(x-1)^2=4$

b. $(x+5)^2=81$

c. $(x-2)^2=0$

d. $(x+11)^2=121$

e. $(x-7)^2=\frac{64}{49}$

4 from Unit 8, Lesson 8

Explain or show why the product of a sum and a difference, such as $(2x+1)(2x-1)$, has no linear term when written in standard form.

5 To solve the equation $(x+3)^2 = 4$, Han first expanded the squared expression. Here is his incomplete work:

$$\begin{aligned}(x+3)^2 &= 4\\ (x+3)(x+3) &= 4\\ x^2+3x+3x+9 &= 4\\ x^2+6x+9 &= 4\end{aligned}$$

a. Complete Han's work, and solve the equation.

b. Jada saw the equation $(x+3)^2 = 4$ and thought, "There are two numbers, 2 and -2, that equal 4 when squared. This means $x+3$ is either 2 or -2. I can find the values of x from there."

Use Jada's reasoning to solve the equation.

c. Can Jada use her reasoning to solve $(x+3)(x-3) = 5$? Explain your reasoning.

6 from Unit 5, Lesson 13

A jar full of marbles is displayed. The following table shows the guesses for 10 people. The actual number of marbles in the jar is 145. Calculate the absolute guessing error for all 10 guesses using $|\text{guess} - 145|$.

guess	190	150	125	133	167	160	148	200	170	115
absolute guessing error										

Sec C

Unit 8, Lesson 12
Addressing CA CCSSM A-REI.4a, A-REI.4b, A-SSE.2; building toward A-REI.4a, A-REI.4b; practicing MP7

Completing the Square (Part 1)

A1

Let's learn a new method for solving quadratic equations.

12.1 Perfect or Imperfect?

Select **all** expressions that are perfect squares. Explain how you know.

1. $(x + 5)(5 + x)$
2. $(x + 5)(x - 5)$
3. $(x - 3)^2$
4. $x - 3^2$
5. $x^2 + 8x + 16$
6. $x^2 + 10x + 20$

Sec C

12.2 Building Perfect Squares

Complete the table so that each row has equivalent expressions that are perfect squares.

standard form	factored form
$x^2 + 6x + 9$	
$x^2 - 10x + 25$	
	$(x - 7)^2$
$x^2 - 20x +$ _______	$(x -$ _______$)^2$
$x^2 + 16x +$ _______	$(x +$ _______$)^2$
$x^2 + 7x +$ _______	$(x +$ _______$)^2$
$x^2 + bx +$ _______	$(x +$ _______$)^2$

12.3 Dipping Our Toes in Completing the Square

One technique for solving quadratic equations is called **completing the square**. Here are two examples of how Diego and Mai completed the square to solve the same equation.

Diego:

$$\begin{aligned} x^2 + 10x + 9 &= 0 \\ x^2 + 10x &= -9 \\ x^2 + 10x + 25 &= -9 + 25 \\ x^2 + 10x + 25 &= 16 \\ (x+5)^2 &= 16 \\ x + 5 = 4 \quad &\text{or} \quad x + 5 = -4 \\ x = -1 \quad &\text{or} \quad x = -9 \end{aligned}$$

Mai:

$$\begin{aligned} x^2 + 10x + 9 &= 0 \\ x^2 + 10x + 9 + 16 &= 16 \\ x^2 + 10x + 25 &= 16 \\ (x+5)^2 &= 16 \\ x + 5 = 4 \quad &\text{or} \quad x + 5 = -4 \\ x = -1 \quad &\text{or} \quad x = -9 \end{aligned}$$

Study the examples, then solve these equations by completing the square:

1. $x^2 + 6x + 8 = 0$

2. $x^2 + 12x = 13$

3. $0 = x^2 - 10x + 21$

Sec C

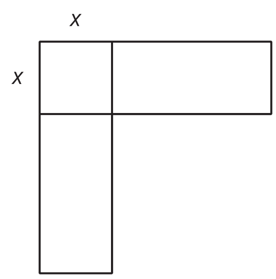
4. $x^2 - 2x + 3 = 83$

5. $x^2 + 40 = 14x$

Sec C

Are you ready for more?

Here is a diagram made of a square and two congruent rectangles. Its total area is $x^2 + 35x$ square units.



1. What is the length of the unlabeled side of each of the two rectangles?
2. If we add lines to make the figure a square, what is the area of the entire figure?
3. How is the process of finding the area of the entire figure like the process of building perfect squares for expressions like $x^2 + bx$?

Lesson 12 Summary

Turning an expression into a perfect square can be a good way to solve a quadratic equation. Suppose we wanted to solve $x^2 - 14x + 10 = \text{-}30$.

The expression on the left, $x^2 - 14x + 10$, is not a perfect square, but $x^2 - 14x + 49$ *is* a perfect square. Let's transform that side of the equation into a perfect square (while keeping the equality of the two sides).

- One helpful way to start is by first moving the constant that is not a perfect square out of the way. Let's subtract 10 from each side:

$$x^2 - 14x + 10 - 10 = \text{-}30 - 10$$
$$x^2 - 14x = \text{-}40$$

- And then add 49 to each side:

$$x^2 - 14x + 49 = \text{-}40 + 49$$
$$x^2 - 14x + 49 = 9$$

- The left side is now a perfect square because it's equivalent to $(x-7)(x-7)$ or $(x-7)^2$. Let's rewrite it:

$$(x-7)^2 = 9$$

- If a number squared is 9, the number has to be 3 or -3. Solve to finish up:

$$x - 7 = 3 \quad \text{or} \quad x - 7 = \text{-}3$$
$$x = 10 \quad \text{or} \quad x = 4$$

This method of solving quadratic equations is called **completing the square**. In general, perfect squares in standard form look like $x^2 + bx + \left(\frac{b}{2}\right)^2$, so to complete the square, take half of the coefficient of the linear term and square it.

In the example, half of -14 is -7, and $(\text{-}7)^2$ is 49. We wanted to make the left side $x^2 - 14x + 49$. To keep the equation true and maintain equality of the two sides of the equation, we added 49 to each side.

Glossary

- completing the square

Lesson 12 Practice Problems

1 Add the number that would make the expression a perfect square. Next, write an equivalent expression in factored form.

a. $x^2 - 6x$

b. $x^2 + 2x$

c. $x^2 + 14x$

d. $x^2 - 4x$

e. $x^2 + 24x$

2 Mai is solving the equation $x^2 + 12x = 13$. She writes:

$$x^2 + 12x = 13$$
$$(x+6)^2 = 49$$
$$x = 1 \text{ or } x = -13$$

Jada looks at Mai's work and is confused. She doesn't see how Mai got her answer.

Complete Mai's missing steps to help Jada see how Mai solved the equation.

3 Match each equation to an equivalent equation with a perfect square on one side.

A. $x^2 + 8x = 2$

B. $x^2 + 10x = -13$

C. $x^2 - 14x = 5$

D. $x^2 + 2x = 0$

E. $x^2 + 4x - 5 = 0$

F. $x^2 - 20x = -9$

1. $(x-7)^2 = 54$
2. $(x+5)^2 = 12$
3. $(x-10)^2 = 91$
4. $(x+4)^2 = 18$
5. $(x+1)^2 = 1$
6. $(x+2)^2 = 9$

4 Solve each equation by completing the square.

$x^2 - 6x + 5 = 12$

$x^2 - 2x = 8$

$11 = x^2 + 4x - 1$

$x^2 - 18x + 60 = -21$

Sec C

5 from Unit 8, Lesson 8

Rewrite each expression in standard form.

a. $(x + 3)(x - 3)$

b. $(7 + x)(x - 7)$

c. $(2x - 5)(2x + 5)$

d. $(x + \frac{1}{8})(x - \frac{1}{8})$

6 from Unit 8, Lesson 8

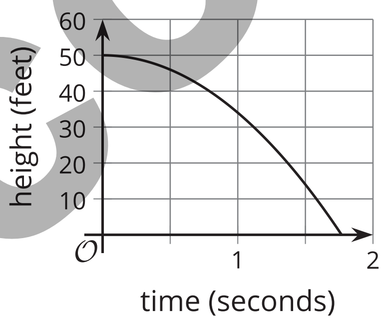
To find the product $203 \cdot 197$ without a calculator, Priya wrote $(200+3)(200-3)$. Very quickly, and without writing anything else, she arrived at 39,991. Explain how writing the two factors as a sum and a difference may have helped Priya.

7 from Unit 7, Lesson 5

A basketball is dropped from the roof of a building, and its height, in feet, is modeled by the function h.

Here is a graph representing h.

Select **all** the true statements about this situation.



A. When $t = 0$, the height is 0 feet.

B. The basketball falls at a constant speed.

C. The expression that defines h is linear.

D. The expression that defines h is quadratic.

E. When $t = 0$, the ball is about 50 feet above the ground.

F. The basketball lands on the ground about 1.75 seconds after it is dropped.

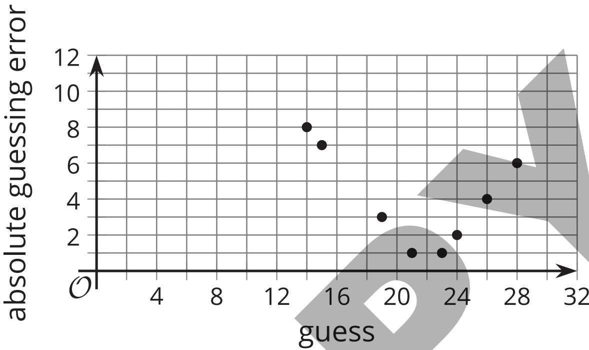
Sec C

8 from Unit 5, Lesson 13

A group of students are guessing the number of paper clips in a small box.

The guesses and the guessing errors are plotted on a coordinate plane.

What is the actual number of paper clips in the box?



Sec C

Unit 8, Lesson 13
Addressing CA CCSSM A-REI.1, A-REI.4b; practicing MP3, MP6, MP7

Completing the Square (Part 2)

A1

Let's solve some harder quadratic equations.

13.1 Math Talk: Equations with Fractions

Solve each equation mentally.

- $x + x = \frac{1}{4}$
- $(\frac{3}{2})^2 = x$
- $\frac{3}{5} + x = \frac{9}{5}$
- $\frac{1}{12} + x = \frac{1}{4}$

Sec C

13.2 Spot Those Errors!

Here are four equations, followed by worked solutions of the equations. Each solution has at least one error.

- Solve one or more of these equations by completing the square.
- Then, look at the worked solution of the same equation as the one you solved. Find and describe the error or errors in the worked solution.

1. $x^2 + 14x = -24$

2. $x^2 - 10x + 16 = 0$

3. $x^2 + 2.4x = -0.8$

4. $x^2 - \frac{6}{5}x + \frac{1}{5} = 0$

Worked solutions (with errors):

1.

$$\begin{aligned} x^2 + 14x &= -24 \\ x^2 + 14x + 28 &= 4 \\ (x+7)^2 &= 4 \end{aligned}$$

$$x + 7 = 2 \quad \text{or} \quad x + 7 = -2$$
$$x = -5 \quad \text{or} \quad x = -9$$

2.

$$\begin{aligned} x^2 - 10x + 16 &= 0 \\ x^2 - 10x + 25 &= 9 \\ (x-5)^2 &= 9 \end{aligned}$$

$$x - 5 = 9 \quad \text{or} \quad x - 5 = -9$$
$$x = 14 \quad \text{or} \quad x = -4$$

3.

$$\begin{aligned} x^2 + 2.4x &= -0.8 \\ x^2 + 2.4x + 1.44 &= 0.64 \\ (x+1.2)^2 &= 0.64 \\ x + 1.2 &= 0.8 \\ x &= -0.4 \end{aligned}$$

4.

$$\begin{aligned} x^2 - \frac{6}{5}x + \frac{1}{5} &= 0 \\ x^2 - \frac{6}{5}x + \frac{9}{25} &= \frac{9}{25} \\ \left(x - \frac{3}{5}\right)^2 &= \frac{9}{25} \end{aligned}$$

$$x - \frac{3}{5} = \frac{3}{5} \quad \text{or} \quad x - \frac{3}{5} = -\frac{3}{5}$$
$$x = \frac{6}{5} \quad \text{or} \quad x = 0$$

Sec C

13.3 Solving Some More Quadratic Equations

Solve these equations by completing the square.

1. $(x-3)(x+1)=5$

Sec C

2. $x^2+\frac{1}{2}x=\frac{3}{16}$

3. $x^2+3x+\frac{8}{4}=0$

4. $(7 - x)(3 - x) + 3 = 0$

5. $x^2 + 1.6x + 0.63 = 0$

Sec C

Are you ready for more?

1. Show that the equation $x^2 + 10x + 9 = 0$ is equivalent to $(x + 3)^2 + 4x = 0$.

2. Write an equation that is equivalent to $x^2 + 9x + 16 = 0$ and that includes $(x + 4)^2$.

3. Does this method help you find solutions to the equations? Explain your reasoning.

Lesson 13 Summary

Completing the square can be a useful method for solving quadratic equations in cases in which it is not easy to rewrite an expression in factored form. For example, let's solve this equation:

$$x^2 + 5x - \frac{75}{4} = 0$$

First, we'll add $\frac{75}{4}$ to each side to make things easier on ourselves.

$$\begin{aligned} x^2 + 5x - \frac{75}{4} + \frac{75}{4} &= 0 + \frac{75}{4} \\ x^2 + 5x &= \frac{75}{4} \end{aligned}$$

To complete the square, take $\frac{1}{2}$ of the coefficient of the linear term, 5, which is $\frac{5}{2}$, and square it, which is $\frac{25}{4}$. Add this to each side:

$$\begin{aligned} x^2 + 5x + \frac{25}{4} &= \frac{75}{4} + \frac{25}{4} \\ x^2 + 5x + \frac{25}{4} &= \frac{100}{4} \end{aligned}$$

Notice that $\frac{100}{4}$ is equal to 25, and rewrite it:

$$x^2 + 5x + \frac{25}{4} = 25$$

Since the left side is now a perfect square, let's rewrite it:

$$\left(x + \frac{5}{2}\right)^2 = 25$$

For this equation to be true, one of these equations must true:

$$x + \frac{5}{2} = 5 \quad \text{or} \quad x + \frac{5}{2} = -5$$

To finish up, we can subtract $\frac{5}{2}$ from each side of the equal sign in each equation.

$$\begin{aligned} x = 5 - \frac{5}{2} \quad &\text{or} \quad x = -5 - \frac{5}{2} \\ x = \frac{5}{2} \quad &\text{or} \quad x = -\frac{15}{2} \\ x = 2\frac{1}{2} \quad &\text{or} \quad x = -7\frac{1}{2} \end{aligned}$$

It takes some practice to become proficient at completing the square, but it makes it possible to solve many more equations than we could by methods we learned previously.

Lesson 13 Practice Problems

1 Add the number that would make the expression a perfect square. Next, write an equivalent expression in factored form.

a. $x^2 + 3x$

b. $x^2 + 0.6x$

c. $x^2 - 11x$

d. $x^2 - \frac{5}{2}x$

e. $x^2 + x$

Sec C

2 Noah is solving the equation $x^2 + 8x + 15 = 3$. He begins by rewriting the expression on the left in factored form and writes $(x + 3)(x + 5) = 3$. He does not know what to do next.

Noah knows that the solutions are $x = -2$ and $x = -6$, but is not sure how to get to these values from his equation.

Solve the original equation by completing the square.

3 An equation and its solutions are given. Explain or show how to solve the equation by completing the square.

a. $x^2 + 20x + 50 = 14$. The solutions are $x = -18$ and $x = -2$.

b. $x^2 + 1.6x = 0.36$. The solutions are $x = -1.8$ and $x = 0.2$.

c. $x^2 - 5x = \frac{11}{4}$. The solutions are $x = \frac{11}{2}$ and $x = \frac{-1}{2}$.

Sec C

4 Solve each equation.

a. $x^2 - 0.5x = 0.5$

b. $x^2 + 0.8x = 0.09$

c. $x^2 + \frac{13}{3}x = \frac{56}{36}$

Sec C

5 from Unit 8, Lesson 8

Match each quadratic expression given in factored form with an equivalent expression in standard form. One expression in standard form has no match.

A. $(2+x)(2-x)$

B. $(x+9)(x-9)$

C. $(2+x)(x-2)$

D. $(x+y)(x-y)$

1. $x^2 - 4$
2. $81 - x^2$
3. $x^2 - y^2$
4. $4 - x^2$
5. $x^2 - 81$

6 from Unit 8, Lesson 9

Four students solved the equation $x^2 + 225 = 0$. Their work is shown here. Only one student solved it correctly.

Student A:

$$x^2 + 225 = 0$$
$$x^2 = \text{-}225$$
$$x = 15 \quad \text{or} \quad x = \text{-}15$$

Student B:

$$x^2 + 225 = 0$$
$$x^2 = \text{-}225$$
$$\text{No solutions}$$

Student C:

$$x^2 + 225 = 0$$
$$(x - 15)(x + 15) = 0$$
$$x = 15 \quad \text{or} \quad x = \text{-}15$$

Student D:

$$x^2 + 225 = 0$$
$$x^2 = 225$$
$$x = 15 \quad \text{or} \quad x = \text{-}15$$

Determine which student solved the equation correctly. For each of the incorrect solutions, explain the mistake.

Sec C

Addressing CA CCSSM A-REI.4a, A-REI.4b, A-SSE.2; building toward A-REI.4a, A-REI.4b, A-SSE.2; practicing MP7 and MP8

Completing the Square (Part 3)

Let's complete the square for some more complicated expressions.

14.1 Perfect Squares in Two Forms

Previously, we saw that $(x+3)^2$ can be expanded to standard form as $x^2 + 2 \cdot 3x + 3^2$.

1. Expand $(5x+3)^2$ into standard form.
2. Be prepared to share a conjecture about the relationship between the coefficients 5 and 3 in the factored form and the values in standard form.

14.2 Perfect in a Different Way

1. Write each expression in standard form:
   a. $(4x+1)^2$

   b. $(5x-2)^2$

   c. $(\frac{1}{2}x+7)^2$

   d. $(3x+n)^2$

   e. $(kx+m)^2$

2. Decide if each expression is a perfect square. If so, write an equivalent expression of the form $(kx+m)^2$. If not, suggest one change to turn it into a perfect square.
   a. $4x^2+12x+9$
   b. $4x^2+8x+25$

14.3 When All the Stars Align

1. Find the value of c to make each expression in the left column a perfect square in standard form. Then, write an equivalent expression in the form of squared factor. In the last row, write your own pair of equivalent expressions.

standard form $(ax^2 + bx + c)$	squared factor $(kx + m)^2$
$100x^2 + 80x + c$	
$36x^2 - 60x + c$	
$25x^2 + 40x + c$	
$0.25x^2 - 14x + c$	

2. Solve each equation by completing the square:

$25x^2 + 40x = -12$

$36x^2 - 60x + 10 = -6$

Sec C

14.4 Putting Stars into Alignment

Here are three methods for solving $3x^2 + 8x + 5 = 0$.

Try to make sense of each method.

Method 1:

$$\begin{aligned} 3x^2 + 8x + 5 &= 0 \\ (3x+5)(x+1) &= 0 \end{aligned}$$

$$x = -\frac{5}{3} \quad \text{or} \quad x = -1$$

Method 2:

$$\begin{aligned} 3x^2 + 8x + 5 &= 0 \\ 9x^2 + 24x + 15 &= 0 \\ (3x)^2 + 8(3x) + 15 &= 0 \\ U^2 + 8U + 15 &= 0 \\ (U+5)(U+3) &= 0 \end{aligned}$$

$$\begin{aligned} U = -5 \quad &\text{or} \quad U = -3 \\ 3x = -5 \quad &\text{or} \quad 3x = -3 \\ x = -\frac{5}{3} \quad &\text{or} \quad x = -1 \end{aligned}$$

Method 3:

$$\begin{aligned} 3x^2 + 8x + 5 &= 0 \\ 9x^2 + 24x + 15 &= 0 \\ 9x^2 + 24x + 16 &= 1 \\ (3x+4)^2 &= 1 \end{aligned}$$

$$\begin{aligned} 3x + 4 = 1 \quad &\text{or} \quad 3x + 4 = -1 \\ x = -1 \quad &\text{or} \quad x = -\frac{5}{3} \end{aligned}$$

Once you understand the methods, use each method at least one time to solve these equations.

1. $5x^2 + 17x + 6 = 0$

2. $6x^2 + 19x = -10$

Sec C

3. $8x^2 - 33x + 4 = 0$

4. $8x^2 - 26x = -21$

5. $10x^2 + 37x = 36$

6. $12x^2 + 20x - 77 = 0$

Sec C

Are you ready for more?

Find the solutions to $3x^2 - 6x + \frac{9}{4} = 0$. Explain your reasoning.

Sec C

Lesson 14 Summary

In earlier lessons, we worked with perfect squares such as $(x+1)^2$ and $(x-5)(x-5)$. We learned that their equivalent expressions in standard form follow a predictable pattern:

- In general, $(x+m)^2$ can be written as $x^2+2mx+m^2$.
- If a quadratic expression of the form ax^2+bx+c is a perfect square, and the value of a is 1, then the value of b is $2m$, and the value of c is m^2 for some value of m.

In this lesson, the variables in the factors being squared had coefficients other than 1, for example $(3x+1)^2$ and $(2x-5)(2x-5)$. Their equivalent expressions in standard form also followed the same pattern we saw earlier.

squared factor	standard form
$(3x+1)^2$	$(3x)^2+2(3x)(1)+1^2$ or $9x^2+6x+1$
$(2x-5)^2$	$(2x)^2+2(2x)(\text{-}5)+(\text{-}5)^2$ or $4x^2-20x+25$

In general, $(kx+m)^2$ can be written as:

$(kx)^2+2(kx)(m)+m^2$ or $k^2x^2+2kmx+m^2$

If a quadratic expression is of the form ax^2+bx+c, then:

- The value of a is k^2.
- The value of b is $2km$.
- The value of c is m^2.

We can use this pattern to help us complete the square and solve equations when the squared term x^2 has a coefficient other than 1—for example, $16x^2+40x=11$.

What constant term c can we add to make the expression on the left of the equal sign a perfect square? And how do we write this expression as a squared factor?

- $a=16$, which is 4^2, so $k=4$, and the squared factor could be $(4x+m)^2$.
- 40 is equal to $2(4m)$, so $2(4m)=40$, or $8m=40$. This means that $m=5$.
- If c is m^2, then $c=5^2$, or $c=25$.
- So the expression $16x^2+40x+25$ is a perfect square and is equivalent to $(4x+5)^2$.

Sec C

Let's solve the equation $16x^2 + 40x = 11$ by completing the square!

$$\begin{gathered}16x^2 + 40x = 11\\16x^2 + 40x + 25 = 11 + 25\\(4x+5)^2 = 36\end{gathered}$$

$$\begin{array}{rcl}4x + 5 = 6 & \text{or} & 4x + 5 = -6\\4x = 1 & \text{or} & 4x = -11\\x = \frac{1}{4} & \text{or} & x = -\frac{11}{4}\end{array}$$

.

Sec C

Lesson 14 Practice Problems

1 Select **all** expressions that are perfect squares.

A. $9x^2 + 24x + 16$

B. $2x^2 + 20x + 100$

C. $(7 - 3x)^2$

D. $(5x + 4)(5x - 4)$

E. $(1 - 2x)(-2x + 1)$

F. $4x^2 + 6x + \frac{9}{4}$

Sec C

2 Find the missing number that makes the expression a perfect square. Next, write the expression in factored form.

a. $49x^2 -$ _______ $x + 16$

b. $36x^2 +$ _______ $x + 4$

c. $4x^2 -$ _______ $x + 25$

d. $9x^2 +$ _______ $x + 9$

e. $121x^2 +$ _______ $x + 9$

3 Find the missing number that makes the expression a perfect square. Next, write the expression in factored form.

a. $9x^2 + 42x +$ _______

b. $49x^2 - 28x +$ _______

c. $25x^2 + 110x +$ _______

d. $64x^2 - 144x +$ _______

e. $4x^2 + 24x +$ _______

4 a. Find the value of c to make the expression a perfect square. Then, write an equivalent expression in factored form.

standard form $ax^2 + bx + c$	factored form $(kx + m)^2$
$4x^2 + 4x +$ _______	
$25x^2 - 30x +$ _______	

b. Solve each equation by completing the square.

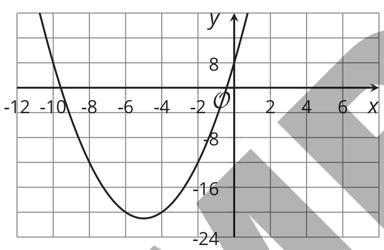
$4x^2 + 4x = 3$

$25x^2 - 30x + 8 = 0$

5 from Unit 8, Lesson 5

For each function f, decide if the equation $f(x) = 0$ has 0, 1, or 2 solutions. Explain how you know.

A



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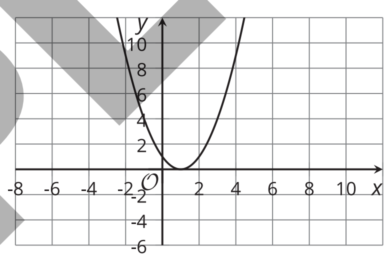

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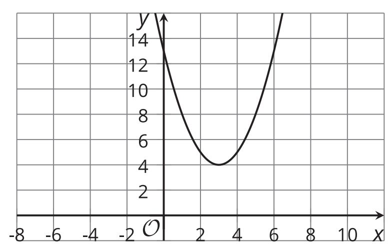

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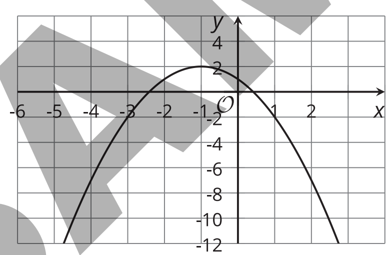

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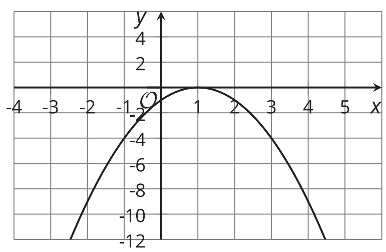
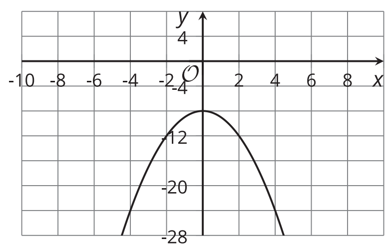


F



6 from Unit 8, Lesson 9

Solve each equation.

$p^2 + 10 = 7p$

$x^2 + 11x + 27 = 3$

$(y + 2)(y + 6) = -3$

7 from Unit 7, Lesson 6

Which function could represent the height in meters of an object thrown upwards from a height of 25 meters above the ground t seconds after being launched?

A. $f(t) = -5t^2$

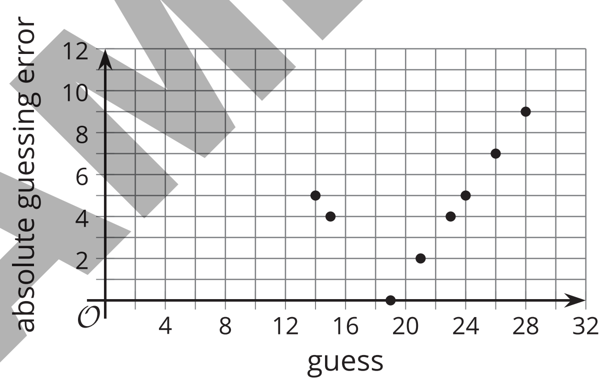
B. $f(t) = -5t^2 + 25$

C. $f(t) = -5t^2 + 25t + 50$

D. $f(t) = -5t^2 + 50t + 25$

8 from Unit 5, Lesson 13

A group of children are guessing the number of pebbles in a glass jar. The guesses and the guessing errors are plotted on a coordinate plane.



a. Which guess is furthest away from the actual number?

b. How far is the furthest guess away from the actual number?

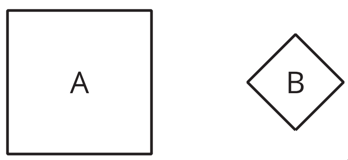
Addressing CA CCSSM A-REI.4a, A-REI.4b, A-REI.11; building on 8.EE.2; building toward N-RN.3; practicing MP6

Quadratic Equations with Irrational Solutions

Let's find exact solutions to quadratic equations, even if the solutions are irrational.

15.1 Roots of Squares

Here are two squares. Square A has an area of 9 square units. Square B has an area of 2 square units.



1. What is the side length of Square A?
2. How does that side length compare to the solutions to the equation $s^2 = 9$?
3. What is the side length of Square B?
4. How does that side length compare to the solutions to the equation $x^2 = 2$?

15.2 Solutions Written as Square Roots

Solve each equation. Use the $\pm$ notation when appropriate.

1. $x^2 - 13 = -12$

2. $(x - 6)^2 = 0$

3. $x^2 + 9 = 0$

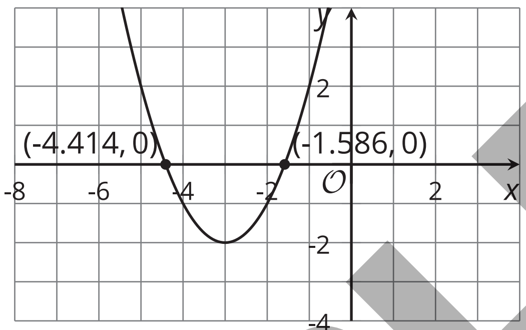
4. $x^2 = 18$

5. $x^2 + 1 = 18$

6. $(x + 1)^2 = 18$

15.3 Finding Irrational Solutions by Completing the Square

Here is an example of an equation being solved by graphing and by completing the square.



$$\begin{aligned} x^2 + 6x + 7 &= 0 \\ x^2 + 6x + 9 &= 2 \\ (x + 3)^2 &= 2 \\ x + 3 &= \pm\sqrt{2} \\ x &= -3 \pm \sqrt{2} \end{aligned}$$

Verify: $\sqrt{2}$ is approximately 1.414.
So $-3 + \sqrt{2} \approx -1.586$ and $-3 - \sqrt{2} \approx -4.414$.

For each equation, find the exact solutions by completing the square and the approximate solutions by graphing. Then, verify that the solutions found using the two methods are close. If you get stuck, study the example.

1. $x^2 + 4x + 1 = 0$

2. $x^2 - 10x + 18 = 0$

Sec C

3. $x^2 + 5x + \frac{1}{4} = 0$

4. $x^2 + \frac{8}{3}x + \frac{14}{9} = 0$

Are you ready for more?

Sec C

Write a quadratic equation of the form $ax^2 + bx + c = 0$ whose solutions are $x = 5 - \sqrt{2}$ and $x = 5 + \sqrt{2}$.

Lesson 15 Summary

When solving quadratic equations, it is important to remember that:

- Any positive number has two square roots, one positive and one negative, because there are two numbers that can be squared to make that number. (For example, 6^2 and $(-6)^2$ both equal 36, so 6 and -6 are both square roots of 36.)
- The square root symbol ($\sqrt{\ }$) can be used to express the positive square root of a number. For example, the square root of 36 is 6, but it can also be written as $\sqrt{36}$ because $\sqrt{36} \cdot \sqrt{36} = 36$.
- To express the negative square root of a number, say 36, we can write -6 or $-\sqrt{36}$.
- When a number is not a perfect square—for example, 40—we can express its square roots by writing $\sqrt{40}$ and $-\sqrt{40}$.

How could we write the solutions to an equation like $(x + 4)^2 = 11$? This equation is saying, "something squared is 11." To make the equation true, that something must be $\sqrt{11}$ or $-\sqrt{11}$. We can write:

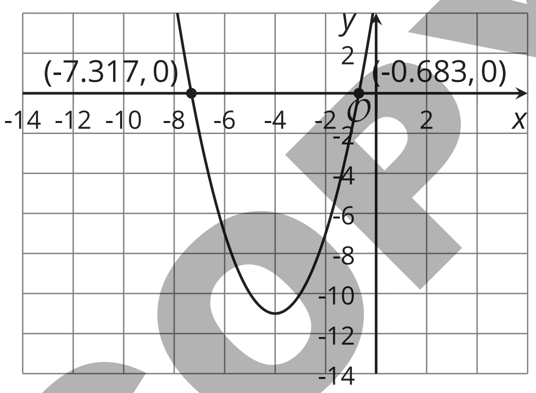
$$\begin{aligned} x + 4 &= \sqrt{11} \quad \text{or} \quad x + 4 = -\sqrt{11} \\ x &= -4 + \sqrt{11} \quad \text{or} \quad x = -4 - \sqrt{11} \end{aligned}$$

A more compact way to write the two solutions to the equation is $x = \text{-}4 \pm \sqrt{11}$.

About how large or small are those numbers? Are they positive or negative? We can use a calculator to compute the approximate values of both expressions:

$$\text{-}4 + \sqrt{11} \approx \text{-}0.683 \quad \text{or} \quad \text{-}4 - \sqrt{11} \approx \text{-}7.317$$

We can also approximate the solutions by graphing. The equation $(x+4)^2 = 11$ is equivalent to $(x+4)^2 - 11 = 0$, so we can graph the function $y = (x+4)^2 - 11$ and find its zeros by locating the x-intercepts of the graph.



Sec C

Lesson 15 Practice Problems

1 Solve each equation, and write the solutions using $\pm$ notation.

a. $x^2 = 144$

b. $x^2 = 5$

c. $4x^2 = 28$

d. $x^2 = \frac{25}{4}$

e. $2x^2 = 22$

f. $7x^2 = 16$

2 Match each expression to an equivalent expression.

A. 4 ± 1

B. $10 \pm \sqrt{4}$

C. -6 ± 11

D. $4 \pm \sqrt{10}$

E. $\sqrt{16} \pm \sqrt{2}$

1. -17 and 5
2. $4 + \sqrt{2}$ and $4 - \sqrt{2}$
3. 8 and 12
4. 3 and 5
5. $4 + \sqrt{10}$ and $4 - \sqrt{10}$

3

a. Is $\sqrt{4}$ a positive or negative number? Explain your reasoning.

b. Is $\sqrt{5}$ a positive or negative number? Explain your reasoning.

c. Explain the difference between $\sqrt{9}$ and the solutions to $x^2 = 9$.

4 *Technology required.* For each equation, find the exact solutions by completing the square and the approximate solutions by graphing. Then, verify that the solutions found using the two methods are close.

$x^2 + 10x + 8 = 0$

$x^2 - 4x - 11 = 0$

Sec C

5 from Unit 8, Lesson 10

Which expression in factored form is equivalent to $30x^2 + 31x + 5$?

A. $(6x + 5)(5x + 1)$

B. $(5x + 5)(6x + 1)$

C. $(10x + 5)(3x + 1)$

D. $(30x + 5)(x + 1)$

6 from Unit 7, Lesson 6

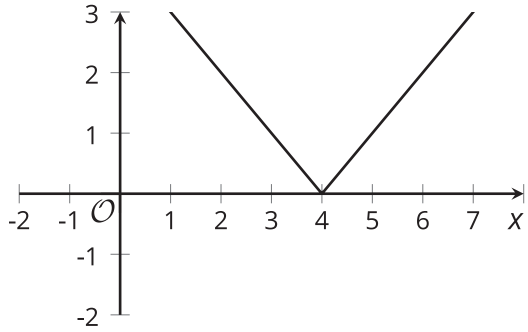
Two rocks are launched straight up in the air. The height of Rock A is given by the function f, where $f(t) = 4 + 30t - 16t^2$. The height of Rock B is given by g, where $g(t) = 5 + 20t - 16t^2$. In both functions, t is time measured in seconds after the rocks are launched, and height is measured in feet above the ground.

a. Which rock is launched from a higher point?

b. Which rock is launched with a greater velocity?

7 from Unit 5, Lesson 14

a. Describe how the graph of $f(x) = |x|$ has to be shifted to match the given graph.



b. Find an equation for the function represented by the graph.

Sec C

Addressing CA CCSSM A-REI.4b, A-SSE.2; building on 8.EE.2; building toward A-REI.4b; practicing MP7

The Quadratic Formula

Let's learn a formula for finding solutions to quadratic equations.

16.1 Evaluate It

Each expression represents two numbers. Evaluate the expressions and find the two numbers.

1. $1 \pm \sqrt{49}$
2. $\dfrac{8 \pm 2}{5}$
3. $\pm\sqrt{(-5)^2 - 4 \cdot 4 \cdot 1}$
4. $\dfrac{-18 \pm \sqrt{36}}{2 \cdot 3}$

16.2 Pesky Equations

Choose one equation to solve, either by rewriting it in factored form or by completing the square. Be prepared to explain your choice of method.

1. $x^2 - 2x - 1.25 = 0$
2. $5x^2 + 9x - 44 = 0$
3. $x^2 + 1.25x = 0.375$
4. $4x^2 - 28x + 29 = 0$

16.3 Meet the Quadratic Formula

Here is a formula called the **quadratic formula**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The formula can be used to find the solutions to any quadratic equation in the form of $ax^2 + bx + c = 0$, where a, b, and c are numbers and a is not 0.

This example shows how it is used to solve $x^2 - 8x + 15 = 0$, in which $a = 1$, $b = -8$, and $c = 15$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{original equation}$$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(15)}}{2(1)} \qquad \text{Substitute the values of } a, b, \text{ and } c.$$

$$x = \frac{8 \pm \sqrt{64 - 60}}{2} \qquad \text{Evaluate each part of the expression.}$$

$$x = \frac{8 \pm \sqrt{4}}{2}$$

$$x = \frac{8 \pm 2}{2}$$

$$x = \frac{10}{2} \quad \text{or} \quad x = \frac{6}{2}$$

$$x = 5 \quad \text{or} \quad x = 3$$

Here are some quadratic equations and their solutions. Use the quadratic formula to show that the solutions are correct.

1. $x^2 + 4x - 5 = 0$. The solutions are $x = -5$ and $x = 1$.

2. $x^2 + 7x + 12 = 0$. The solutions are $x = -3$ and $x = -4$.

3. $x^2 + 10x + 18 = 0$. The solutions are $x = -5 \pm \frac{\sqrt{28}}{2}$.

4. $x^2 - 8x + 11 = 0$. The solutions are $x = 4 \pm \frac{\sqrt{20}}{2}$.

5. $9x^2 - 6x + 1 = 0$. The solution is $x = \frac{1}{3}$.

6. $6x^2 + 9x - 15 = 0$. The solutions are $x = -\frac{5}{2}$ and $x = 1$.

Sec D

Are you ready for more?

1. Use the quadratic formula to solve $ax^2 + c = 0$. Let's call the resulting equation P.

2. Solve the equation $3x^2 - 27 = 0$ in two ways, showing your reasoning for each:

   - Without using any formulas
   - Using equation P

3. Check that you got the same solutions using each method.

4. Use the quadratic formula to solve $ax^2 + bx = 0$. Let's call the resulting equation Q.

5. Solve the equation $2x^2 + 5x = 0$ in two ways, showing your reasoning for each:

   - Without using any formulas
   - Using equation Q

6. Check that you got the same solutions using each method.

Lesson 16 Summary

We have learned a couple of methods for solving quadratic equations algebraically:

- By rewriting the equation as so that one side is 0 and the other side is in factored form, then using the zero product property
- By completing the square

Some equations can be solved quickly with one of these methods, but many cannot. Here is an example: $5x^2 - 3x - 1 = 0$. The expression on the left cannot be rewritten in factored form with rational coefficients. Because the coefficient of the squared term is not a perfect square, and the coefficient of the linear term is an odd number, completing the square would be inconvenient and would result in a perfect square with fractions.

The **quadratic formula** can be used to find the solutions to any quadratic equation, including those that are tricky to solve with other methods.

For an equation of the form $ax^2 + bx + c = 0$, where a, b, and c are numbers and $a \neq 0$, the solutions are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For the equation $5x^2 - 3x - 1 = 0$, we see that $a = 5$, $b = -3$, and $c = -1$. Let's solve it!

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{the quadratic formula}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(5)(-1)}}{2(5)} \quad \text{Substitute the values of } a, b, \text{ and } c.$$

$$x = \frac{3 \pm \sqrt{9 + 20}}{10} \quad \text{Evaluate each part of the expression.}$$

$$x = \frac{3 \pm \sqrt{29}}{10}$$

A calculator gives approximate solutions of 0.84 and -0.24 for $\frac{3+\sqrt{29}}{10}$ and $\frac{3-\sqrt{29}}{10}$.

We can also use the formula for simpler equations like $x^2 - 9x + 8 = 0$, but it may not be the most efficient way. If the quadratic expression can be easily rewritten in factored form or made into a perfect square, those methods may be preferable. For example, rewriting $x^2 - 9x + 8 = 0$ as $(x-1)(x-8) = 0$ immediately tells us that the solutions are 1 and 8.

Glossary

- quadratic formula

Lesson 16 Practice Problems

1 For each equation, identify the values of a, b, and c that you would substitute into the quadratic formula to solve the equation.

a. $3x^2 + 8x + 4 = 0$

b. $2x^2 - 5x + 2 = 0$

c. $-9x^2 + 13x - 1 = 0$

d. $x^2 + x - 11 = 0$

e. $-x^2 + 16x + 64 = 0$

2 Use the quadratic formula to show that the given solutions are correct.

a. $x^2 + 9x + 20 = 0$. The solutions are $x = -4$ and $x = -5$.

b. $x^2 - 10x + 21 = 0$. The solutions are $x = 3$ and $x = 7$.

c. $3x^2 - 5x + 1 = 0$. The solutions are $x = \frac{5}{6} \pm \frac{\sqrt{13}}{6}$.

3 from Unit 8, Lesson 14

Select **all** the equations that are equivalent to $81x^2 + 180x - 200 = 100$.

A. $81x^2 + 180x - 100 = 0$

B. $81x^2 + 180x + 100 = 200$

C. $81x^2 + 180x + 100 = 400$

D. $(9x + 10)^2 = 400$

E. $(9x + 10)^2 = 0$

F. $(9x - 10)^2 = 10$

G. $(9x - 10)^2 = 20$

Sec D

4 from Unit 7, Lesson 6

Technology required.

Two objects are launched upward. Each function gives the distance from the ground, in meters, as a function of time, t, in seconds.

Object A: $f(t) = 25 + 20t - 5t^2$ Object B: $g(t) = 30 + 10t - 5t^2$

Use graphing technology to graph each function.

a. Which object reaches the ground first? Explain how you know.

b. What is the maximum height of each object?

5 Identify the values of a, b, and c that you would substitute into the quadratic formula to solve the equation.

a. $x^2 + 9x + 18 = 0$

b. $4x^2 - 3x + 11 = 0$

c. $81 - x + 5x^2 = 0$

d. $\frac{4}{5}x^2 + 3x = \frac{1}{3}$

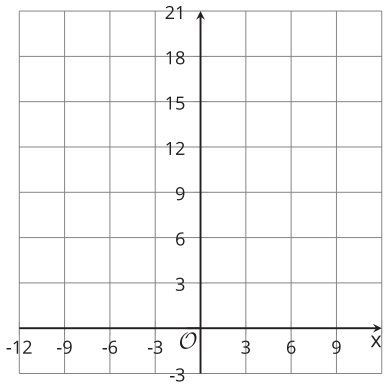
e. $121 = x^2$

f. $7x + 14x^2 = 42$

6 from Unit 5, Lesson 14

On the same coordinate plane, sketch a graph of each function.

- Function v, defined by $v(x) = |x + 6|$
- Function z, defined by $z(x) = |x| + 9$



Unit 8, Lesson 17

Addressing CA CCSSM A-CED.1, A-REI.1, A-REI.4b, F-IF.5, F-LE.6; building toward A-REI.4; practicing MP2

Applying the Quadratic Formula (Part 1)

Let's use the quadratic formula to solve some problems.

17.1 No Solutions for You!

Here is an example of someone solving a quadratic equation that has no solutions:

$$(x+3)^2 + 9 = 0$$
$$(x+3)^2 = -9$$
$$x + 3 = \pm\sqrt{-9}$$

1. Study the example. At what point did you realize the equation had no solutions?

2. Explain how you know the equation $49 + x^2 = 0$ has no solutions.

17.2 The Potato and the Pumpkin

Answer each question without graphing. Explain or show your reasoning.

1. The equation $h(t) = -16t^2 + 80t + 64$ represents the height, in feet, of a potato t seconds after it has been launched.

   a. Write an equation that can be solved to find when the potato hits the ground. Then solve the equation.

   b. Write an equation that can be solved to find when the potato is 40 feet off the ground. Then solve the equation.

2. The equation $g(t) = 2 + 23.7t - 4.9t^2$ models the height, in meters, of a pumpkin t seconds after it has been launched from a catapult.

   a. Is the pumpkin still in the air 8 seconds later? Explain or show how you know.

   b. At what value of t does the pumpkin hit the ground? Show your reasoning.

17.3 Back to the Framer

1. Solve this equation without graphing. $(7 + 2x)(4 + 2x) = 38$.

   Pause for a discussion about the equation.

2. Suppose you have another picture that is 10 inches by 5 inches, and are now using a fancy paper that is 8.5 inches by 4 inches to frame the picture. Again, the frame is to be uniform in thickness all the way around. No fancy framing paper is to be wasted!

   Find out how thick the frame should be.

Are you ready for more?

Suppose that your border paper is 6 inches by 8 inches. You want to use all the paper to make a half-inch border around a rectangular picture.

1. Find two possible pairs of length and width of a rectangular picture that could be framed with a half-inch border and no leftover materials.
2. What must be true about the length and width of any rectangular picture that can be framed this way? Explain how you know.

Lesson 17 Summary

Quadratic equations that represent situations cannot always be neatly put into factored form or easily solved by finding square roots. Completing the square is a workable strategy, but for some equations, it may involve many cumbersome steps. Graphing is also a handy way to solve the equations, but it doesn't always give us precise solutions.

With the quadratic formula, we can solve these equations more readily and precisely.

Here's an example: Function h models the height of an object, in meters, t seconds after it is launched into the air. It is is defined by $h(t) = -5t^2 + 25t$.

To know how much time it would take the object to reach 15 meters, we could solve the equation $15 = -5t^2 + 25t$. How should we do it?

- Rewriting it in standard form gives $-5t^2 + 25t - 15 = 0$. The expression on the left side of the equation cannot be written in factored form, however.
- Completing the square isn't convenient because the coefficient of the squared term is not a perfect square and the coefficient the linear term is an odd number.
- Let's use the quadratic formula, using $a = -5$, $b = 25$, and $c = -15$!

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-25 \pm \sqrt{25^2 - 4(-5)(-15)}}{2(-5)}$$

$$t = \frac{-25 \pm \sqrt{325}}{-10}$$

The expression $\frac{-25 \pm \sqrt{325}}{-10}$ represents the two exact solutions of the equation.

We can also get approximate solutions by using a calculator, or by reasoning that $\sqrt{325} \approx 18$.

The solutions tell us that there are two times after the launch when the object is at a height of 15 meters: at about 0.7 second (as the object is going up) and 4.3 seconds (as it comes back down).

Lesson 17 Practice Problems

1 Select **all** the equations that have 2 solutions.

A. $(x+3)^2 = 9$

B. $(x-5)^2 = -5$

C. $(x+2)^2 - 6 = 0$

D. $(x-9)^2 + 25 = 0$

E. $(x+10)^2 = 1$

F. $(x-8)^2 = 0$

G. $5 = (x+1)(x+1)$

2 A frog jumps in the air. The height, in inches, of the frog is modeled by the function $h(t) = 60t - 75t^2$, where t is the time after it jumped, measured in seconds.

Solve $60t - 75t^2 = 0$. What do the solutions tell us about the jumping frog?

3 A tennis ball is hit straight up in the air, and its height, in feet above the ground, is modeled by the equation $f(t) = 4 + 12t - 16t^2$, where t is measured in seconds since the ball was thrown.

a. Find the solutions to the equation $0 = 4 + 12t - 16t^2$.

b. What do the solutions tell us about the tennis ball?

4 from Unit 8, Lesson 10

Rewrite each quadratic expression in standard form.

a. $(x + 1)(7x + 2)$

b. $(8x + 1)(x - 5)$

c. $(2x + 1)(2x - 1)$

d. $(4 + x)(3x - 2)$

5 from Unit 8, Lesson 10

Find the missing expression in parentheses so that each pair of quadratic expressions is equivalent. Show that your expression meets this requirement.

a. $(4x - 1)($______________$)$ and $16x^2 - 8x + 1$

b. $(9x + 2)($______________$)$ and $9x^2 - 16x - 4$

c. $($______________$)(\text{-}x + 5)$ and $\text{-}7x^2 + 36x - 5$

Sec D

6 from Unit 6, Lesson 12

The number of downloads of a song during a week is a function, f, of the number of weeks, w, since the song was released. The equation $f(w) = 100{,}000 \cdot \left(\frac{9}{10}\right)^w$ defines this function.

a. What does the number 100,000 tell you about the downloads? What about the $\frac{9}{10}$?

b. Is $f(\text{-}1)$ meaningful in this situation? Explain your reasoning.

7 from Unit 8, Lesson 16

Consider the equation $4x^2 - 4x - 15 = 0$.

a. Identify the values of a, b, and c that you would substitute into the quadratic formula to solve the equation.

b. Evaluate each expression using the values of a, b, and c.

$-b$ $\quad$ b^2 $\quad$ $4ac$ $\quad$ $b^2 - 4ac$

$\sqrt{b^2 - 4ac}$ $\quad$ $-b \pm \sqrt{b^2 - 4ac}$ $\quad$ $2a$ $\quad$ $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

c. The solutions to the equation are $x = -\frac{3}{2}$ and $x = \frac{5}{2}$. Do these match the values of the last expression you evaluated in the previous question?

8 from Unit 7, Lesson 13

a. Describe the graph of $y = -x^2$. (Does it open upward or downward? Where is its y-intercept? What about its x-intercepts?)

b. Without graphing, describe how adding $16x$ to $-x^2$ would change each feature of the graph of $y = -x^2$. (If you get stuck, consider writing the expression in factored form.)

i. the x-intercepts
ii. the vertex
iii. the y-intercept
iv. the direction of opening of the *U*-shape graph

Sec D

Addressing CA CCSSM A-CED.1, A-REI.4b, F-IF.2, F-LE.6; building on 6.EE.2c, 8.EE.2; building toward A-REI.4b; practicing MP3 and MP5

Applying the Quadratic Formula (Part 2)

Let's use the quadratic formula and solve quadratic equations with care.

18.1 Bits and Pieces

Evaluate each expression for $a = 9$, $b = -5$, and $c = -2$

1. $-b$
2. b^2
3. $b^2 - 4ac$
4. $-b \pm \sqrt{a}$

18.2 Using the Formula with Care

Here are four equations, followed by attempts to solve them using the quadratic formula. Each attempt contains at least one error.

- Solve 1–2 equations by using the quadratic formula.
- Then, find and describe the error(s) in the worked solutions of the same equations as the ones you solved.

Equation 1: $2x^2 + 3 = 8x$

Equation 2: $x^2 + 3x = 10$

Equation 3: $9x^2 - 2x - 1 = 0$

Equation 4: $x^2 - 10x + 23 = 0$

Here are the worked solutions with errors:

Equation 1: $2x^2 + 3 = 8x$

$a = 2,\ b = -8,\ c = 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(2)(3)}}{2(2)}$$
$$x = \frac{8 \pm \sqrt{64 - 24}}{4}$$
$$x = \frac{8 \pm \sqrt{40}}{4}$$
$$x = 2 \pm \sqrt{10}$$

Equation 2: $x^2 + 3x = 10$

$a = 1,\ b = 3,\ c = 10$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(10)}}{2(1)}$$
$$x = \frac{-3 \pm \sqrt{9 - 40}}{2}$$
$$x = \frac{-3 \pm \sqrt{-31}}{2}$$

No solutions

Equation 3: $9x^2 - 2x - 1 = 0$

$a = 9,\ b = -2,\ c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(9)(-1)}}{2}$$
$$x = \frac{2 \pm \sqrt{4 + 36}}{2}$$
$$x = \frac{2 \pm \sqrt{40}}{2}$$

Equation 4: $x^2 - 10x + 23 = 0$

$a = 1,\ b = -10,\ c = 23$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-10 \pm \sqrt{(-10)^2 - 4(1)(23)}}{2}$$
$$x = \frac{-10 \pm \sqrt{-100 - 92}}{2}$$
$$x = \frac{-10 \pm \sqrt{-192}}{2}$$

No solutions

18.3 Sure about That?

1. The equation $h(t) = 2 + 30t - 5t^2$ represents the height, as a function of time, of a pumpkin that was catapulted up in the air. Height is measured in meters, and time is measured in seconds.

   a. The pumpkin reached a maximum height of 47 meters. How many seconds after launch did that happen? Show your reasoning.

   b. Suppose someone was unconvinced by your solution. Find another way (besides the steps you already took) to show your solution is correct.

2. The equation $r(p) = 80p - p^2$ models the revenue a band expects to collect as a function of the price of one concert ticket. Ticket prices and revenues are in dollars.

   A band member says that a ticket price of either \$15.50 or \$74.50 would generate approximately \$1,000 in revenue. Do you agree? Show your reasoning.

Are you ready for more?

Function g is defined by the equation $g(t) = 2 + 30t - 5t^2 - 47$. Its graph opens downward.

1. Find the zeros of function g without graphing. Show your reasoning.

2. Explain or show how the zeros you found can tell us the vertex of the graph of g.

3. Study the expressions that define functions g and h (which defined the height of the pumpkin). Explain how the maximum of function h, once we know it, can tell us the maximum of g.

Lesson 18 Summary

The quadratic formula has many parts in it. A small error in any one part can lead to incorrect solutions.

Suppose we are solving $2x^2 - 6 = 11x$. To use the formula, let's rewrite it in the form of $ax^2 + bx + c = 0$, which gives $2x^2 - 11x - 6 = 0$.

Here are some things to keep in mind:

- Use the correct values for a, b, and c in the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-11 \pm \sqrt{(-11)^2 - 4(2)(-6)}}{2(2)}$$

Nope! b is -11, so -b is -(-11), which is 11, not -11.

$$x = \frac{11 \pm \sqrt{(-11)^2 - 4(2)(-6)}}{2(2)}$$

That's better!

- Multiply 2 by a for the denominator in the formula.

$$x = \frac{11 \pm \sqrt{(-11)^2 - 4(2)(-6)}}{2}$$

Nope! The denominator is $2a$, which is $2(2)$, or 4.

$$x = \frac{11 \pm \sqrt{(-11)^2 - 4(2)(-6)}}{2(2)}$$

That's better!

- Remember that squaring a negative number produces a positive number.

$$x = \frac{11 \pm \sqrt{-121 - 4(2)(-6)}}{4}$$

Nope! $(-11)^2$ is 121, not -121.

$$x = \frac{11 \pm \sqrt{121 - 4(2)(-6)}}{4}$$

That's better!

- Remember that a negative number times a positive number is a negative number.

$$x = \frac{11 \pm \sqrt{121 - 48}}{4}$$

Nope! $4(2)(-6) = -48$, and $121 - (-48)$ is $121 + 48$.

$$x = \frac{11 \pm \sqrt{121 + 48}}{4}$$

That's better!

- Follow the properties of algebra.

$$x = \frac{11 \pm \sqrt{169}}{4}$$

$$x = 11 \pm \sqrt{42.25}$$

$$x = \frac{11 \pm 13}{4}$$

That's better!

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Nope! Both parts of the numerator, 11 and $\sqrt{169}$, get divided by 4. Also, $\frac{\sqrt{169}}{4}$ is not $\sqrt{42.25}$.

Let's finish by evaluating $\frac{11 \pm 13}{4}$ correctly:

$$x = \frac{11+13}{4} \qquad \text{or} \qquad x = \frac{11-13}{4}$$
$$x = \frac{24}{4} \qquad \text{or} \qquad x = -\frac{2}{4}$$
$$x = 6 \qquad \text{or} \qquad x = -\frac{1}{2}$$

To make sure our solutions are correct, we can substitute each solution back into the original equation and see whether it results in a true equation.

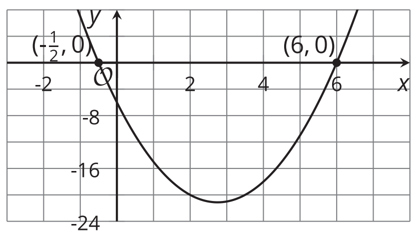
Checking 6 as a solution:

$$\begin{aligned} 2x^2 - 6 &= 11x \\ 2(6)^2 - 6 &= 11(6) \\ 2(36) - 6 &= 66 \\ 72 - 6 &= 66 \\ 66 &= 66 \end{aligned}$$

Checking $-\frac{1}{2}$ as a solution:

$$\begin{aligned} 2x^2 - 6 &= 11x \\ 2\left(-\frac{1}{2}\right)^2 - 6 &= 11\left(-\frac{1}{2}\right) \\ 2\left(\frac{1}{4}\right) - 6 &= -\frac{11}{2} \\ \frac{1}{2} - 6 &= -5\frac{1}{2} \\ -5\frac{1}{2} &= -5\frac{1}{2} \end{aligned}$$

We can also graph the equation $y = 2x^2 - 11x - 6$ and find its x-intercepts to see whether our solutions to $2x^2 - 11x - 6 = 0$ are accurate (or close to accurate).



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Lesson 18 Practice Problems

1 Mai and Jada both solve the equation $2x^2 - 7x = 15$ using the quadratic formula but find different solutions.

Mai writes:

$$x = \frac{-7 \pm \sqrt{7^2 - 4(2)(-15)}}{2(2)}$$
$$x = \frac{-7 \pm \sqrt{49 - (-120)}}{4}$$
$$x = \frac{-7 \pm \sqrt{169}}{4}$$
$$x = \frac{-7 \pm 13}{4}$$
$$x = -5 \quad \text{or} \quad x = \frac{3}{2}$$

Jada writes:

$$x = \frac{-(-7) \pm \sqrt{-7^2 - 4(2)(-15)}}{2(2)}$$
$$x = \frac{7 \pm \sqrt{-49 - (-120)}}{4}$$
$$x = \frac{7 \pm \sqrt{71}}{4}$$

a. If this equation is written in standard form, $ax^2 + bx + c = 0$, what are the values of a, b, and c?

b. Do you agree with either of them? Explain your reasoning.

2 The equation $h(t) = -16t^2 + 80t + 64$ represents the height, in feet, of a potato t seconds after it was launched from a mechanical device.

a. Write an equation that would allow us to find the time the potato hits the ground.

b. Solve the equation without graphing. Show your reasoning.

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3 Priya found $x = 3$ and $x = -1$ as solutions to $3x^2 - 6x - 9 = 0$. Is she correct? Show how you know.

4 from Unit 8, Lesson 11

Lin says she can tell that $25x^2 + 40x + 16$ and $49x^2 - 112x + 64$ are perfect squares because each expression has the following characteristics, which she saw in other perfect squares in standard form:

- The first term is a perfect square. The last term is also a perfect square.
- If we multiply a square root of the first term and a square root of the last term and then double the product, the result is the middle term.

a. Show that each expression has the characteristics Lin described.

b. Write each expression in factored form.

5 from Unit 8, Lesson 16

What are the solutions to the equation $2x^2 - 5x - 1 = 0$?

A. $x = \frac{-5 \pm \sqrt{17}}{4}$

B. $x = \frac{5 \pm \sqrt{17}}{4}$

C. $x = \frac{-5 \pm \sqrt{33}}{4}$

D. $x = \frac{5 \pm \sqrt{33}}{4}$

6 from Unit 8, Lesson 16

Solve each equation by rewriting the quadratic expression in factored form and using the zero product property, or by completing the square. Then, check if your solutions are correct by using the quadratic formula.

a. $x^2 + 11x + 24 = 0$

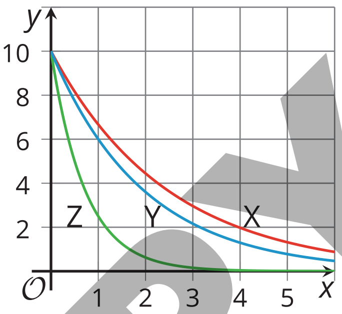
b. $4x^2 + 20x + 25 = 0$

c. $x^2 + 8x = 5$

7 from Unit 6, Lesson 15

Here are the graphs of three equations.

Match each graph with the appropriate equation.



A. $y = 10\left(\frac{2}{3}\right)^x$

B. $y = 10\left(\frac{1}{4}\right)^x$

C. $y = 10\left(\frac{3}{5}\right)^x$

1. X
2. Y
3. Z

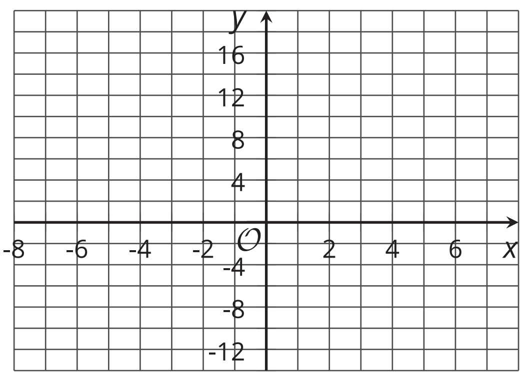
8 from Unit 7, Lesson 11

The function f is defined by $f(x) = (x + 1)(x + 6)$.

a. What are the x-intercepts of the graph of f?

b. Find the coordinates of the vertex of the graph of f. Show your reasoning.

c. Sketch a graph of f.



Sec D

Addressing CA CCSSM A-REI.4a, A-SSE.2; building toward A-REI.4a; practicing MP3 and MP7

Deriving the Quadratic Formula

Let's find out where the quadratic formula comes from.

19.1 Studying Structure

Here are some perfect squares in factored and standard forms, and expressions showing how the two forms are related.

1. Complete the table.

factored form		standard form
$(3x-4)^2$	$(3x)^2+2(\_\_x)(\_\_)+(\_\_)^2$	$9x^2-24x+16$
$(5x+\_\_\_\_)^2$	$(\_\_x)^2+2(\_\_x)(\_\_)+(\_\_)^2$	$25x^2+30x+\_\_\_\_$
$(kx+m)^2$	$(\_\_x)^2+2(\_\_x)(\_\_)+(\_\_)^2$	$\_\_x^2+\_\_x+\_\_$

2. Look at the expression in the last row of the table. If ax^2+bx+c is equivalent to $(kx+m)^2$, how are a, b, and c related to k and m?

19.2 Complete the Square Using a Placeholder

1. One way to solve the quadratic equation $x^2 + 5x + 3 = 0$ is by completing the square. A partially solved equation is shown here. Study the steps.

   Then, knowing that P is a placeholder for $2x$, continue to solve for x without evaluating any part of the expression. Be prepared to explain each step.

$x^2 + 5x + 3 = 0$	original equation
$4x^2 + 20x + 12 = 0$	Multiply each side by 4.
$4x^2 + 20x = -12$	Subtract 12 from each side.
$(2x)^2 + 10(2x) = -12$	Rewrite $4x^2$ as $(2x)^2$ and $20x$ as $10(2x)$.
$P^2 + 10P = -12$	Use P as a placeholder for $2x$.

$$P^2 + 10P + \underline{\hspace{2em}}^2 = -12 + \underline{\hspace{2em}}^2$$

$$(P + \underline{\hspace{2em}})^2 = -12 + \underline{\hspace{2em}}^2$$

$$P + \underline{\hspace{2em}} = \pm\sqrt{-12 + \underline{\hspace{2em}}^2}$$

$$P = \underline{\hspace{2em}} \pm \sqrt{-12 + \underline{\hspace{2em}}^2}$$

$$P = \underline{\hspace{2em}} \pm \sqrt{\underline{\hspace{2em}}^2 - 12}$$

$$2x = \underline{\hspace{2em}} \pm \sqrt{\underline{\hspace{2em}}^2 - 12}$$

$$x =$$

2. Explain how the solution is related to the quadratic formula.

19.3 Decoding the Quadratic Formula

Here is one way to make sense of how the quadratic formula came about. Study the derivation until you can explain what happens in each step. Record your explanation next to each step.

$$ax^2 + bx + c = 0$$

$$4a^2x^2 + 4abx + 4ac = 0$$

$$4a^2x^2 + 4abx = -4ac$$

$$(2ax)^2 + 2b(2ax) = -4ac$$

$$M^2 + 2bM = -4ac$$

$$M^2 + 2bM + b^2 = -4ac + b^2$$

$$(M + b)^2 = b^2 - 4ac$$

$$M + b = \pm\sqrt{b^2 - 4ac}$$

$$M = -b \pm \sqrt{b^2 - 4ac}$$

$$2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Are you ready for more?

Here is another way to derive the quadratic formula by completing the square.

- First, divide each side of the equation $ax^2 + bx + c = 0$ by a to get $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.
- Then, complete the square for $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

1. The beginning steps of this approach are shown here. Briefly explain what happens in each step.

$$
\begin{array}{rcll}
x^2 + \frac{b}{a}x + \frac{c}{a} &=& 0 & \text{original equation} \\
x^2 + \frac{b}{a}x &=& -\frac{c}{a} & [1] \\
x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 &=& -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 & [2] \\
\left(x + \frac{b}{2a}\right)^2 &=& -\frac{c}{a} + \frac{b^2}{4a^2} & [3] \\
\left(x + \frac{b}{2a}\right)^2 &=& -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} & [4] \\
\left(x + \frac{b}{2a}\right)^2 &=& \frac{b^2 - 4ac}{4a^2} & [5] \\
x + \frac{b}{2a} &=& \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} & [6] \\
x + \frac{b}{2a} &=& \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} & [7]
\end{array}
$$

2. Continue the solving process until you have the equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

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Lesson 19 Summary

Recall that any quadratic equation can be solved by completing the square. The quadratic formula is essentially what we get when we put all the steps taken to complete the square for $ax^2 + bx + c = 0$ into a single expression.

When we expand a squared factor like $(3x + 5)^2$, the result is $(3x)^2 + 2(5)(3x) + 25$. Notice how $(3x)$ appears in two places. If we replace $(3x)$ with another letter, like P, we have $P^2 + 10P + 25$, which is a recognizable perfect square.

Likewise, if we expand $(kx + m)^2$, we have $(kx)^2 + 2m(kx) + m^2$. Replacing kx with P gives $P^2 + 2mP + m^2$, also a recognizable perfect square.

To complete the square is essentially to make one side of the equation have the same structure as $(kx)^2 + 2m(kx) + m^2$. Substituting a letter for (kx) makes it easier to see what is needed to complete the square. Let's complete the square for $ax^2 + bx + c = 0$!

- Start by subtracting c from each side.

$$ax^2 + bx = -c$$

- Next, let's multiply both sides by $4a$, which is allowed because $a \neq 0$. On the left, this gives $4a^2$, a perfect square for the coefficient of x^2.

$$4a^2x^2 + 4abx = -4ac$$

- $4a^2x^2$ can be written $(2ax)^2$, and $4abx$ can be written $2b(2ax)$.

$$(2ax)^2 + 2b(2ax) = -4ac$$

- Let's replace $(2ax)$ with the letter P.

$$P^2 + 2bP = -4ac$$

- b^2 is the constant term that completes the square, so let's add b^2 to each side.

$$P^2 + 2bP + b^2 = -4ac + b^2$$

- The left side is now a perfect square and can be written as a squared factor.

$$(P + b)^2 = b^2 - 4ac$$

- The square roots of the expression on the right are the values of $P + b$.

$$P + b = \pm\sqrt{b^2 - 4ac}$$

  Once P is isolated, we can write $2ax$ in its place and solve for x.

$$P = -b \pm \sqrt{b^2 - 4ac}$$

$$2ax = -b \pm \sqrt{b^2 - 4ac}$$

- The solution is the quadratic formula!

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Lesson 19 Practice Problems

1

a. The quadratic equation $x^2 + 7x + 10 = 0$ is in the form of $ax^2 + bx + c = 0$. What are the values of a, b, and c?

b. Some steps for solving the equation by completing the square have been started here. In the third line, what might be a good reason for multiplying each side of the equation by 4?

$x^2 + 7x + 10 = 0$	original equation
$x^2 + 7x = -10$	Subtract 10 from each side.
$4x^2 + 4(7x) = 4(-10)$	Multiply each side by 4.
$(2x)^2 + 2(7)2x + \_\_\_\_^2 = \_\_\_\_^2 - 4(10)$	Rewrite $4x^2$ as $(2x)^2$ and $4(7x)$ as $2(7)2x$.
$(2x + \_\_\_\_)^2 = \_\_\_\_^2 - 4(10)$	
$2x + \_\_\_\_ = \pm\sqrt{\_\_\_\_^2 - 4(10)}$	
$2x = \_\_\_\_ \pm \sqrt{\_\_\_\_^2 - 4(10)}$	
$x =$	

c. Complete the unfinished steps, and explain what happens in these steps.

d. Substitute the values of a, b, and c into the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, but do not evaluate any of the expressions. Explain how the expression on the right side of the equal sign is related to solving $x^2 + 7x + 10 = 0$ by completing the square.

2 Consider the equation $x^2 - 39 = 0$.

a. Does the quadratic formula work to solve this equation? Explain or show how you know.

b. Can you solve this equation using square roots? Explain or show how you know.

3 Clare is deriving the quadratic formula by solving $ax^2 + bx + c = 0$ by completing the square.

She arrived at this equation. $(2ax + b)^2 = b^2 - 4ac$

Briefly describe what she needs to do to finish solving for x, and then show the steps.

4 from Unit 8, Lesson 12

Tyler is solving the quadratic equation $x^2 + 8x + 11 = 4$.

Study his work and explain the mistake he made. Then, solve the equation correctly.

$$\begin{aligned} x^2 + 8x + 11 &= 4 \\ x^2 + 8x + 16 &= 4 \\ (x+4)^2 &= 4 \\ x = \text{-}8 \quad \text{or} \quad x &= 0 \end{aligned}$$

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5 from Unit 8, Lesson 16

Solve the equation by using the quadratic formula. Then, check if your solutions are correct by rewriting the quadratic expression in factored form and using the zero product property.

a. $2x^2 - 3x - 5 = 0$

b. $x^2 - 4x = 21$

c. $3 - x - 4x^2 = 0$

6 from Unit 8, Lesson 17

A tennis ball is hit straight up in the air, and its height, in feet above the ground, is modeled by the equation $f(t) = 4 + 12t - 16t^2$, where t is measured in seconds since the ball was thrown.

a. Find the solutions to $6 = 4 + 12t - 16t^2$ without graphing. Show your reasoning.

b. What do the solutions say about the tennis ball?

7 from Unit 7, Lesson 11

Sec D

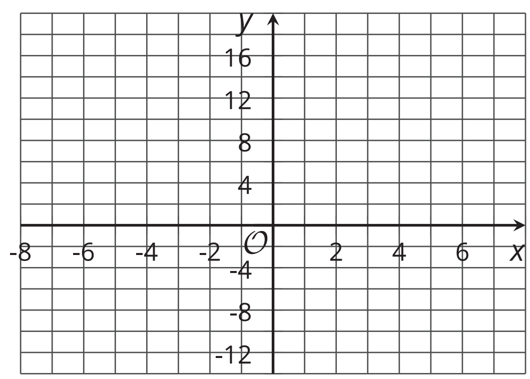
Consider the equation $y = 2x(6 - x)$.

a. What are the x-intercepts of the graph of this equation? Explain how you know.

b. What is the x-coordinate of the vertex of the graph of this equation? Explain how you know.

c. What is the y-coordinate of the vertex? Show your reasoning.

d. Sketch the graph of this equation.



Addressing CA CCSSM A-REI.4b, F-IF.7a, N-RN.3; building on 8.NS.1; building toward N-RN.3; practicing MP3, MP6, MP8

Rational and Irrational Solutions

Let's consider the kinds of numbers we get when solving quadratic equations.

20.1 Rational or Irrational?

Numbers like -1.7, $\sqrt{16}$, and $\frac{5}{3}$ are known as *rational numbers.*

Numbers like $\sqrt{12}$ and $\sqrt{\frac{5}{9}}$ are known as *irrational numbers.*

Here is a list of numbers. Sort them into rational and irrational.

$97 \quad -8.2 \quad \sqrt{5} \quad -\frac{3}{7} \quad \sqrt{100} \quad \sqrt{\frac{9}{4}} \quad -\sqrt{18}$

20.2 Suspected Irrational Solutions

1. Graph each quadratic equation using graphing technology. Identify the zeros of the function that the graph represents, and say whether you think they might be rational or irrational. Be prepared to explain your reasoning.

equation	zeros	rational or irrational?
$y = x^2 - 8$		
$y = (x-5)^2 - 1$		
$y = (x-7)^2 - 2$		
$y = \left(\frac{x}{4}\right)^2 - 5$		

2. Find exact solutions (not approximate solutions) to each equation and show your reasoning. Then, say whether you think each solution is rational or irrational. Be prepared to explain your reasoning.

   a. $x^2 - 8 = 0$

   b. $(x-5)^2 = 1$

   c. $(x-7)^2 = 2$

   d. $\left(\frac{x}{4}\right)^2 - 5 = 0$

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20.3 Experimenting with Rational and Irrational Numbers

Here is a list of numbers:

$$2 \quad 3 \quad \frac{1}{3} \quad 0 \quad \sqrt{2} \quad \sqrt{3} \quad -\sqrt{3} \quad \frac{1}{\sqrt{3}}$$

Here are some statements about the sums and products of numbers. For each statement, decide whether it is *always* true, true for *some* numbers but not others, or *never* true.

1. Sums:
   a. The sum of two rational numbers is rational.
   b. The sum of a rational number and an irrational number is irrational.
   c. The sum of two irrational numbers is irrational.
2. Products:
   a. The product of two rational numbers is rational.
   b. The product of a rational number and an irrational number is irrational.
   c. The product of two irrational numbers is irrational.

Experiment with sums and products of two numbers in the given list to help you decide.

Are you ready for more?

It can be quite difficult to show that a number is irrational. To do so, we have to explain why the number is impossible to write as a ratio of two integers. It took mathematicians thousands of years before they were finally able to show that π is irrational, and they still don't know whether or not π^π is irrational.

Here is a way we could show that $\sqrt{2}$ can't be rational, and is therefore irrational.

- Let's assume that $\sqrt{2}$ is rational and can be written as a fraction $\frac{a}{b}$, where a and b are nonzero integers.
- Let's also assume that a and b are integers that no longer have any common factors. For example, to express 0.4 as $\frac{a}{b}$, we write $\frac{2}{5}$ instead of $\frac{4}{10}$ or $\frac{200}{500}$. That is, we assume that a and b are 2 and 5, rather than 4 and 10, or 200 and 500.

1. If $\sqrt{2} = \frac{a}{b}$, then $2 = \frac{\square}{\square}$.

2. Explain why a^2 must be an even number.

3. Explain why if a^2 is an even number, then a itself is also an even number. (If you get stuck, consider squaring a few different integers.)

4. Because a is an even number, then a is 2 times another integer, say, k. We can write $a = 2k$. Substitute $2k$ for a in the equation you wrote in the first question. Then, solve for b^2.

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5. Explain why the resulting equation shows that b^2, and therefore b, are also even numbers.

6. We just arrived at the conclusion that a and b are even numbers, but given our assumption about a and b, it is impossible for this to be true. Explain why this is.

If a and b cannot both be even, $\sqrt{2}$ must be equal to some number other than $\frac{a}{b}$.

Because our original assumption that we could write $\sqrt{2}$ as a fraction $\frac{a}{b}$ led to a false conclusion, that assumption must be wrong. In other words, we must not be able to write $\sqrt{2}$ as a fraction. This means $\sqrt{2}$ is irrational!

Lesson 20 Summary

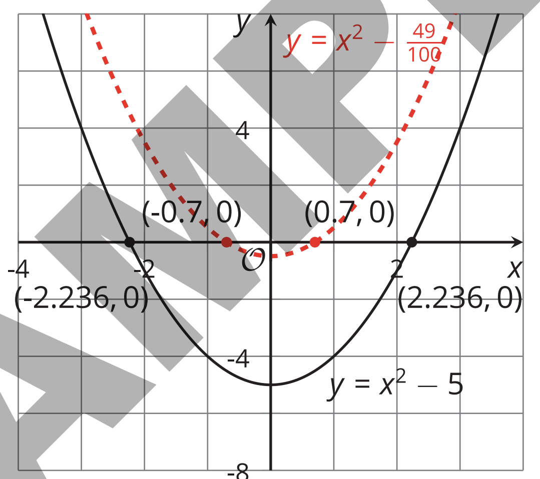
The solutions to quadratic equations can be rational or irrational. Recall that:

- *Rational numbers* can be written as positive or negative fractions. Numbers like 12, -3, $\frac{5}{3}$, $\sqrt{25}$, -4.79, and $\sqrt{\frac{9}{16}}$ are rational. ($\sqrt{25}$ can be written as a fraction, because it's equal to $\frac{5}{1}$. The number -4.79 is the opposite of 4.79, which is $\frac{479}{100}$.)
- Any number that is not rational is *irrational*. Some examples are $\sqrt{2}, \pi, -\sqrt{5}$, and $\sqrt{\frac{7}{2}}$. When an irrational number is written as a decimal, its digits go on forever without eventually making a repeating pattern, so a decimal can only approximate the value of the number.

How do we know if the solutions to a quadratic equation are rational or irrational?

If we solve a quadratic equation $ax^2 + bx + c = 0$ by graphing a corresponding function ($y = ax^2 + bx + c$), sometimes we can tell from the x-coordinates of the x-intercepts. Other times, we can't be sure.

Let's solve $x^2 - \frac{49}{100} = 0$ and $x^2 - 5 = 0$ by graphing $y = x^2 - \frac{49}{100}$ and $y = x^2 - 5$.



The graph of $y = x^2 - \frac{49}{100}$ crosses the x-axis at -0.7 and 0.7. There are no digits after 7, suggesting that the x-values are exactly $-\frac{7}{10}$ and $\frac{7}{10}$, which are rational.

To verify that these numbers are exact solutions to the equation, we can see if they make the original equation true.

$(0.7)^2 - \frac{49}{100} = 0$ and $(-0.7)^2 - \frac{49}{100} = 0$, so ± 0.7 are exact solutions.

The graph of $y = x^2 - 5$, created using graphing technology, is shown to cross the x-axis at -2.236 and 2.236. It is unclear if the x-coordinates stop at three decimal places or if they continue. If they stop or eventually make a repeating pattern, the solutions are rational. If they never stop or make a repeating pattern, the solutions are irrational.

We can tell, though, that 2.236 is not an exact solution to the equation. Substituting 2.236 for x in the original equation gives $2.236^2 - 5$, which we can tell is close to 0 but is not exactly 0. This means ± 2.236 are not exact solutions, and the solutions may be irrational.

To be certain whether the solutions are rational or irrational, we can solve the equations.

- The solutions to $x^2 - \frac{49}{100} = 0$ are ± 0.7, which are rational.
- The solutions to $x^2 - 5 = 0$ are $\pm\sqrt{5}$, which are irrational. (2.236 is an approximation of $\sqrt{5}$, not equal to $\sqrt{5}$.)

Lesson 20 Practice Problems

1 Decide whether each number is rational or irrational.

10 $\frac{4}{5}$ $\sqrt{4}$ $\sqrt{10}$ -3 $\sqrt{\frac{25}{4}}$ $\sqrt{0.6}$

2 Here are the solutions to some quadratic equations. Select **all** solutions that are rational.

A. 5 ± 2

B. $\sqrt{4} \pm 1$

C. $\frac{1}{2} \pm 3$

D. $10 \pm \sqrt{3}$

E. $\pm\sqrt{25}$

F. $1 \pm \sqrt{2}$

3 Solve each equation. Then, determine if the solutions are rational or irrational.

a. $(x + 1)^2 = 4$

b. $(x - 5)^2 = 36$

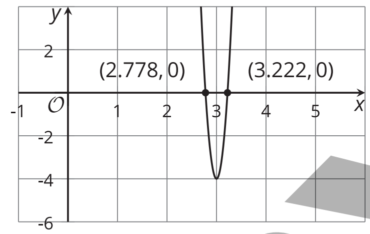
c. $(x + 3)^2 = 11$

d. $(x - 4)^2 = 6$

Sec D

4 Here is a graph of the equation $y = 81(x - 3)^2 - 4$.

a. According to the graph, what are the solutions to the equation $81(x - 3)^2 = 4$?



b. Can you tell whether they are rational or irrational? Explain how you know.

c. Solve the equation using a different method and say whether the solutions are rational or irrational. Explain or show your reasoning.

5 from Unit 8, Lesson 13

Match each equation to an equivalent equation with a perfect square on one side.

A. $x^2 - 9x = \frac{1}{2}$

B. $x^2 + 6.4x - 8.9 = 0$

C. $x^2 - 5x = 11$

D. $x^2 + 0.1x + 0.0005 = 0$

E. $x^2 - \frac{6}{7}x = \frac{1}{49}$

F. $x^2 + 1.21x = 6.28$

1. $(x - 2.5)^2 = 17.25$
2. $(x - \frac{9}{2})^2 = \frac{83}{4}$
3. $(x - \frac{3}{7})^2 = \frac{10}{49}$
4. $(x + 0.05)^2 = 0.002$
5. $(x + 3.2)^2 = 19.14$
6. $(x + 0.605)^2 = 6.646025$

6 from Unit 8, Lesson 19

To derive the quadratic formula, we can multiply $ax^2 + bx + c = 0$ by an expression so that the coefficient of x^2 is a perfect square and the coefficient of x is an even number.

a. Which expression, a, $2a$, or $4a$, would you multiply $ax^2 + bx + c = 0$ by to get started deriving the quadratic formula?

b. What does the equation $ax^2 + bx + c = 0$ look like when you multiply both sides by your answer?

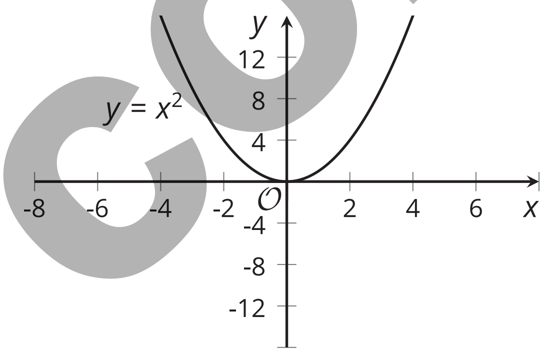
7 from Unit 7, Lesson 12

Here is a graph that represents $y = x^2$.

On the same coordinate plane, sketch and label the graph that represents each equation:

a. $y = -x^2 - 4$

b. $y = 2x^2 + 4$



8 from Unit 7, Lesson 15

Which quadratic expression is in vertex form?

A. $x^2 - 6x + 8$

B. $(x - 6)^2 + 3$

C. $(x - 3)(x - 6)$

D. $(8 - x)x$

9 from Unit 5, Lesson 10

Function f is defined by the expression $\frac{5}{x-2}$.

a. Evaluate $f(12)$.

b. Explain why $f(2)$ is undefined.

c. Give a possible domain for f.

Sec D

Addressing CA CCSSM A-REI.4b, N-RN.3; building on 7.NS.1; building toward N-RN.3; practicing MP1, MP2, MP3, MP5, MP7, MP8

Sums and Products of Rational and Irrational Numbers

Let's make convincing arguments about why the sums and products of rational numbers and irrational numbers are always certain kinds of numbers.

21.1 Operations on Integers

Here are some examples of integers:

-25 -10 -2 -1 0 5 9 40

1. Experiment with adding any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:
   a. add up to another integer.
   b. add up to a number that is *not* an integer.
2. Experiment with multiplying any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:
   a. multiply to make another integer.
   b. multiply to make a number that is *not* an integer.

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21.2 Sums and Products of Rational Numbers

1. Here are a few examples of adding two rational numbers. Is each sum a rational number? Be prepared to explain how you know.

   a. $4 + 0.175 = 4.175$

   b. $\frac{1}{2} + \frac{4}{5} = \frac{5}{10} + \frac{8}{10} = \frac{13}{10}$

   c. $-0.75 + \frac{14}{8} = \frac{-6}{8} + \frac{14}{8} = \frac{8}{8} = 1$

   d. a is an integer: $\frac{2}{3} + \frac{a}{15} = \frac{10}{15} + \frac{a}{15} = \frac{10+a}{15}$

2. Here is a way to explain why the sum of two rational numbers is rational:

   Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are fractions. That means that a, b, c, and d are integers, and b and d are not 0.

   a. Find the sum of $\frac{a}{b}$ and $\frac{c}{d}$. Show your reasoning.

   b. In the sum, are the numerator and the denominator integers? How do you know?

   c. Use your responses to explain why the sum of $\frac{a}{b} + \frac{c}{d}$ is a rational number.

3. Use the same reasoning as in the previous question to explain why the product of two rational numbers, $\frac{a}{b} \cdot \frac{c}{d}$, must be rational.

Are you ready for more?

Consider numbers that are of the form $a + b\sqrt{5}$, where a and b are integers. Let's call such numbers *quintegers*.

Here are some examples of quintegers:

- $3 + 4\sqrt{5}$ $(a = 3, b = 4)$
- $7 - 2\sqrt{5}$ $(a = 7, b = -2)$
- $-5 + \sqrt{5}$ $(a = -5, b = 1)$
- 3 $(a = 3, b = 0)$.

1. When we add two quintegers, will we always get another quinteger? Either prove this or find two quintegers whose sum is not a quinteger.

2. When we multiply two quintegers, will we always get another quinteger? Either prove this or find two quintegers whose product is not a quinteger.

Sec D

21.3 Sums and Products of Rational and Irrational Numbers

1. Here is a way to explain why $\sqrt{2} + \frac{1}{9}$ is irrational.

   - Let s be the sum of $\sqrt{2}$ and $\frac{1}{9}$, or $s = \sqrt{2} + \frac{1}{9}$.
   - Suppose s is rational.

   a. Is $s + -\frac{1}{9}$ rational or irrational? Explain how you know.

   b. Evaluate $s + -\frac{1}{9}$. Is the sum rational or irrational?

   c. Use your responses to explain why s cannot be a rational number, and therefore $\sqrt{2} + \frac{1}{9}$ cannot be rational.

Sec D

2. Use a similar reasoning as in the earlier question to explain why $\sqrt{2} \cdot \frac{1}{9}$ is irrational. Here are some assumptions to get you started.

   - Let p be the product of $\sqrt{2}$ and $\frac{1}{9}$, or $p = \sqrt{2} \cdot \frac{1}{9}$.
   - Suppose p is rational.

21.4 Equations with Different Kinds of Solutions

1. Consider the equation $4x^2 + bx + 9 = 0$. Find a value of b so that the equation has:

   a. 2 rational solutions

   b. 2 irrational solutions

   c. 1 solution

   d. no solutions

2. Describe all the values of b that produce 2 solutions, 1 solution, and no solutions.

3. Write a new quadratic equation with each type of solution. Be prepared to explain how you know that your equation has the specified type and number of solutions.

   a. no solutions

   b. 2 irrational solutions

   c. 2 rational solutions

   d. 1 solution

Lesson 21 Summary

We know that quadratic equations can have rational solutions or irrational solutions. For example, the solutions to $(x+3)(x-1)=0$ are -3 and 1, which are rational. The solutions to $x^2-8=0$ are $\pm\sqrt{8}$, which are irrational.

Sometimes solutions to equations combine two numbers by addition or multiplication—for example, $\pm 4\sqrt{3}$ and $1+\sqrt{12}$. What kind of numbers are these expressions?

When we add or multiply two rational numbers, is the result rational or irrational?

- The sum of two rational numbers is rational. Here is one way to explain why it is true:
  - Any two rational numbers can be written $\frac{a}{b}$ and $\frac{c}{d}$, where $a, b, c,$ and d are integers, and b and d are not zero.
  - The sum of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad+bc}{bd}$. The denominator is not zero because neither b nor d is zero.
  - Multiplying or adding two integers always gives an integer, so we know that ad, bc, bd and $ad+bc$ are all integers.
  - If the numerator and denominator of $\frac{ad+bc}{bd}$ are integers, then the number is a fraction, which is rational.
- The product of two rational numbers is rational. We can show why in a similar way:
  - For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, where $a, b, c,$ and d are integers, and b and d are not zero, the product is $\frac{ac}{bd}$.
  - Multiplying two integers always results in an integer, so both ac and bd are integers. Therefore, $\frac{ac}{bd}$ is a rational number.

What about two irrational numbers?

- The sum of two irrational numbers could be either rational or irrational. We can show this through examples:
  - $\sqrt{3}$ and -$\sqrt{3}$ are both irrational, but their sum is 0, which is rational.
  - $\sqrt{3}$ and $\sqrt{5}$ are both irrational, and their sum is irrational.
- The product of two irrational numbers could be either rational or irrational. We can show this through examples:
  - $\sqrt{2}$ and $\sqrt{8}$ are both irrational, but their product is $\sqrt{16}$, or 4, which is rational.
  - $\sqrt{2}$ and $\sqrt{7}$ are both irrational, and their product is $\sqrt{14}$, which is not a perfect square and is therefore irrational.

What about a rational number and an irrational number?

- The sum of a rational number and an irrational number is irrational. To explain why requires a slightly different argument:
    - Let R be a rational number and I an irrational number. We want to show that $R + I$ is irrational.
    - Suppose s represents the sum of R and I ($s = R + I$), and suppose s is rational.
    - If s is rational, then $s + -R$ would also be rational, because the sum of two rational numbers is rational.
    - $s + -R$ is not rational, however, because $(R + I) + -R = I$.
    - $s + -R$ cannot be both rational and irrational, which means that our original assumption that s is rational was incorrect. s, which is the sum of a rational number and an irrational number, must be irrational.
- The product of a nonzero rational number and an irrational number is irrational. We can show why this is true in a similar way:
    - Let R be rational and I irrational. We want to show that $R \cdot I$ is irrational.
    - Suppose p is the product of R and I ($p = R \cdot I$), and suppose p is rational.
    - If p is rational, then $p \cdot \frac{1}{R}$ would also be rational because the product of two rational numbers is rational.
    - $p \cdot \frac{1}{R}$ is not rational, however, because $R \cdot I \cdot \frac{1}{R} = I$.
    - $p \cdot \frac{1}{R}$ cannot be both rational and irrational, which means our original assumption that p is rational was false. p, which is the product of a rational number and an irrational number, must be irrational.

Sec D

Lesson 21 Practice Problems

1 from Unit 8, Lesson 15

Match each expression to an equivalent expression.

A. $\sqrt{5} \pm \sqrt{3}$

B. $1 \pm \sqrt{3}$

C. $\sqrt{3} \pm 1$

D. 5 ± -2

E. -3 ± -3

1. 3 and 7
2. $\sqrt{5} + \sqrt{3}$ and $\sqrt{5} - \sqrt{3}$
3. -6 and 0
4. $\sqrt{3} + 1$ and $\sqrt{3} - 1$
5. $1 + \sqrt{3}$ and $1 - \sqrt{3}$

2 Consider the statement: "An irrational number multiplied by an irrational number always makes an irrational product."

Select **all** the examples that show that this statement is false.

A. $\sqrt{4} \cdot \sqrt{5}$

B. $\sqrt{4} \cdot \sqrt{4}$

C. $\sqrt{7} \cdot \sqrt{7}$

D. $\frac{1}{\sqrt{5}} \cdot \sqrt{5}$

E. $\sqrt{0} \cdot \sqrt{7}$

F. $-\sqrt{5} \cdot \sqrt{5}$

G. $\sqrt{5} \cdot \sqrt{7}$

3 from Unit 7, Lesson 15

a. Where is the vertex of the graph that represents $y = (x-3)^2 + 5$?

b. Does the graph open up or down? Explain how you know.

4 Here are the solutions to some quadratic equations. Decide if the solutions are rational or irrational.

$3 \pm \sqrt{2}$ $\quad$ $\sqrt{9} \pm 1$ $\quad$ $\frac{1}{2} \pm \frac{3}{2}$ $\quad$ 10 ± 0.3

$\frac{1 \pm \sqrt{8}}{2}$ $\quad$ $-7 \pm \sqrt{\frac{4}{9}}$

5 Find an example that shows that each statement is false.

a. An irrational number multiplied by an irrational number always makes an irrational product.

b. A rational number multiplied by an irrational number never gives a rational product.

c. Adding an irrational number to an irrational number always gives an irrational sum.

6 from Unit 8, Lesson 13

Which equation is equivalent to $x^2 - 3x = \frac{7}{4}$ but has a perfect square on one side?

A. $x^2 - 3x + 3 = \frac{19}{4}$

B. $x^2 - 3x + \frac{3}{4} = \frac{10}{4}$

C. $x^2 - 3x + \frac{9}{4} = \frac{16}{4}$

D. $x^2 - 3x + \frac{9}{4} = \frac{7}{4}$

7 from Unit 8, Lesson 18

A student who used the quadratic formula to solve $2x^2 - 8x = 2$ said that the solutions are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

a. What equations can we graph to check those solutions? What features of the graph do we analyze?

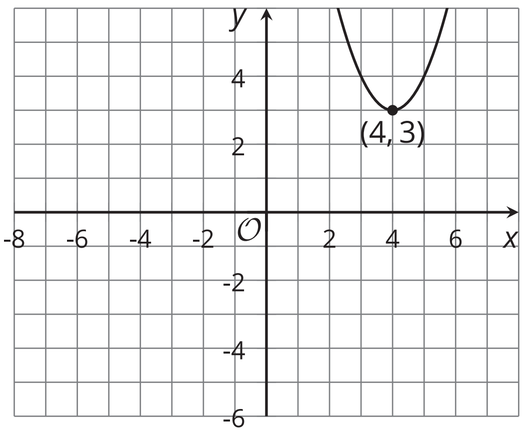
b. How do we look for $2 + \sqrt{5}$ and $2 - \sqrt{5}$ on a graph?

Sec D

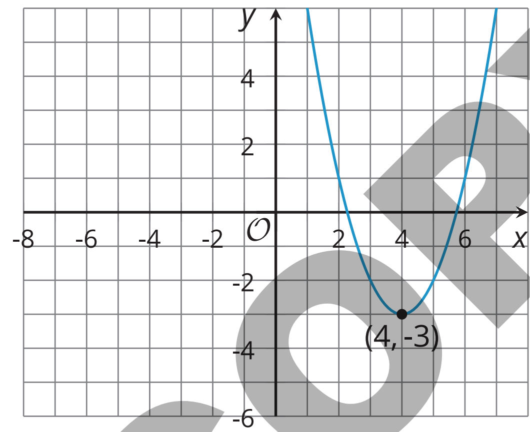
8 from Unit 7, Lesson 15

Here are four graphs. Match each graph with a quadratic equation that it represents.

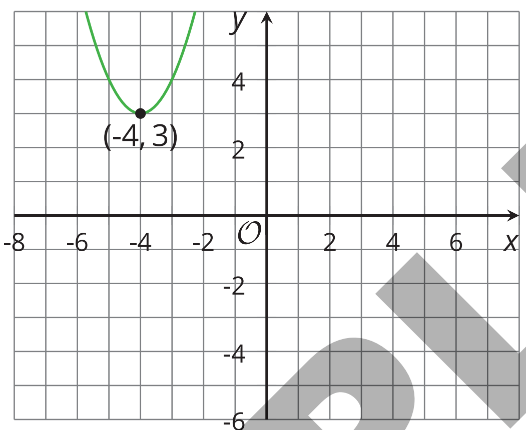
Graph A



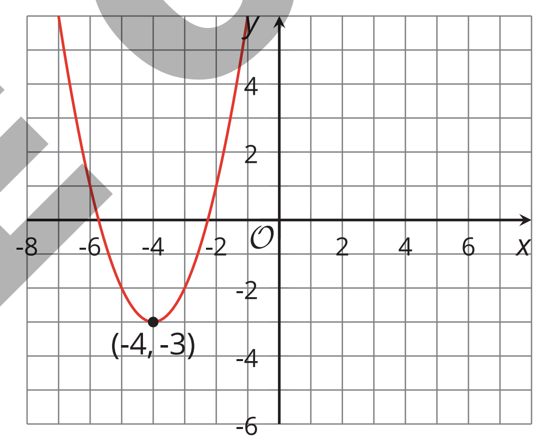
Graph B



Graph C



Graph D



A. Graph A

B. Graph B

C. Graph C

D. Graph D

1. $y = (x + 4)^2 - 3$
2. $y = (x - 4)^2 - 3$
3. $y = (x + 4)^2 + 3$
4. $y = (x - 4)^2 + 3$

Addressing CA CCSSM A-SSE.2, A-SSE.3, A-SSE.3b, F-IF.8a, F-BF.1b; building on F-IF.7a; building toward A-SSE.3b; practicing MP1, MP6, MP7

Rewriting Quadratic Expressions in Vertex Form

Let’s see what else completing the square can help us do.

22.1 Three Expressions, One Function

Each of these expressions defines the same function.

$$x^2 + 6x + 8 \qquad (x + 2)(x + 4) \qquad (x + 3)^2 - 1$$

Without graphing or doing any calculations, determine where the following features would be on a graph that represents the function.

1. the vertex
2. the x-intercepts
3. the y-intercept

22.2 Back and Forth

1. Here are two expressions in vertex form. Rewrite each expression in standard form. Show your reasoning.
   a. $(x + 5)^2 + 1$
   b. $(x - 3)^2 - 7$
2. Think about the steps you took and how to reverse them. Convert one or both of the expressions in standard form back into vertex form. Explain how you go about converting the expressions.

3. Test your strategy by rewriting $x^2 + 10x + 9$ in vertex form.

4. Let's check the expression that you have rewritten in vertex form.

   a. Use graphing technology to graph both $y = x^2 + 10x + 9$ and your new expression. Does it appear that they define the same function?

   b. If you convert your expression in vertex form back into standard form, do you get $x^2 + 10x + 9$?

22.3 Inconvenient Coefficients

1. a. Here is one way to rewrite $3x^2 + 12x + 9$ in vertex form. Study the steps, and write a brief explanation of what is happening at each step.

$$3x^2 + 12x + 9 \qquad \text{original expression}$$

$$3(x^2 + 4x + 3)$$

$$3(x^2 + 4x + 3 + 1 - 1)$$

$$3(x^2 + 4x + 4 - 1)$$

$$3\left((x + 2)^2 - 1\right)$$

$$3(x + 2)^2 - 3$$

   b. What is the vertex of the graph that represents this expression?

   c. Does the graph open upward or downward? Explain how you know.

2. Rewrite each expression in vertex form. Show your reasoning.

a. $-2x^2 - 4x + 6$

b. $4x^2 + 24x + 20$

c. $-x^2 + 20x$

Are you ready for more?

1. Write $f(x) = 2(x - 3)(x - 9)$ in vertex form without completing the square. (Hint: Think about finding the zeros of the function.) Explain your reasoning.

2. Write $g(x) = 2(x - 3)(x - 9) + 21$ in vertex form without completing the square. Explain your reasoning.

Sec E

22.4 Info Gap: Features of Functions

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the problem card:

1. Silently read your card and think about what information you need to answer the question.
2. Ask your partner for the specific information that you need. “Can you tell me _____?”
3. Explain to your partner how you are using the information to solve the problem. “I need to know _____ because . . . .”
   Continue to ask questions until you have enough information to solve the problem.
4. Once you have enough information, share the problem card with your partner, and solve the problem independently.
5. Read the data card, and discuss your reasoning.

If your teacher gives you the data card:

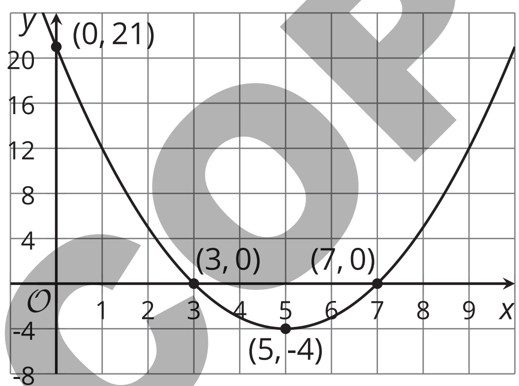
1. Silently read your card. Wait for your partner to ask for information.
2. Before telling your partner any information, ask, “Why do you need to know _____?”
3. Listen to your partner’s reasoning, and ask clarifying questions. Only give information that is on your card. Do not figure out anything for your partner!
   These steps may be repeated.
4. Once your partner says they have enough information to solve the problem, read the problem card, and solve the problem independently.
5. Share the data card, and discuss your reasoning.

Lesson 22 Summary

Remember that a quadratic function can be defined by equivalent expressions in different forms, which enable us to see different features of its graph. For example, these expressions define the same function:

$$\begin{array}{rl} (x-3)(x-7) & \text{factored form} \\ x^2-10x+21 & \text{standard form} \\ (x-5)^2-4 & \text{vertex form} \end{array}$$

- From factored form, we can tell that the x-intercepts are $(3, 0)$ and $(7, 0)$.
- From standard form, we can tell that the y-intercept is $(0, 21)$.
- From vertex form, we can tell that the vertex is $(5, \text{-}4)$.



Recall that a function expressed in vertex form is written as $a(x-h)^2+k$. The values of h and k reveal the vertex of the graph: (h, k) are the coordinates of the vertex. In this example, a is 1, h is 5, and k is -4.

- If we have an expression in vertex form, we can rewrite it in standard form by using the distributive property and combining like terms.

  Let's say we want to rewrite $(x-1)^2-4$ in standard form.

$$\begin{array}{l} (x-1)^2-4 \\ (x-1)(x-1)-4 \\ x^2-2x+1-4 \\ x^2-2x-3 \end{array}$$

- If we have an expression in standard form, we can rewrite it in vertex form by completing the square.

  Let's rewrite $x^2+10x+24$ in vertex form.

  A perfect square would be $x^2+10x+25$, so we need to add 1. Adding 1, however, would change the expression. To keep the new expression equivalent to the original one, we will need to both add 1 and subtract 1.

$$x^2 + 10x + 24$$
$$x^2 + 10x + 24 + 1 - 1$$
$$x^2 + 10x + 25 - 1$$
$$(x + 5)^2 - 1$$

- Let's rewrite another expression in vertex form: $-2x^2 + 12x - 30$.

  To make it easier to complete the square, we can use the distributive property to rewrite the expression with -2 as a factor, which gives $-2(x^2 - 6x + 15)$.

  For the expression in the parentheses to be a perfect square, we need $x^2 - 6x + 9$. We have 15 in the expression, so we can subtract 6 from it to get 9, and then add 6 again to keep the value of the expression unchanged. Then, we can rewrite $x^2 - 6x + 9$ in factored form.

$$-2x^2 + 12x - 30$$
$$-2(x^2 - 6x + 15)$$
$$-2(x^2 - 6x + 15 - 6 + 6)$$
$$-2(x^2 - 6x + 9 + 6)$$
$$-2\left((x - 3)^2 + 6\right)$$

  This expression is not yet in vertex form, however. To finish up, we need to apply the distributive property again so that the expression is of the form $a(x - h)^2 + k$:

$$-2\left((x - 3)^2 + 6\right)$$
$$-2(x - 3)^2 - 12$$

  When written in this form, we can see that the vertex of the graph representing $-2(x - 3)^2 - 12$ is $(3, -12)$.

Lesson 22 Practice Problems

1 The following quadratic expressions all define the same function.

$(x+5)(x+3)$ $\qquad$ $x^2+8x+15$ $\qquad$ $(x+4)^2-1$

Select **all** of the statements that are true about the graph of this function.

A. The y-intercept is $(0, \text{-}15)$.

B. The vertex is $(\text{-}4, \text{-}1)$.

C. The x-intercepts are $(\text{-}5, 0)$ and $(\text{-}3, 0)$.

D. The x-intercepts are $(0, 5)$ and $(0, 3)$.

E. The x-intercept is $(0, 15)$.

F. The y-intercept is $(0, 15)$.

G. The vertex is $(4, \text{-}1)$.

2 The following expressions all define the same quadratic function.

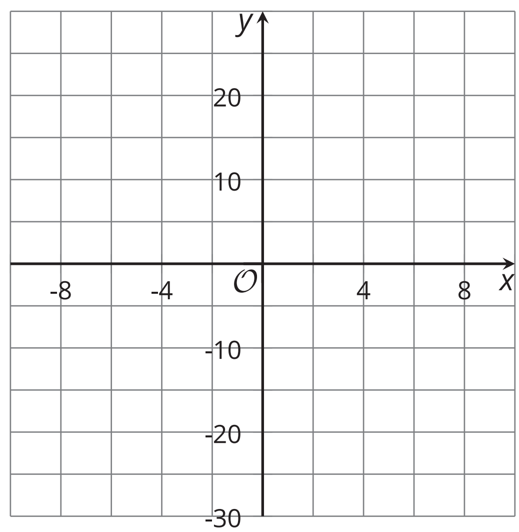
$(x-4)(x+6)$ $\qquad$ $x^2+2x-24$ $\qquad$ $(x+1)^2-25$

a. What is the y-intercept of the graph of the function?

b. What are the x-intercepts of the graph?

c. What is the vertex of the graph?

d. Sketch a graph of the function without using graphing technology. Make sure the x-intercepts, y-intercept, and vertex are plotted accurately.



3 Here is one way an expression in standard form is rewritten into vertex form.

$x^2 - 7x + 6$	original expression
$x^2 - 7x + \left(\text{-}\frac{7}{2}\right)^2 + 6 - \left(\text{-}\frac{7}{2}\right)^2$	step 1
$\left(x - \frac{7}{2}\right)^2 + 6 - \frac{49}{4}$	step 2
$\left(x - \frac{7}{2}\right)^2 + \frac{24}{4} - \frac{49}{4}$	step 3
$\left(x - \frac{7}{2}\right)^2 - \frac{25}{4}$	step 4

a. In step 1, where does the number $\text{-}\frac{7}{2}$ come from?

b. In step 1, why is $\left(\text{-}\frac{7}{2}\right)^2$ added and then subtracted?

c. What happens in step 2?

d. What happens in step 3?

e. What does the last expression tell us about the graph of a function defined by this expression?

4 Rewrite each quadratic expression in vertex form.

a. $d(x) = x^2 + 12x + 36$

b. $f(x) = x^2 + 10x + 21$

c. $g(x) = 2x^2 - 20x + 32$

Sec E

5 from Unit 8, Lesson 21

a. Give an example that shows that the sum of two irrational numbers can be rational.

b. Give an example that shows that the sum of two irrational numbers can be irrational.

6 from Unit 8, Lesson 21

a. Give an example that shows that the product of two irrational numbers can be rational.

b. Give an example that shows that the product of two irrational numbers can be irrational.

7 from Unit 8, Lesson 15

Select **all** the equations with irrational solutions.

A. $36 = x^2$

B. $x^2 = \frac{1}{4}$

C. $x^2 = 8$

D. $2x^2 = 8$

E. $x^2 = 0$

F. $x^2 = 40$

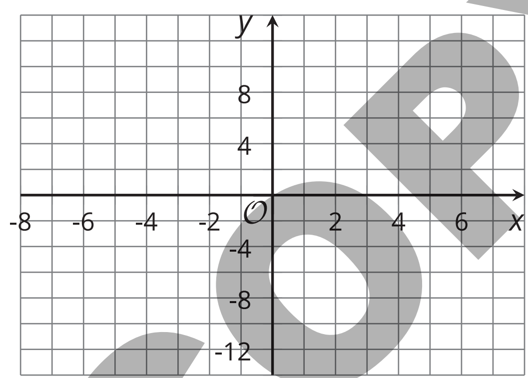
G. $9 = x^2 - 1$

8 from Unit 7, Lesson 16

a. What are the coordinates of the vertex of the graph of the function defined by $f(x) = 2(x+1)^2 - 4$?

b. Find the coordinates of two other points on the graph.

c. Sketch the graph of f.



9 from Unit 7, Lesson 17

How is the graph of the equation $y = (x-1)^2 + 4$ related to the graph of the equation $y = x^2$?

A. The graph of $y = (x-1)^2 + 4$ is the same as the graph of $y = x^2$ but is shifted 1 unit to the right and 4 units up.

B. The graph of $y = (x-1)^2 + 4$ is the same as the graph of $y = x^2$ but is shifted 1 unit to the left and 4 units up.

C. The graph of $y = (x-1)^2 + 4$ is the same as the graph of $y = x^2$ but is shifted 1 unit to the right and 4 units down.

D. The graph of $y = (x-1)^2 + 4$ is the same as the graph of $y = x^2$ but is shifted 1 unit to the left and 4 units down.

Addressing CA CCSSM A-SSE.3b, F-IF.7a, F-IF.9, F-BF.1b; building on F-IF.2; building toward A-SSE.3b; practicing MP3 and MP7

Using Quadratic Expressions in Vertex Form to Solve Problems

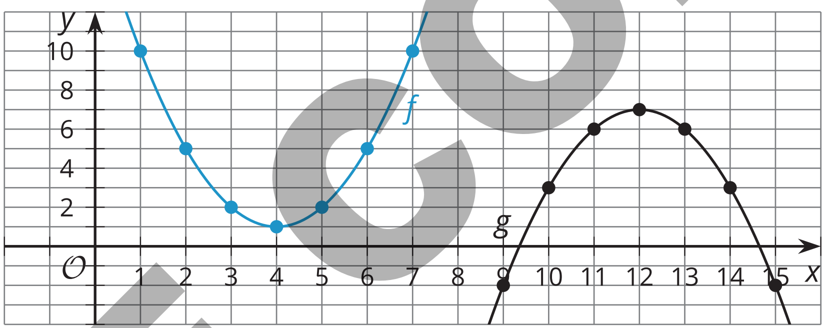
Let's find the maximum or minimum value of a quadratic function.

23.1 Values of a Function

Here are graphs that represent two functions, f and g, defined by these equations:

$$f(x) = (x - 4)^2 + 1$$

$$g(x) = -(x - 12)^2 + 7$$



1. $f(1)$ can be expressed in words as "the value of f when x is 1." Find or compute:
   a. the value of f when x is 1
   b. $f(3)$
   c. $f(10)$
2. Does f have a maximum, minimum, or neither? If it has a maximum or minimum, what is the greatest or least value $f(x)$ can have?
3. $g(9)$ can be expressed in words as "the value of g when x is 9." Find or compute:
   a. the value of g when x is 9
   b. $g(13)$
   c. $g(2)$
4. Does g have a maximum, minimum, or neither? If it has a maximum or minimum, what is the greatest or least value $g(x)$ can have?

23.2 Maximums and Minimums

1. The graph that represents $p(x) = (x - 8)^2 + 1$ has its vertex at $(8, 1)$. Here is one way to show, without graphing, that $(8, 1)$ corresponds to the *minimum* value of p.

   - When $x = 8$, the value of $(x - 8)^2$ is 0, because $(8 - 8)^2 = 0^2 = 0$.
   - Squaring any nonzero number always results in a positive number, so when x is any value other than 8, $(x - 8)$ will be a number other than 0, and when the expression is squared, $(x - 8)^2$, it will be positive.
   - Any positive number is greater than 0, so when $x \neq 8$, the value of $(x - 8)^2$ will be greater than when $x = 8$. In other words, p has the least value when $x = 8$.

   Use similar reasoning to explain why the point $(4, 1)$ corresponds to the *maximum* value of q, defined by $q(x) = -2(x - 4)^2 + 1$.

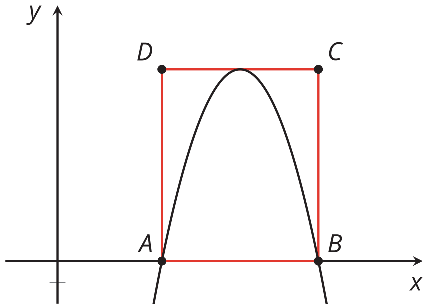
Sec E

2. Here are some quadratic functions and the coordinates of the vertex of the graph of each. Determine if the vertex corresponds to the maximum or the minimum value of the function. Be prepared to explain how you know.

equation	coordinates of the vertex	maximum or minimum?
$f(x) = -(x - 4)^2 + 6$	$(4, 6)$	
$g(x) = (x + 7)^2 - 3$	$(-7, -3)$	
$h(x) = 4(x + 5)^2 + 7$	$(-5, 7)$	
$k(x) = x^2 - 6x - 3$	$(3, -12)$	
$m(x) = -x^2 + 8x$	$(4, 16)$	

Are you ready for more?

Here is a portion of the graph of function q, defined by $q(x) = -x^2 + 14x - 40$.



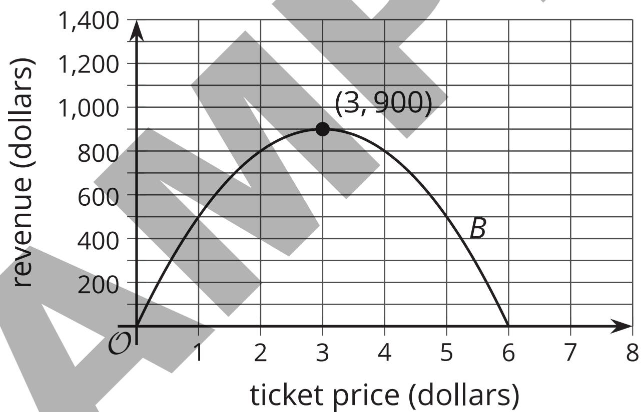
$ABCD$ is a rectangle. Points A and B coincide with the x-intercepts of the graph, and segment CD just touches the vertex of the graph.

Find the area of $ABCD$. Show your reasoning.

23.3 All the World's a Stage

Function A, defined by $p(600 - 75p)$, describes the revenue collected from the sales of tickets for Performance A, a musical.

The graph represents a function, B, that models the revenue collected from the sales of tickets for Performance B, a Shakespearean comedy.



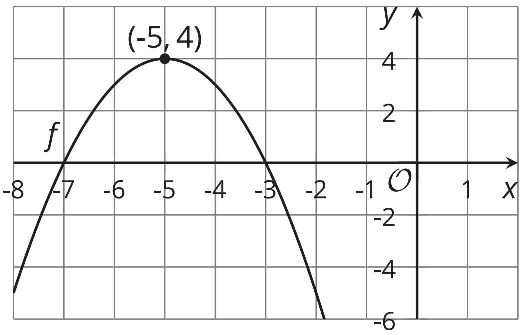
In both functions, p represents the price of one ticket, and both revenues and prices are measured in dollars.

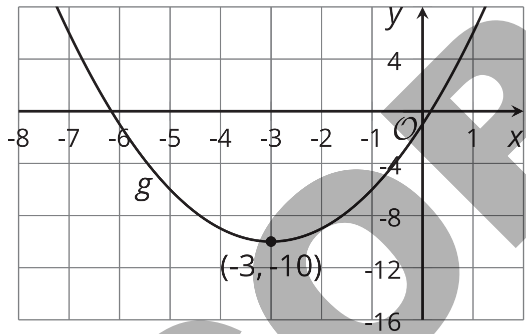
Without creating a graph of A, determine which performance gives the greater maximum revenue when tickets are p dollars each. Explain or show your reasoning.

Lesson 23 Summary

Any quadratic function has either a *maximum* or a *minimum* value. We can tell whether a quadratic function has a maximum or a minimum by observing the vertex of its graph.

Here are graphs representing functions f and g, defined by $f(x) = -(x + 5)^2 + 4$ and $g(x) = x^2 + 6x - 1$.





- The vertex of the graph of f is $(-5, 4)$, and the graph is a parabola that opens downward.
- No other points on the graph of f (no matter how much we zoom out) are higher than $(-5, 4)$, so we can say that f has a maximum of 4, and that this occurs when $x = -5$.

- The vertex of the graph of g is at $(-3, -10)$, and the graph is a parabola that opens upward.
- No other points on the graph (no matter how much we zoom out) are lower than $(-3, -10)$, so we can say that g has a minimum of -10, and that this occurs when $x = -3$.

We know that a quadratic expression in vertex form can reveal the vertex of the graph, so we don't actually have to graph the expression. But how do we know, without graphing, if the vertex corresponds to a maximum or a minimum value of a function?

The vertex form can give us that information as well!

To see if $(-3, -10)$ is a minimum or maximum of g, we can rewrite $x^2 + 6x - 1$ in vertex form, which is $(x + 3)^2 - 10$. Let's look at the squared term in $(x + 3)^2 - 10$.

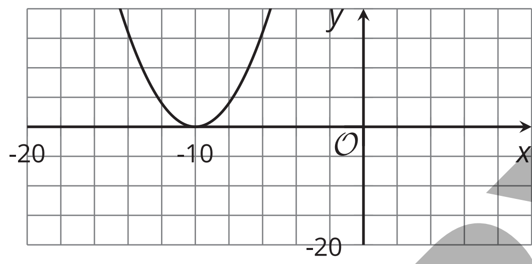
- When $x = -3$, $(x + 3)$ is 0, so $(x + 3)^2$ is also 0.
- When x is not -3, the expression $(x + 3)$ is a nonzero number, and $(x + 3)^2$ is positive.
- Because a squared number cannot have a value less than 0, $(x + 3)^2$ has the least value when $x = -3$.

To see if $(-5, 4)$ is a minimum or maximum of f, let's look at the squared term in $-(x+5)^2+4$.

- When $x = -5$, $(x+5)$ is 0, so $(x+5)^2$ is also 0.
- When x is not -5, the expression $(x+5)$ is nonzero, so $(x+5)^2$ is positive. The expression $-(x+5)^2$ has a coefficient of -1, however. Multiplying $(x+5)^2$ (which is positive when $x \neq -5$) by a negative number results in a negative number.
- Because a negative number is always less than 0, the value of $-(x+5)^2+4$ will always be less when $x \neq -5$ than when $x = -5$. This means $x = -5$ gives the greatest value of f.

Lesson 23 Practice Problems

1 Here is a graph of a quadratic function $f(x)$. What is the minimum value of $f(x)$?



2 The graph that represents $f(x) = (x + 1)^2 - 4$ has its vertex at $(-1, -4)$.

Explain how we can tell from the expression $(x + 1)^2 - 4$ that -4 is the minimum value rather than the maximum value of f.

3 Each expression here defines a quadratic function. Find the vertex of the graph of the function. Then, state whether the vertex corresponds to the maximum or the minimum value of the function.

a. $(x - 5)^2 + 6$

b. $(x + 5)^2 - 1$

c. $-2(x + 3)^2 - 10$

d. $3(x - 7)^2 + 11$

e. $-(x - 2)^2 - 2$

f. $(x + 1)^2$

Sec E

4 from Unit 8, Lesson 19

Consider the equation $x^2 = 12x$.

a. Can we use the quadratic formula to solve this equation? Explain or show how you know.

b. Is it easier to solve this equation by completing the square or by rewriting it in factored form and using the zero product property? Explain or show your reasoning.

5 from Unit 8, Lesson 17

Match each equation to the number of solutions it has.

A. $(x-1)(x-5)=5$

B. $x^2-2x=-1$

C. $(x-5)^2=-25$

1. no solutions
2. 1 solution
3. 2 solutions

6 from Unit 8, Lesson 20

Which equation has irrational solutions?

A. $100x^2=9$

B. $9(x-1)^2=4$

C. $4x^2-1=0$

D. $9(x+3)^2=27$

7 from Unit 8, Lesson 21

Let I represent an irrational number and let R represent a nonzero rational number. Decide if each statement is true or false. Explain your thinking.

a. $R \cdot I$ can be rational.

b. $I \cdot I$ can be rational.

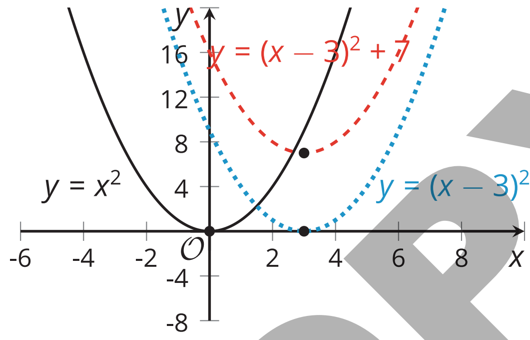
c. $R \cdot R$ can be rational.

Sec E

8 from Unit 7, Lesson 17

Here are graphs of the equations $y = x^2$, $y = (x - 3)^2$, and $y = (x - 3)^2 + 7$.

a. How do the three graphs compare?



b. How does the -3 in $(x - 3)^2$ affect the graph when compared to $y = x^2$?

c. How does the +7 in $(x - 3)^2 + 7$ affect the graph when compared to $y = (x - 3)^2$?

9 from Unit 6, Lesson 18

Three $5,000 loans have different annual interest rates. Loan A charges 10.5% annual interest, Loan B charges 15.75%, and Loan C charges 18.25%.

a. If we graph the amount owed as a function of years without payment, what would the three graphs look like? Describe or sketch your prediction.

b. Use technology to graph each function. According to your graphs, if no payments are made, about how many years will it take for the unpaid balance of each loan to triple?

Addressing CA CCSSM A-REI.4b, A-REI.7, F-IF.8a, F-LE.6, F-BF.1b; building on F-LE.2; practicing MP1

Using Quadratic Equations to Model Situations and Solve Problems

Let's analyze a situation modeled by a quadratic equation.

24.1 Equations of Two Lines and a Curve

1. Write an equation representing the line that passes through each pair of points.
   a. $(3,3)$ and $(5,5)$

   b. $(0,4)$ and $(-4,0)$

2. Solve this equation: $x+1=(x-2)^2-3$. Show your reasoning.

24.2 The Dive

The function h, defined by $h(t)=-5t^2+10t+7.5$, models the height of a diver above the water (in meters), t seconds after the diver leaves the board. For each question, explain how you know.

1. How high above the water is the diving board?

2. When does the diver hit the water?

Sec F

3. At what point during her descent toward the water is the diver at the same height as the diving board?

4. When does the diver reach the maximum height of the dive?

5. What is the maximum height the diver reaches during the dive?

Are you ready for more?

Another diver jumps off a platform, rather than a springboard. The platform is also 7.5 meters above the water, but this diver hits the water after about 1.5 seconds.

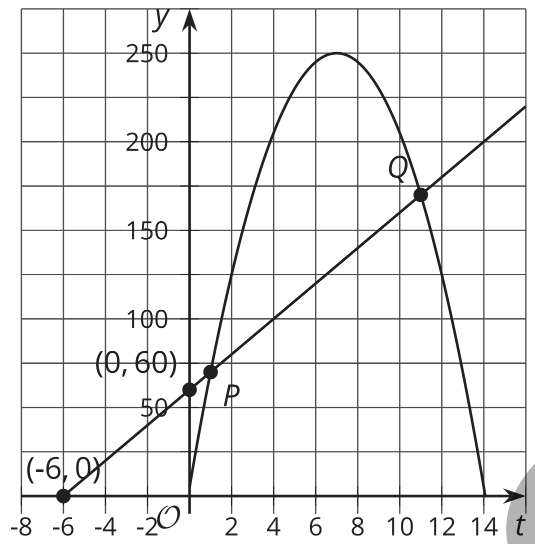
Write an equation that would approximately model her height over the water, h, in meters, t seconds after she has left the platform. Include the term $-5t^2$, which accounts for the effect of gravity.

24.3 A Linear Function and a Quadratic Function

Sec F

A golf ball is shot straight up into the air so that its height above the ground, in meters, is given by $y = -5t^2 + 70t + 5$, where t represents the number of seconds after the ball is launched.

A camera is on a device that was on the ground 6 seconds before the ball was launched, and it rises at a constant rate so that it is 60 meters above the ground when the ball is hit.



1. Write an equation that gives y, the height of the camera above the ground, in meters, as a function of t, seconds after the ball is launched.
2. Find the coordinates of points P and Q, where the two graphs intersect. Explain or show your reasoning.
3. What do points P and Q mean in this situation?

Lesson 24 Summary

Certain real-world situations can be modeled by quadratic functions, and these functions can be represented by equations. Sometimes, all the skills we have developed are needed to make sense of these situations. When we have a mathematical model and the skills to use the model to answer questions, we are able to gain useful or interesting insights about the situation.

Suppose we have a model for the height of a launched object, h, as a function of time since it was launched, t, defined by $h(t) = -4.9t^2 + 28t + 2.1$. We can answer questions such as these about the object's flight:

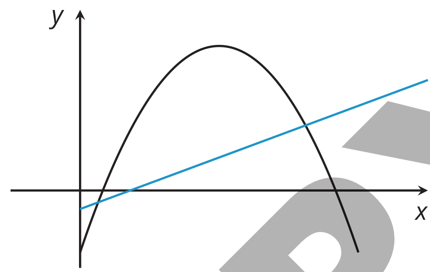
- From what height is the object launched? (An expression in standard form can help us with this question. Or, we can evaluate $h(0)$ to find the answer.)
- At what time does it hit the ground? (When an object hits the ground, its height is 0, so we can find the zeros using one of the methods we learned: graphing, rewriting the equation in factored form, completing the square, or using the quadratic formula.)

Sec F

- What is its maximum height, and at what time does it reach the maximum height?

(We can rewrite the expression in vertex form, or we can use the zeros or a graph of the function to find the vertex.)

Sometimes, relationships between quantities can be effectively communicated with graphs and expressions rather than with words. For example, these graphs represent a linear function, f, and a quadratic function, g, with the same variables for their inputs and outputs.



If we know the expressions that define these functions, we can use our knowledge of quadratic equations to answer questions such as:

- Will the two functions ever have the same value?

(Yes. We can see that their graphs intersect at a couple of places.)

- If so, at what input values does that happen? What are the output values they have in common?

(To find out, we can write and solve this equation: $f(x) = g(x)$. The solution provides the x-values for the intersection points, and the y-values can be found by substituting the solutions for x in either original function.)

Sec F

Lesson 24 Practice Problems

1 The function h represents the height of an object t seconds after it is launched into the air. The function is defined by $h(t) = -5t^2 + 20t + 18$. Height is measured in meters.

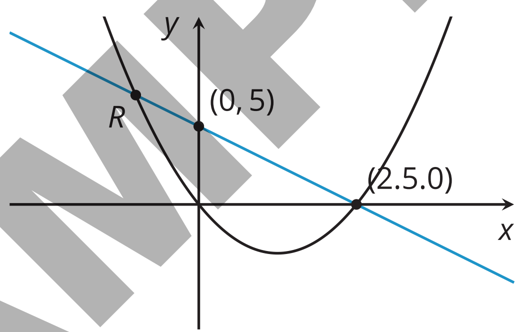
Answer each question without graphing. Explain or show your reasoning.

a. After how many seconds does the object reach a height of 33 meters?

b. When does the object reach its maximum height?

c. What is the maximum height the object reaches?

2 The graphs that represent a linear function and a quadratic function are shown here.



The quadratic function is defined by $2x^2 - 5x$.

Find the coordinates of point R without using graphing technology. Show your reasoning.

3 Diego finds his neighbor's baseball in his yard, about 10 feet away from a five-foot fence. He wants to return the ball to his neighbors, so he tosses the baseball in the direction of the fence.

Function h, defined by $h(x) = -0.078x^2 + 0.7x + 5.5$, gives the height of the ball as a function of the horizontal distance away from Diego.

Does the ball clear the fence? Explain or show your reasoning.

4 from Unit 8, Lesson 21

Clare says, “I know that $\sqrt{3}$ is an irrational number because its decimal never terminates or forms a repeating pattern. I also know that $\frac{2}{9}$ is a rational number because its decimal forms a repeating pattern. But I don’t know how to add or multiply these decimals, so I am not sure if $\sqrt{3} + \frac{2}{9}$ and $\sqrt{3} \cdot \frac{2}{9}$ are rational or irrational."

a. Here is an argument that explains why $\sqrt{3} + \frac{2}{9}$ is irrational. Complete the missing parts of the argument.

   i. Let $x = \sqrt{3} + \frac{2}{9}$. If x were rational, then $x - \frac{2}{9}$ would also be rational because . . . .

   ii. But $x - \frac{2}{9}$ is not rational because . . . .

   iii. Since x is not rational, it must be . . . .

b. Use the same type of argument to explain why $\sqrt{3} \cdot \frac{2}{9}$ is irrational.

5 from Unit 8, Lesson 22

The following expressions all define the same quadratic function.

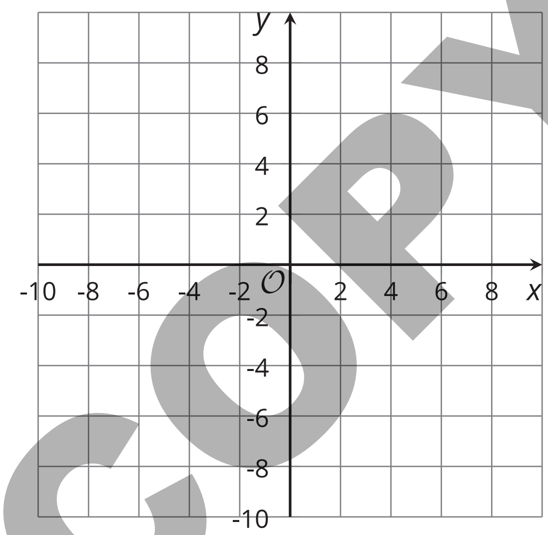
$x^2 + 2x - 8$ $\quad$ $(x+4)(x-2)$ $\quad$ $(x+1)^2 - 9$

a. What is the y-intercept of the graph of the function?

b. What are the x-intercepts of the graph?

c. What is the vertex of the graph?

d. Sketch a graph of the quadratic function without using technology. Make sure the x-intercepts, y-intercept, and vertex are plotted accurately.



6 from Unit 8, Lesson 23

Here are two quadratic functions: $f(x) = (x+5)^2 + \frac{1}{2}$ and $g(x) = (x+5)^2 + 1$.

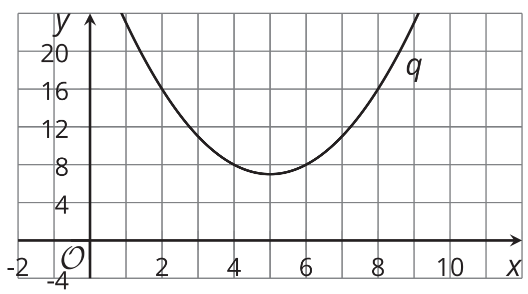
Andre says that both f and g have a minimum value, and that the minimum value of f is less than that of g. Do you agree? Explain your reasoning.

7 from Unit 8, Lesson 23

Function p is defined by the equation $p(x) = (x+10)^2 - 3$.

Function q is represented by this graph.

Which function has the smaller minimum? Explain your reasoning.

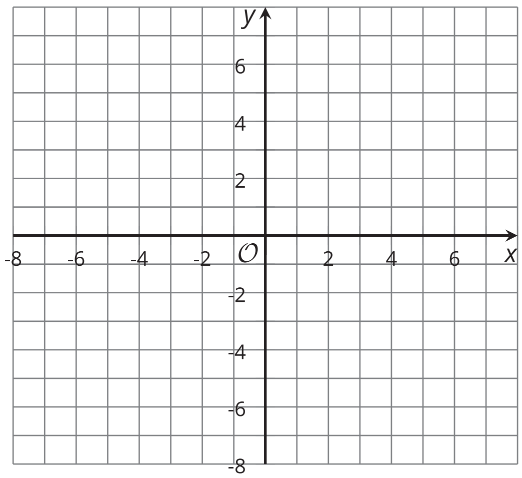


Sec F

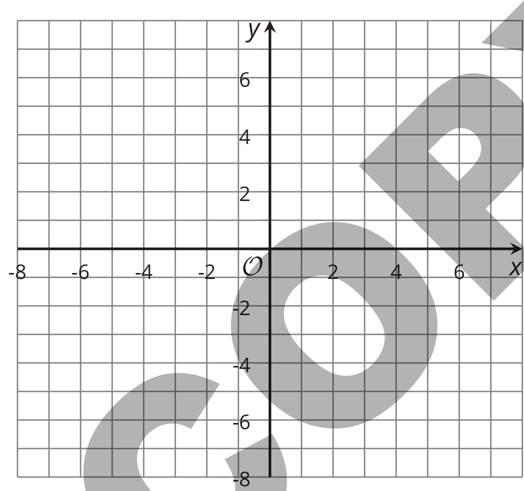
8 from Unit 8, Lesson 22

Without using graphing technology, sketch a graph that represents each quadratic function. Make sure the vertices and any intercepts that fit on the given grids are plotted accurately.

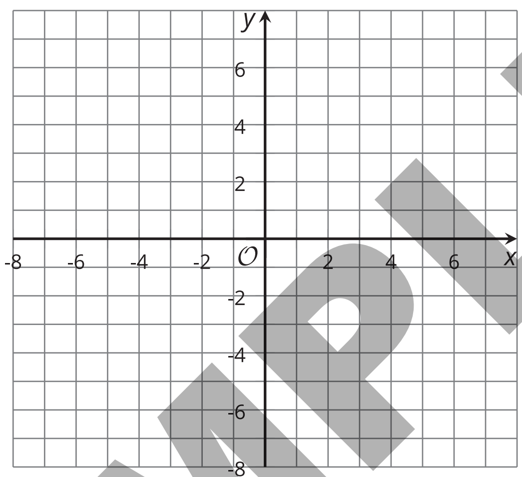
$f(x) = x^2 + 4x + 3$



$g(x) = x^2 - 4x + 3$



$h(x) = x^2 - 11x + 28$



Learning Targets

Lesson 1 Finding Unknown Inputs

- I can explain the meaning of a solution to an equation in terms of a situation.
- I can write a quadratic equation that represents a situation.

Lesson 2 When and Why Do We Write Quadratic Equations?

- I can recognize the factored form of a quadratic expression and know when it can be useful for solving problems.
- I can use a graph to find the solutions to a quadratic equation but also know its limitations.

Lesson 3 Solving Quadratic Equations by Reasoning

- I can find solutions to quadratic equations by reasoning about the values that make the equation true.
- I know that quadratic equations may have two solutions.

Lesson 4 Solving Quadratic Equations with the Zero Product Property

- I can explain the meaning of the “zero product property.”
- I can find solutions to quadratic equations when one side is a product of factors and the other side is zero.

Lesson 5 How Many Solutions?

- I can explain why dividing by a variable to solve a quadratic equation is not a good strategy.
- I know that quadratic equations can have no solutions and can explain why when there are none.

Lesson 6 Rewriting Quadratic Expressions in Factored Form (Part 1)

- I can explain how the numbers in a quadratic expression in factored form relate to the numbers in an equivalent expression in standard form.
- When given quadratic expressions in factored form, I can rewrite them in standard form.
- When given quadratic expressions in the form of $x^2 + bx + c$, I can rewrite them in factored form.

Lesson 7 Rewriting Quadratic Expressions in Factored Form (Part 2)

- I can explain how the numbers and signs in a quadratic expression in factored form relate to the numbers and signs in an equivalent expression in standard form.
- When given a quadratic expression given in standard form with a negative constant term, I

can write an equivalent expression in factored form.

Lesson 8 Rewriting Quadratic Expressions in Factored Form (Part 3)

- I can explain why multiplying a sum and a difference, $(x+m)(x-m)$, results in a quadratic expression with no linear term.
- When given quadratic expressions in the form of x^2+bx+c, I can rewrite them in factored form.

Lesson 9 Solving Quadratic Equations by Using Factored Form

- I can rearrange a quadratic equation to be written as an expression in factored form equal to zero and find the solutions.
- I can recognize the number of solutions for a quadratic equation from the factored form.

Lesson 10 Rewriting Quadratic Expressions in Factored Form (Part 4)

- I can use the factored form of a quadratic expression or a graph of a quadratic function to answer questions about a situation.
- When given quadratic expressions of the form ax^2+bx+c, where a is not 1, I can write equivalent expressions in factored form.

Lesson 11 What Are Perfect Squares?

- I can recognize perfect-square expressions written in different forms.
- I can recognize quadratic equations that have a perfect-square expression and solve the equations.

Lesson 12 Completing the Square (Part 1)

- I can explain what it means to "complete the square" and describe how to do it.
- I can solve quadratic equations by completing the square and finding square roots.

Lesson 13 Completing the Square (Part 2)

- When given a quadratic equation in which the coefficient of the squared term is 1, I can solve it by completing the square.

Lesson 14 Completing the Square (Part 3)

- I can complete the square for quadratic expressions of the form ax^2+bx+c, where a is not 1, and explain the process.
- I can solve quadratic equations in which the squared term coefficient is not 1 by completing the square.

Lesson 15 Quadratic Equations with Irrational Solutions

- I can use the radical and “plus-minus” symbols to represent solutions to quadratic equations.
- I know why the plus-minus symbol is used when solving quadratic equations by finding square roots.

Lesson 16 The Quadratic Formula

- I can use the quadratic formula to solve quadratic equations.
- I know some methods for solving quadratic equations can be more convenient than others.

Lesson 17 Applying the Quadratic Formula (Part 1)

- I can use the quadratic formula to solve an equation and interpret the solutions in terms of a situation.

Lesson 18 Applying the Quadratic Formula (Part 2)

- I can identify common errors when using the quadratic formula.
- I know some ways to tell if a number is a solution to a quadratic equation.

Lesson 19 Deriving the Quadratic Formula

- I can explain the steps and complete some missing steps for deriving the quadratic formula.
- I know how the quadratic formula is related to the process of completing the square for a quadratic equation $ax^2 + bx + c = 0$.

Lesson 20 Rational and Irrational Solutions

- I can explain why adding a rational number and an irrational number produces an irrational number.
- I can explain why multiplying a rational number (except 0) and an irrational number produces an irrational number.
- I can explain why the sum or product of two rational numbers is rational.

Lesson 21 Sums and Products of Rational and Irrational Numbers

- I can explain why adding a rational number and an irrational number produces an irrational number.
- I can explain why multiplying a rational number (except 0) and an irrational number produces an irrational number.
- I can explain why the sum or product of two rational numbers is rational.

Lesson 22 Rewriting Quadratic Expressions in Vertex Form

- I can identify the vertex of the graph of a quadratic function when the expression that defines it is written in vertex form.

- I know the meaning of the term "vertex form" and can recognize examples of quadratic expressions written in this form.
- When given a quadratic expression in standard form, I can rewrite it in vertex form.

Lesson 23 Using Quadratic Expressions in Vertex Form to Solve Problems

- I can find the maximum or minimum of a function by writing, in vertex form, the quadratic expression that defines it.
- When given a quadratic function in vertex form, I can explain why the vertex is a maximum or minimum.

Lesson 24 Using Quadratic Equations to Model Situations and Solve Problems

- I can interpret information about a quadratic function given its equation or a graph.
- I can rewrite quadratic functions in different but equivalent forms of my choosing and use that form to solve problems.
- In situations modeled by quadratic functions, I can decide which form to use depending on the questions being asked.

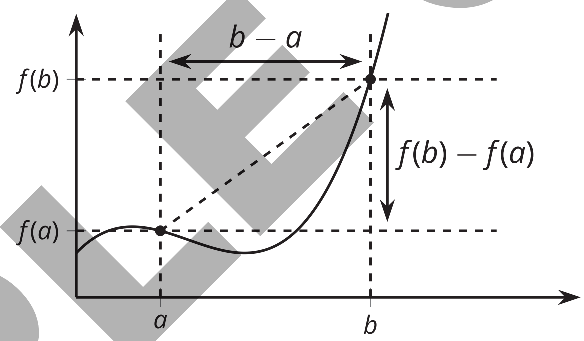
Glossary

- absolute value
  The absolute value of a number is its distance from 0 on the number line.

- association
  There is an association between two variables if they are statistically related to each other. This means that the value of one variable can be used to estimate the value of the other. An association can apply to categorical data or numerical data.

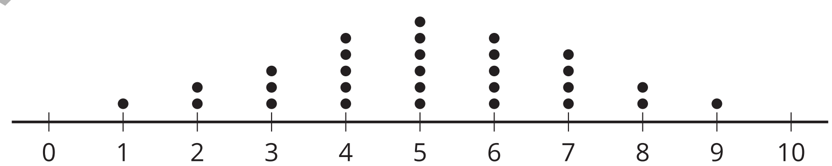
- average rate of change
  The average rate of change of a function is a ratio that describes how fast one quantity changes with respect to another.

  The average rate of change for function f between inputs a and b is the change in the outputs divided by the change in the inputs: $\frac{f(b)-f(a)}{b-a}$. It is the slope of the line that connects $(a, f(a))$ and $(b, f(b))$ on the graph.



- bell-shaped distribution
  In a bell-shaped distribution, most of the data cluster near the center and fewer points are farther from the center. The dot plot or histogram for the data has the form of a bell.

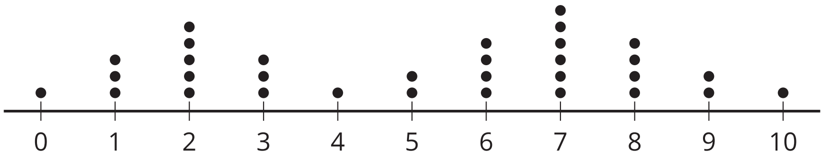
  This dot plot shows a bell-shaped distribution.



- bimodal distribution
  In a bimodal distribution, there are two very common data values. The dot plot or histogram

for the data has two distinct peaks.

This dot plot shows a bimodal distribution. The two common data values are 2 and 7.



- categorical data
  Categorical data are data where the values are divided into groups, or categories.

  For example, the breeds of 10 different dogs are categorical data. Another example is the colors of 100 different flowers.

- categorical variable
  A categorical variable is a variable that takes on values that are divided into groups, or categories.

  For example, color is a categorical variable that can take on the values, red, blue, green, and so on.

- causal relationship
  In a causal relationship, a change in one of the variables causes a change in the other variable.

- completing the square
  Completing the square is a method of rewriting a quadratic expression or equation.

  - A quadratic expression is rewritten in the form $a(x+p)^2 - q$, where a, p, and q are constants and $a \neq 0$.
  - A quadratic equation is rewritten in the form $a(x+p)^2 = q$.

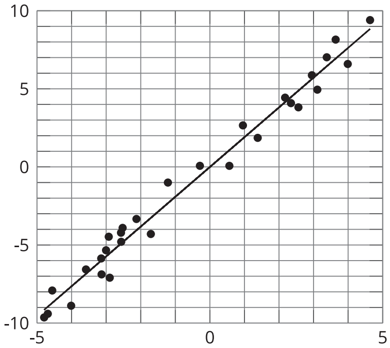
- constraint
  A constraint is a limitation on the possible values of variables in a model. It is often expressed by an equation or inequality or by specifying that the value must be an integer.

  For example, distance above the ground d, in meters, might be constrained to be non-negative, expressed by $d \geq 0$.

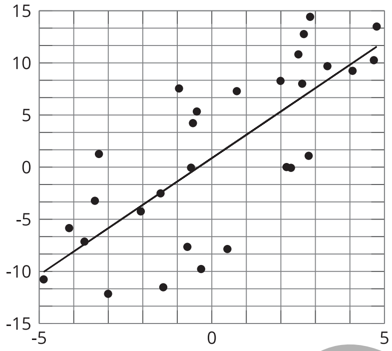
- correlation coefficient
  A correlation coefficient is a number between -1 and 1 that describes the strength and

direction of a linear relationship between two numerical variables.

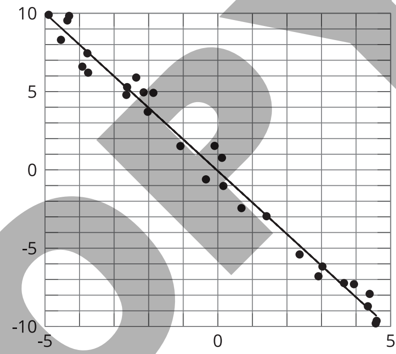
- The sign of the correlation coefficient is the same as the sign of the slope of the best-fit line.
- The closer the correlation coefficient is to 0, the weaker the linear relationship.
- When the correlation coefficient is closer to 1 or -1, the linear model fits the data better.



Correlation coefficient is close to 1.



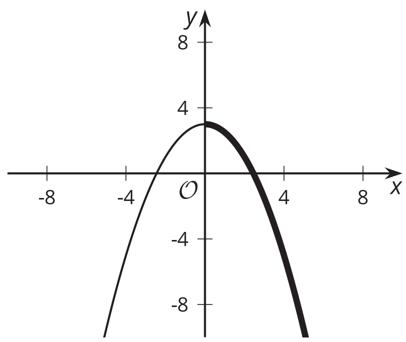
Correlation coefficient is positive, and closer to 0.



Correlation coefficient is close to -1.

- decreasing (function)
  A function is decreasing if its outputs get smaller as the inputs get larger. This results in a downward sloping graph as it goes from left to right. A function can also be decreasing just for a restricted range of inputs.

  This graph shows the function f given by $f(x) = 3 - x^2$. It is decreasing for $x \geq 0$ because the graph slopes downward to the right of the vertical axis.



- dependent variable
  A dependent variable is a variable that represents the output of a function.

  For example, the equation $y = 6 - x$ defines y as a function of x.

  - The variable x is called the *independent variable* because its value is chosen.
  - The variable y is called the *dependent variable* because it depends on x. Once a value is chosen for x, the value of y is determined.

- distribution
  The distribution of a data set tells how many times each value occurs.

- domain
  The domain of a function is the set of all of its possible input values.

- elimination
  Elimination is a method of solving a system of two equations in two variables. A multiple of one equation is added to or subtracted from another to get an equation with only one of the variables. (The other variable is eliminated.)

- equivalent equations
  Equivalent equations are equations that have the exact same solutions.

- equivalent systems
  Equivalent systems are systems that share the exact same solution set.

- exponential function
  An exponential function is a function that has a constant growth factor. This means that it grows by equal factors over equal intervals.

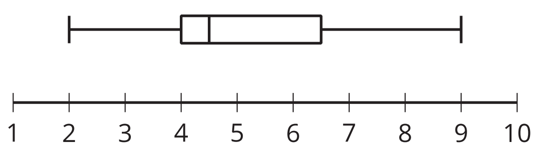
  For example, $f(x) = 2 \cdot 3^x$ defines an exponential function. Any time x increases by 1, $f(x)$ increases by a factor of 3.

- factored form (of a quadratic expression)
  A quadratic expression is in factored form when it is written as the product of a constant times two linear factors.

  - $2x^2 + 4x - 6$ written in factored form is $2(x - 1)(x + 3)$.
  - $15x^2 + x - 2$ written in factored form is $(5x + 2)(3x - 1)$.

- five-number summary
  The five-number summary is one way to describe the distribution of a data set. The five numbers are the minimum, the three quartiles, and the maximum.

  This box plot represents a data set with the following five-number summary: The minimum is 2, the three quartiles are 4, 4.5, and 6.5, and the maximum is 9.



- function

A function is a rule that takes inputs from one set and assigns them to outputs from another set. Each input is assigned exactly one output.

- function notation
  Function notation is a way of writing the relationship between the inputs and outputs of a function.

  For example, a function is named f and x is an input. Then $f(x)$ denotes the corresponding output in function notation.

- growth factor
  In an exponential function, the output is multiplied by the same factor every time the input increases by 1. This multiplier is called the *growth factor*.

- growth rate
  In an exponential function, the growth rate is the fraction or percentage of the output that gets added every time the input is increased by 1.

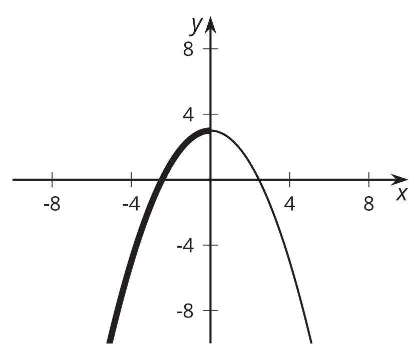
  For example, if the growth rate is 20%, or 0.2, then the growth factor is 1.2.

- horizontal intercept
  A horizontal intercept of a graph is a point where the graph crosses the horizontal axis. If the axis is labeled with the variable x, a horizontal intercept is also called an x-intercept. The term can also refer to only the x-coordinate of the point where the graph crosses the horizontal axis.

  For example, the horizontal intercept of the graph of $2x + 4y = 12$ is $(6, 0)$, or just 6.

- increasing (function)
  A function is increasing if its outputs get larger as the inputs get larger. This results in an upward sloping graph as it goes from left to right. A function can also be increasing just for a restricted range of inputs.

  This graph shows the function f given by $f(x) = 3 - x^2$. It is increasing for $x \leq 0$ because the graph slopes upward to the left of the vertical axis.



- independent variable
  An independent variable is a variable that represents the input of a function.

For example, the equation $y = 6 - x$ defines y as a function of x.

- The variable x is called the *independent variable* because its value is chosen.
- The variable y is called the *dependent variable* because it depends on x. Once a value is chosen for x, the value of y is determined.

- inverse (function)
  Two functions are inverses to each other if their input-output pairs are reversed.

  - If one function takes a as input and gives b as an output, then the other function takes b as an input and gives a as an output.
  - An inverse function can sometimes be found by reversing the processes that define the first function in order to define the second function.

  For example, in the function $w = 52y$, the input is y, the number of years, and the output is w, the number of weeks. The inverse function, $y = \frac{w}{52}$, is the result of reversing the process of multiplying by 52. For this function, the input is w, the number of weeks, and the output is y, the number of years.

- linear function
  A linear function is a function that has a constant rate of change. This means that it grows by equal differences over equal intervals.

  For example, $f(x) = 4x - 3$ defines a linear function. Any time x increases by 1, $f(x)$ increases by 4.

- linear term
  A linear term of an expression has a variable raised to the first power.

  - In the expression, $5x + 2$, the linear term is $5x$.
  - In the expression, $20 - x^2 + x$, the linear term is x.
  - In the expression $ax^2 + bx + c$, where a, b, and c are constants, the linear term is bx.

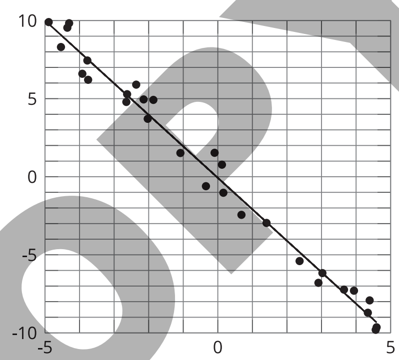
- maximum (of a function)
  A maximum of a function is a value of the function that is greater than or equal to all the other values. The maximum of the function's graph is the highest point on the graph.

- minimum (of a function)
  A minimum of a function is a value of the function that is less than or equal to all the other values. The minimum of the function's graph is the lowest point on the graph.

- model
  A model is a mathematical or statistical representation of information from science, technology, engineering, work, or everyday life, that is used to understand the situation and make decisions.

- negative relationship
  Two numerical variables have a negative relationship if an increase in the data for one variable tends to be paired with a decrease in the data for the other variable.

  This scatter plot shows a negative relationship.



- non-statistical question
  A non-statistical question is a question that can be answered by a specific measurement or procedure where no variability is expected.

  For example:

  - How high is that building?
  - If I run at 2 meters per second, how long will it take me to run 100 meters?
  - Who was the first woman appointed to the Supreme Court?

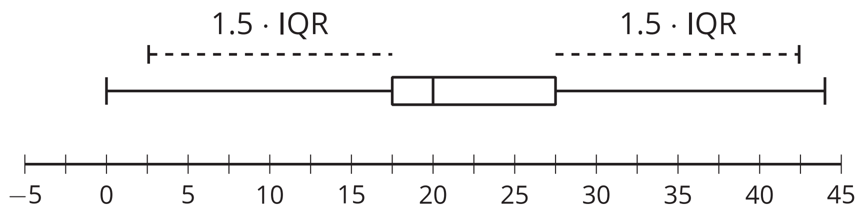
- numerical data
  Numerical data are data where the values are numbers, measurements, or quantities. Numerical data is also called *measurement data* or *quantitative data*.

  For example, the weights of 10 different dogs are numerical data.

- outlier
  An outlier is a data value that is far from the other values in the data set. A value is considered an outlier if it is:

  - More than 1.5 times the interquartile range greater than Q3.
  - More than 1.5 times the interquartile range less than Q1.

  In this box plot, the minimum, 0, and the maximum, 44, are both outliers.

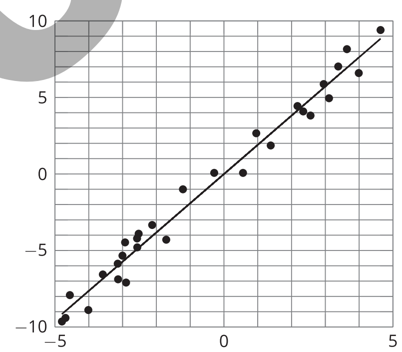


- perfect square
  A perfect square is a number or an expression that is the result of multiplying a number or an expression to itself. In general, the multiplied number is rational and the multiplied expression has rational coefficients.

- piecewise function
  A piecewise function is a function defined using different expressions for different intervals in its domain.

- positive relationship
  Two numerical variables have a positive relationship if an increase in the data for one variable tends to be paired with an increase in the data for the other variable.

  This scatter plot shows a positive relationship.



- quadratic equation
  A quadratic equation is an equation that is equivalent to one of the form $ax^2 + bx + c = 0$, where a, b, and c are constants and $a \neq 0$.

- quadratic expression
  A quadratic expression is an expression that is equivalent to one of the form $ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$.

- quadratic formula
  The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and gives the solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are constants and $a \neq 0$.

- quadratic function

A quadratic function is a function where the output is given by a quadratic expression in the input.

For example, $f(x) = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$, is a quadratic function.

- range (of a function)
  The range of a function is the set of all of its possible output values.

- relative frequency table
  A relative frequency table is a version of a two-way table that shows how often data values occur in relation to a total. Each entry in the table shows the frequency of one response divided by the total number of responses in the entire table or by the total number of responses in a row or a column.

  Each entry in this relative frequency table represents the proportion of all the textbooks that have the characteristics given by its row and column. For example, out of all 1,000 textbooks, the proportion of textbooks that are new and \$10 or less is 0.025, or 2.5%.

  frequency table

	\$10 or less	more than \$10 but less than \$30	\$30 or more	total
new	25	75	225	325
used	275	300	100	675
total	300	375	325	1,000

  relative frequency table

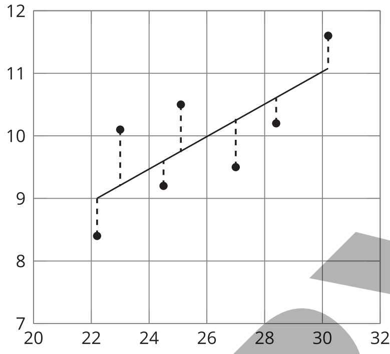
	\$10 or less	more than \$10 but less than \$30	\$30 or more
new	$0.025 = \frac{25}{1000}$	0.300	0.225
used	0.275	0.300	0.100

- residual

A residual is the difference between an actual data value and its value predicted by a model. It can be found by subtracting the y-value predicted by the linear model from the y-value for the data point.

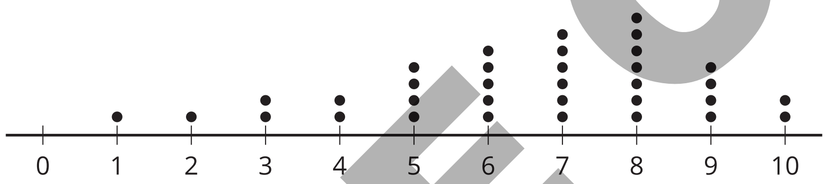
On a scatter plot, the residual can be seen as the vertical distance between a data point and the best-fit line.

The lengths of the dashed segments on this scatter plot show the residuals for each data point.



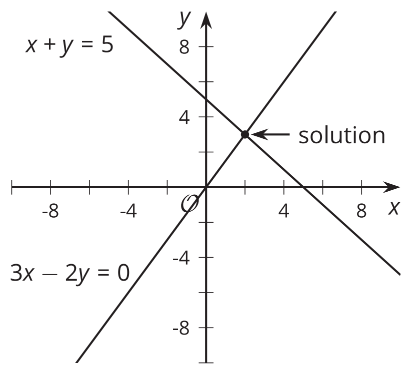
- skewed distribution
  In a skewed distribution, one side has more values farther from the bulk of the data than the other side. The mean is usually not equal to the median. The dot plot or histogram for the data shows only one peak leaning to one side.

  This dot plot shows a skewed distribution. The data values on the left, such as 1, 2, and 3, are farther from the bulk of the data than the data values on the right.



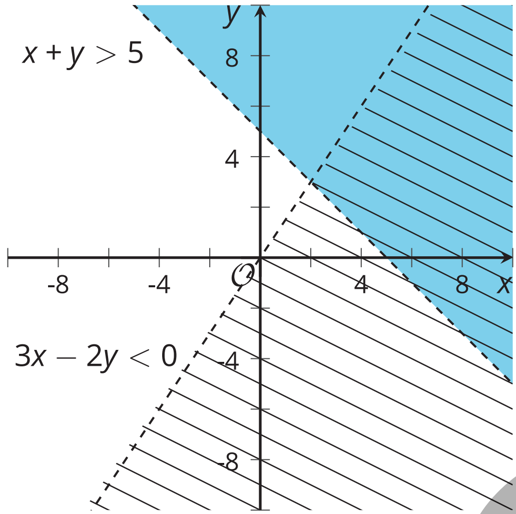
- solution to a system of equations
  A solution to a system of equations is the values for the variables that make all the equations true.

  This graph shows a system of two equations. The solution of the system is a coordinate pair that makes both equations true. On the graph, the solution is shown as the point where the two lines intersect.



- solutions to a system of inequalities
  The solutions to a system of inequalities are all the values for the variables that make all of the inequalities true.

  This graph shows a system of two inequalities. The solutions of the system are all the coordinate pairs that make both inequalities true. On the graph, the solution is shown as all the points in the region where the graphs of the two inequalities overlap.



- standard deviation
  The standard deviation is a measure of the variability, or spread, of a distribution. It is calculated by a method similar to the method for calculating the MAD (mean absolute deviation). The exact method is studied in more advanced courses.

- standard form (of a quadratic expression)
  The standard form of a quadratic expression is $ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$.

- statistic
  A statistic is a quantity that is calculated from sample data, such as mean, median, or MAD (mean absolute deviation).

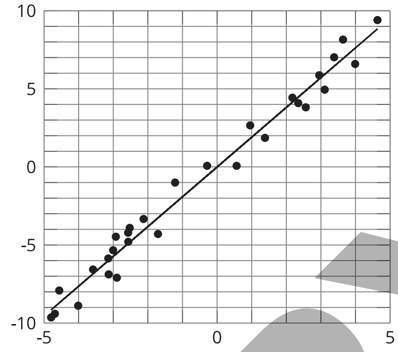
- statistical question
  A statistical question is a question that can only be answered by using data in which variability is expected.

  For example:

  - Who is the most popular musical artist at our school?
  - When do students in our class typically eat dinner?
  - Which classroom in our school has the most books?

- strong relationship

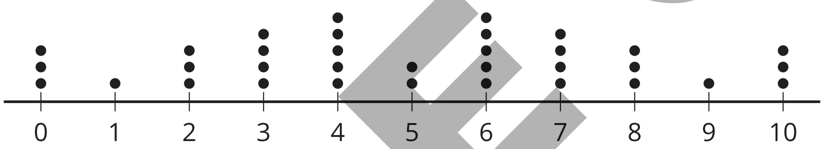
Two numerical variables have a strong relationship if the data is tightly clustered around the best-fit line.



- substitution
  Substitution is the action of replacing a variable with a number or expression it is equal to.

- symmetric distribution
  In a symmetric distribution, the data values on each side of the center mirror each other. The dot plot or histogram for the data has a vertical line of symmetry in the center, where the mean is equal to the median.

  This dot plot shows a distribution that is symmetric about the data value 5.



- system of equations
  Two or more equations that represent the constraints in the same situation form a system of equations.

- system of inequalities
  Two or more inequalities that represent the constraints in the same situation form a system of inequalities.

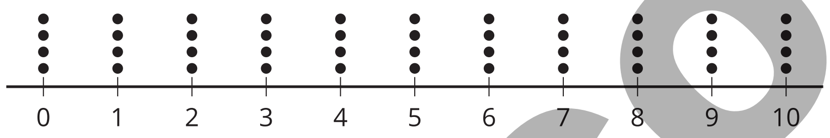
- two-way table
  A two-way table is a way of organizing data from two categorical variables in order to investigate the association between them.

  This two-way table can be used to study the relationship between age group and cell phone ownership.

	has a cell phone	does not have a cell phone
10–12 years old	25	35
13–15 years old	38	12
16–18 years old	52	8

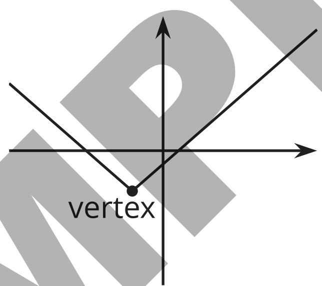
- uniform distribution
  In a uniform distribution, the data values are evenly spread out across the range of the data. The dot plot or histogram for the data shows no peaks.

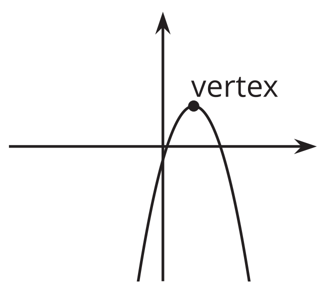
  This dot plot shows a uniform distribution.



- variable (statistics)
  A variable is a characteristic of individuals in a population that can take on different values.

- vertex (of a graph)
  The vertex of the graph of a quadratic function or of an absolute value function is the point where the graph changes from increasing to decreasing, or vice versa. It is the highest or lowest point on the graph.



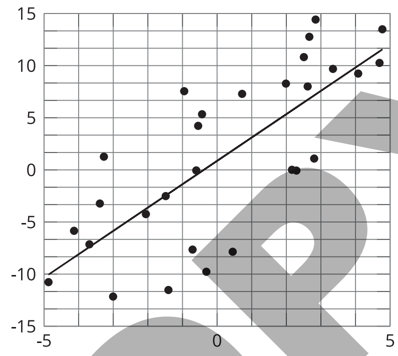


- vertex form (of a quadratic expression)
  The vertex form of a quadratic expression is $a(x-h)^2+k$, where a, h, and k are constants and $a \neq 0$. The vertex of the graph is at the point (h, k).

- vertical intercept
  A vertical intercept of a graph is a point where the graph crosses the vertical axis. If the axis is labeled with the variable y, a vertical intercept is also called a y-intercept. The term can also refer to only the y-coordinate of the point where the graph crosses the vertical axis.

For example, the vertical intercept of the graph of $y = 3x - 5$ is $(0, \text{-}5)$, or just -5.

- weak relationship
  Two numerical variables have a weak relationship if the data is loosely spread around the best-fit line.



- zero (of a function)
  A zero of a function is an input that results in an output of 0. In other words, if $f(a) = 0$, then a is a zero of f.

- zero product property
  The zero product property says that if the product of two numbers is 0, then one of the numbers must be 0.

Attributions

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California Common Core State Standards for Mathematics (CA CCSSM) References

Number and Quantity

The Real Number System (N-RN)

Extend the properties of exponents to rational exponents.

N-RN.1

Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*

N-RN.2

Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

N-RN.3

Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Quantities (N-Q)

Reason quantitatively and use units to solve problems. [Foundation for work with expressions, equations and functions]

N-Q.1

Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.*

N-Q.2

Define appropriate quantities for the purpose of descriptive modeling.*

N-Q.3

Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.*

Algebra

Seeing Structure in Expressions (A-SSE)

Interpret the structure of expressions. [Linear, exponential, and quadratic]

A-SEE.1

Interpret expressions that represent a quantity in terms of its context.*

A-SSE.1a

Interpret parts of an expression, such as terms, factors, and coefficients.*

Note: * Indicates a modeling standard linking mathematics to everyday life, work, and decision-making.

A-SSE.1b

Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret *$P(1 + r)^n$ as the product of P and a factor not depending on P.**

A-SSE.2

Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems. [Quadratic and exponential]

A-SSE.3

Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

A-SSE.3a

Factor a quadratic expression to reveal the zeros of the function it defines.*

A-SSE.3b

Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.*

A-SSE.3c

Use the properties of exponents to transform expressions for exponential functions. *For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.**

Arithmetic with Polynomials and Rational Expressions (A-APR)

Perform arithmetic operations on polynomials. [Linear and quadratic]

A-APR.1

Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Creating Equations (A-CED)

Create equations that describe numbers or relationships. [Linear, quadratic, and exponential (integer inputs only); for A-CED.3 linear only]

A-CED.1

Create equations and inequalities in one variable including ones with absolute value and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.**

A-CED.2

Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.*

A-CED.3

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.**

A-CED.4

Note: * Indicates a modeling standard linking mathematics to everyday life, work, and decision-making.

Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.**

Reasoning with Equations and Inequalities (A-REI)

Understand solving equations as a process of reasoning and explain the reasoning. [Master linear; learn as general principle.]

A-REI.1

Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable. [Linear inequalities; literal equations that are linear in the variables being solved for; quadratics with real solutions]

A-REI.3

Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

A-REI.3.1

Solve one-variable equations and inequalities involving absolute value, graphing the solutions and interpreting them in context.

A-REI.4

Solve quadratic equations in one variable.

A-REI.4a

Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

A-REI.4b

Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations. [Linear-linear and linear-quadratic]

A-REI.5

Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

A-REI.6

Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

A-REI.7

Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

Represent and solve equations and inequalities graphically. [Linear and exponential; learn as general principle.]

A-REI.10

Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

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A-REI.11

Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

A-REI.12

Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Functions

Interpreting Functions (F-IF)

Understand the concept of a function and use function notation. [Learn as general principle; focus on linear and exponential and on arithmetic and geometric sequences.]

F-IF.1

Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

F-IF.2

Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

F-IF.3

Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n + 1) = f(n) + f(n - 1)$ *for* $n \geq 1$.

Interpret functions that arise in applications in terms of the context. [Linear, exponential, and quadratic]

F-IF.4

For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.**

F-IF.5

Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.**

F-IF.6

Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*

Note: * Indicates a modeling standard linking mathematics to everyday life, work, and decision-making.

Analyze functions using different representations. [Linear, exponential, quadratic, absolute value, step, piecewise-defined]

F-IF.7

Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

F-IF.7a

Graph linear and quadratic functions and show intercepts, maxima, and minima.*

F-IF.7b

Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.*

F-IF.7e

Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.*

F-IF.8

Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

F-IF.8a

Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

F-IF.8b

Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, *and* $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.*

F-IF.9

Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Building Functions (F-BF)

Build a function that models a relationship between two quantities. [For F.BF.1, 2, linear, exponential, and quadratic]

F-BF.1

Write a function that describes a relationship between two quantities.*

F-BF.1a

Determine an explicit expression, a recursive process, or steps for calculation from a context.*

F-BF.1b

Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.**

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F-BF.2

Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.*

Build new functions from existing functions. [Linear, exponential, quadratic, and absolute value; for F-BF.4a, linear only]

F-BF.3

Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

F-BF.4

Find inverse functions.

F-BF.4a

Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

Linear, Quadratic, and Exponential Models (F-LE)

Construct and compare linear, quadratic, and exponential models and solve problems.

F-LE.1

Distinguish between situations that can be modeled with linear functions and with exponential functions.*

F-LE.1a

Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.*

F-LE.1b

Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.*

F-LE.1c

Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.*

F-LE.2

Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).*

F-LE.3

Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.*

Interpret expressions for functions in terms of the situation they model.

F-LE.5

Interpret the parameters in a linear or exponential function in terms of a context.* [Linear and exponential of form $f(x) = bx + k$]

Note: * Indicates a modeling standard linking mathematics to everyday life, work, and decision-making.

F-LE.6
Apply quadratic functions to physical problems, such as the motion of an object under the force of gravity. CA*

Statistics and Probability

Interpreting Categorical and Quantitative Data (S-ID)

Summarize, represent, and interpret data on a single count or measurement variable.

S-ID.1
Represent data with plots on the real number line (dot plots, histograms, and box plots).*

S-ID.2
Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.*

S-ID.3
Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).*

Summarize, represent, and interpret data on two categorical and quantitative variables. [Linear focus; discuss general principle.]

S-ID.5
Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.*

S-ID.6
Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.*

S-ID.6a
Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.**

S-ID.6b
Informally assess the fit of a function by plotting and analyzing residuals.*

S-ID.6c
Fit a linear function for a scatter plot that suggests a linear association.*

Interpret linear models.

S-ID.7
Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.*

S-ID.8
Compute (using technology) and interpret the correlation coefficient of a linear fit.*

S-ID.9
Distinguish between correlation and causation.*

Note: * Indicates a modeling standard linking mathematics to everyday life, work, and decision-making.

California Common Core State Standards for Mathematics Standards for Mathematical Practice

These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

MP1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

MP3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

- Students build proofs by induction and proofs by contradiction. CA 3.1 (for higher mathematics only).

MP4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

MP5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

MP6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

MP7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

MP8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Mathematical Practices to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.