LESSON 7.4

Determining Rate of Change

You are in charge of ordering T-shirts for your club. The shirt company provides you with the price list below.

Your club only has 28 members, so you don't want to order 20 or 30 shirts. You only want to order 28 shirts. But you are not sure what that would cost.

You could look at the cost per shirt, but that cost is different at each point in the price list provided. For instance, if you order 20 shirts, the cost is $\frac{\$290}{20} = \$14.50$ for each shirt. What is the cost per shirt if you order 30 shirts?

What is more useful is the rate at which the cost of an order of shirts is increasing. If this were a linear function, then that rate would be called the slope. For a line the slope, or **rate of change,** is the same everywhere. Since this is not a linear function, the rate of change is not constant but instead depends on the interval you are looking at. So the rate of change is found by using the **average rate of change.**

Consider the interval from 20 to 30 shirts. The rate of change for the interval is $\frac{325 - 290}{30 - 20} = \frac{35}{10} = \$3.50$ per shirt. This is not the cost of the shirt; it is the rate at which the total cost is changing. Another way to think about it is that each additional shirt ordered between the total of 20 and 30 shirts will cost \$3.50. Based on this rate, we can estimate the cost of ordering 28 shirts as 8 times \$3.50, or \$28, more than the cost of 20 shirts. So the cost of ordering 28 shirts is about $\$290 + \$28 = \$318$.

How would you estimate the cost of ordering 52 shirts?

Any continuous function will have an average rate of change that can help you understand and describe the behavior of that function. In the next example you will see that increasing the diameter of a rug rapidly changes the time it takes to make the rug for larger rugs.

EXAMPLE A

The time t, in hours, it takes the Round Rug Company to make a braided rug is equal to 7 times the square of the diameter of the rug d, in feet ($t = 7d^2$). Find the average rate of change between a rug with a 4-foot diameter and a rug with a 6-foot diameter, and explain what the value tells you. Then do the same for a rug with an 8-foot diameter and a rug with a 10-foot diameter.

Solution

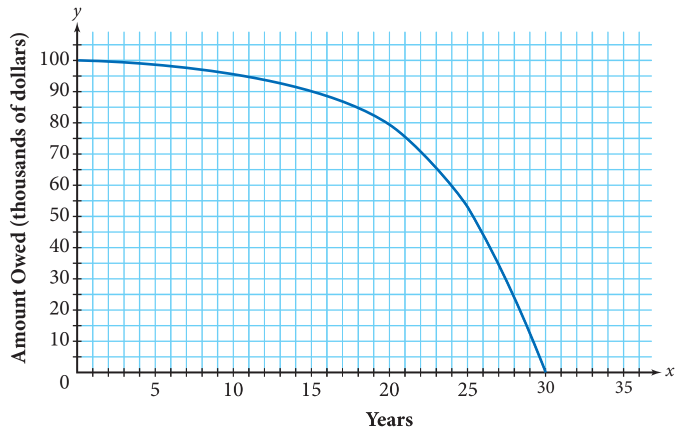
A 4-foot-diameter rug takes $7(4)^2$ or 112 hours, while a 6-foot-diameter rug takes $7(6)^2$ or 252 hours. So the average rate of change is $\frac{252 - 112}{6 - 4} = 70$ hours per foot. The total time is increased an average of 70 hours per foot if you increase the diameter of a rug between 4 and 6 feet by an extra foot.

The average rate of change between 8-foot- and 10-foot-diameter rugs is $\frac{7(10)^2 - 7(8)^2}{10 - 8} = \frac{700 - 448}{2} = \frac{252}{2} = 126$ hours per foot. Adding an extra foot to an 8- to 10-foot-diameter rug could add about 126 hours to the project.

Another way to describe this average rate of change by using graphs is to give the slope of the **secant line.** This is a line that passes through any two points of your graph. You can visually see how the average rate of change depends on the interval you are interested in.

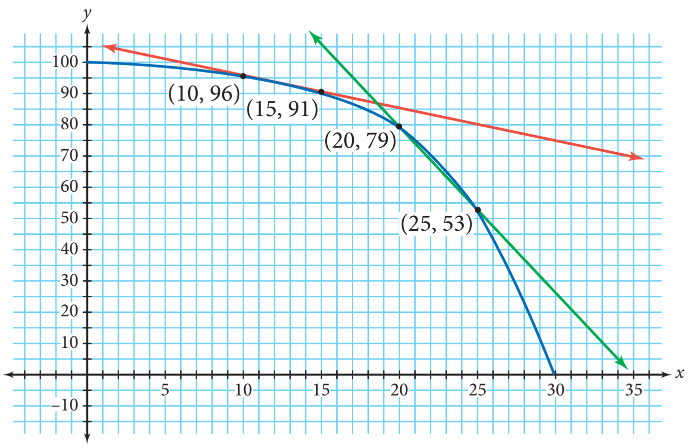
EXAMPLE B

The amount you owe on a \$100,000 home loan for a 30-year mortgage is shown in the graph. The graph shows that after 20 years you still owe about \$80,000. Estimate and compare the average rate of change between years 10 and 15 with the average rate of change between years 20 and 25.



Solution

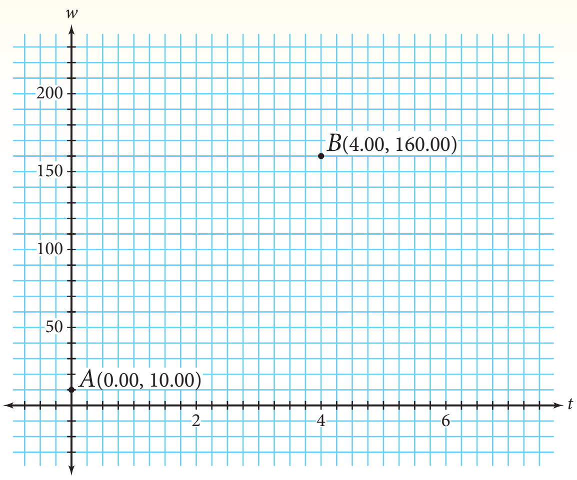
At year 10 you owe about \$96,000, which drops to about \$91,000 by year 15. So this average rate of change is $\frac{91{,}000 - 96{,}000}{15 - 10} = \frac{-5{,}000}{5} = -\$1{,}000$ per year. But in the interval between years 20 and 25 the rate is $\frac{53{,}000 - 79{,}000}{25 - 20} = \frac{-26{,}000}{5} = -\$5{,}200$ per year. So in this interval your mortgage is dropping at more than five times the rate it was in the other interval.



INVESTIGATION

Rate of Change

Over a four-week period, the number of weeds in Angelica's garden increased from 10 to 160. The average rate of change for the weeds during the four weeks is the slope between the two points (0, 10) and (4, 160).



Here are three different models that fit these two data points:

$$w_1 = 10 + 37.5t$$
$$w_2 = 10(2^t)$$
$$w_3 = 10 + 75\sqrt{t}$$

Step 1 Calculate the number of weeds each week for each model, and record your answers in a table.

Week

Model	0	1	2	3	4
$w_1 = 10 + 37.5t$	10	47.5			

Step 2 Find the average rate of change in the number of weeds from each week to the next for each model. Record your answers in a table.

Average Rate of Change

Model	Week 0 to Week 1	Week 1 to Week 2	Week 2 to Week 3	Week 3 to Week 4
$w_1 = 10 + 37.5t$	37.5			

Step 3 Sketch a graph of each model over the interval $t = 0$ to $t = 4$.

Step 4 Look at the numbers in your table, and look at the shapes of the graphs.

a. For the first model, the rate of change should have been the same for each week. How does this show up in the graph?

b. For the second model, how does the rate of change vary from one week to the next? Does it increase or decrease? How does this show up in the graph?

c. For the third model, how does the trend in the rate of change differ from the second model? How does the shape of the graph show this difference?

Step 5 Graph the model $w_4 = -4.6875x^3 + 28.125x^2 + 10$. By looking at the graph, describe how you think the rate of change will vary over the four-week period.

Step 6 Sketch the graph of a function that meets each description:

a. The average rate of change is increasing.

b. The average rate of change is decreasing.

7.4 Exercises

Practice Your Skills

1. Find the slope of the line that passes through each pair of points.

a. (2, 7), (11, 20) **b.** (−3, 5), (6, −4) **c.** (0, 11), (8, 11) @

2. The slope of a line is $\frac{2}{3}$. If the point (6, 1) is on the line, name another point on the same line.

3. Consider the function $y = x^2 - 4$.

a. Find the average rate of change on the interval from $x = 1$ to $x = 3$. @

b. Find the average rate of change on the interval from $x = 3$ to $x = 7$.

c. Write the equation of the secant line containing the points for $x = 1$ and $x = 3$. @

d. Write the equation of the secant line containing the points for $x = 3$ and $x = 7$.

4. Akshay planted a garden and measured the height of one of his bean plants every day for a week. The data he collected are in the table below.

Day	1	2	3	4	5	6	7
Height (cm)	3	4.2	5.3	6.1	7.2	7.8	8.4

a. What is the average rate of change from day 1 to day 7?

b. What is the average rate of change from day 2 to day 5?

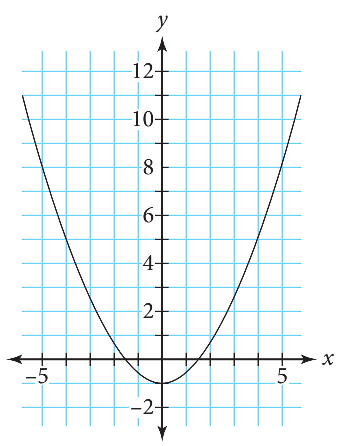
c. What is the average rate of change from day 4 to day 5? @

5. Use the graph to estimate values to answer the questions.

a. Estimate the average rate of change on the interval $0 \leq x \leq 4$. @

b. Estimate the average rate of change on the interval $-2 \leq x \leq 2$.

c. Estimate the average rate of change on the interval $-5 \leq x \leq 0$.



Reason and Apply

6. Consider the function $y = x^2 - 2$.

a. Graph this function, and draw the secant line between the points at $x = -3$ and $x = -1$. Then draw the secant line between the points at $x = 1$ and $x = 3$. ⓐ

b. Find the average rate of change for each of the secant lines in part a. How do they compare?

c. Find another pair of secant lines for this function that will have the same relationship.

d. What property of the function causes this relationship between the secant lines?

7. Some values for a continuous function are shown in the table.

x	1	3	4	8	11
y	23	26	28	37	56

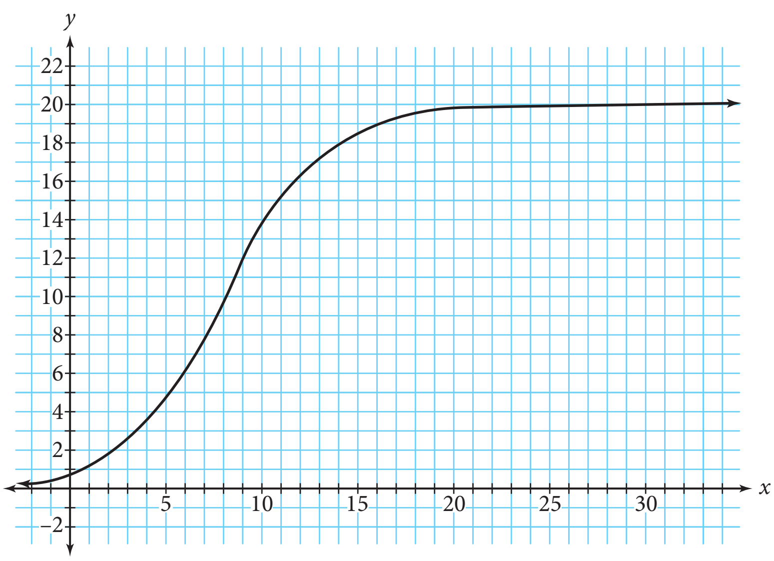
a. What is the average rate of change on the interval $1 \le x \le 3$?

b. What is the average rate of change on the interval $4 \le x \le 11$?

c. Find an interval where the average rate of change is 2.2.

d. If x is measured in ounces and y is measured in dollars, then what are the units of the average rate of change?

8. Given this graph of a continuous function, estimate the values requested.



a. What is the average rate of change on the interval $4 \le x \le 10$?

b. What is the average rate of change on the interval $26 \le x \le 30$?

c. Find an interval where the average rate of change is 1.5.

d. How do you know there is no interval where the average rate of change is -2? ⓗ

e. Where would you find the greatest average rate of change?

9. Suppose the temperature T of a pizza is given by $T = 200 - 140(0.9^t)$, where t is the number of minutes after the pizza is placed in the oven.

a. Find the average rate of change over the first 2 minutes.

b. What is the real-world meaning of this average rate of change?

Review

10. Complete this table of values for $g(x) = |x - 3|$ and $h(x) = (x - 3)^2$.

x	0	1	2	3	4	5	6
$g(x)$							
$h(x)$							

11. Use $f(x) = 2 + 3x$ to find

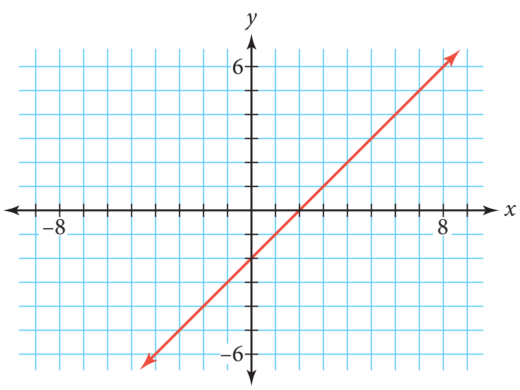
a. $f(5)$ **b.** x when $f(x) = -10$ @ **c.** $f(x + 2)$ **d.** $f(2x - 1)$ @

12. Solve each equation for x.

a. $5 = -3 + 2x$ **b.** $-4 = -8 + 3(x - 2)$ **c.** $7 + 2x = 3 + x$

13. Write an equation to describe each graph.

a.



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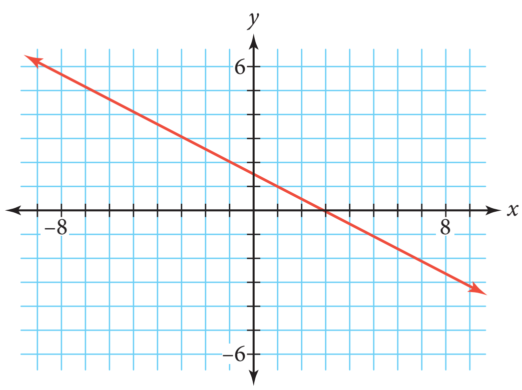
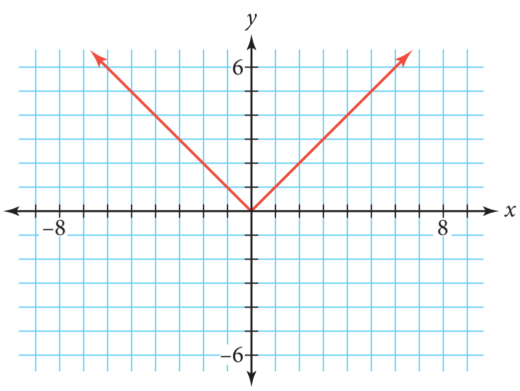


c.



d.