LESSON 1.1

This lesson reviews how arithmetic and geometric sequences can be defined recursively.

COMMON CORE STATE STANDARDS

APPLIED	DEVELOPED	INTRODUCED
	F.IF.3 F.BF.1a F.BF.2	F.LE.2

Objectives

- Discover recursive formulas for sequences
- Define, explore, and use arithmetic and geometric sequences
- Use recursively defined sequences to model real-life situations

Vocabulary

recursion
sequence
term
general term
recursive formula
arithmetic sequence
common difference
spreadsheet
fractal
geometric sequence
common ratio
Fibonacci sequence

Materials

- Calculator Notes: Reentry; Recursion; Making Spreadsheets Using the CellSheet App

Launch

What is the next term in each sequence?

a. 4, 8, 12, 16, 20, …
b. 1, 0.1, 0.01, 0.001, 0.0001, …

Describe in words how each sequence was generated.

a. 24. Answers will vary. Each term is 4 more than the previous term.
b. 0.00001. Answers will vary. Each term is $\frac{1}{10}$ of the previous term.

LESSON 1.1

Recursively Defined Sequences

For every pattern that appears, a mathematician feels he ought to know why it appears.
W. W. SAWYER

Look around! You are surrounded by patterns and influenced by how you perceive them. You have learned to recognize visual patterns in floor tiles, window panes, tree leaves, and flower petals. In every discipline, people discover, observe, re-create, explain, generalize, and use patterns. Artists and architects use patterns that are attractive or practical. Scientists and manufacturing engineers follow patterns and predictable processes that ensure quality, accuracy, and uniformity. Mathematicians frequently encounter patterns in numbers and shapes.

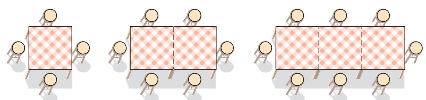
The arches in the Pershore Abbey in Worcestershire, United Kingdom, show an artistic use of repeated patterns.

Scientists use patterns and repetition to conduct experiments, gather data, and analyze results.

You can discover and explain many mathematical patterns by thinking about recursion. **Recursion** is a process in which each step of a pattern is dependent on the step or steps that come before it. It is often easy to define a pattern recursively, and a recursive definition reveals a lot about the properties of the pattern.

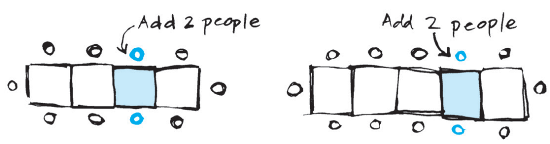
EXAMPLE A

A square table seats 4 people. Two square tables pushed together seat 6 people. Three tables pushed together seat 8 people. How many people can sit at 10 tables pushed together? How many tables are needed to seat 32 people? Write a recursive definition to find the number of people who can sit at any linear arrangement of square tables.



Solution

Sketch the arrangements of four tables and five tables. Notice that when you add another table, you seat two more people than in the previous arrangement.



ELL

It may help ELL students to develop a number sequence consisting of several terms so that the concepts of common difference and ratio become more evident. Students might be confused by the word *table* being used for both the table to sit at and the data table.

Support

As needed, assign the Refreshing Your Skills: Differences and Ratios lesson. Students could draw several steps of numbers of tables and carefully count the number of tables and students, entering the data into a table.

Advanced

Use the **One Step** version of the Investigation. This will aid in your assessment of students' prior skills with recursive problems. Focus on clear understanding of the specific notation of recursive rules.

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You can put this information into a table, and that reveals a clear pattern. You can continue the pattern to find that 10 tables seat 22 people, and 15 tables are needed for 32 people.

Tables	1	2	3	4	5	6	7	8	9	10	. . .	15
People	4	6	8	10	12	14	16	18	20	22	. . .	32

You can also organize the information like this:

number of people at 1 table = 4

number of people at 2 tables = number of people at 1 table + 2

number of people at 3 tables = number of people at 2 tables + 2

If you assume the same pattern continues, then

number of people at 10 tables = number of people at 9 tables + 2.

In general, the pattern is

number of people at n tables = number of people at $(n - 1)$ tables + 2.

This rule shows how to use recursion to find the number of people at any number of tables. In recursion, you use the previous value in the pattern to find the next value.

A **sequence** is an ordered list of numbers. The table in Example A represents the sequence

4, 6, 8, 10, 12, . . .

Each number in the sequence is called a **term.** The first term, u_1 (pronounced "u sub one"), is 4. The second term, u_2, is 6, and so on.

The nth term, u_n, is called the **general term** of the sequence. A **recursive formula,** the formula that defines a sequence, must specify one (or more) starting terms and a **recursive rule** that defines the nth term in relation to a previous term (or terms).

You generate the sequence 4, 6, 8, 10, 12, . . . with this recursive formula:

$u_1 = 4$

$u_n = u_{n-1} + 2 \quad \text{where } n \geq 2$

Because the starting value is $u_1 = 4$, the recursive rule $u_n = u_{n-1} + 2$ is first used to find u_2. This is clarified by saying that n must be greater than or equal to 2 to use the recursive rule.

This means *the first term is 4* and *each subsequent term is equal to the previous term plus 2.* Notice that each term, u_n, is defined in relation to the previous term, u_{n-1}. For example, the 10th term relies on the 9th term, or $u_{10} = u_9 + 2$.

Investigate

To introduce the lesson, ASK "What's a sequence?" Some students may recall the idea from a previous course. One synonym is *list*. Try to caution against using the word *series* in this context.

Example A

This example reviews the notion of recursion in the context of a sequence of numbers. Consider projecting Example A from your ebook and having students work in pairs. Have students present their strategy as well as their solution. Emphasize the use of correct mathematical terminology, asking probing questions to review the vocabulary of the lesson. For example, if students are using the words *next* and *previous* instead of u_n and u_{n-1}, to ease them into the subscript notation ASK "So that would be u sub what?" Whether student responses are correct or incorrect, ask other students if they agree and why. SMP 1, 3, 6

ASK "What's behind the pattern?" [As a new table is inserted, two new people sitting down, one on each side.] Mathematics isn't only about seeing patterns but also about explaining them. SMP 3, 6, 8

ALERT If students are having difficulties with subscript notation, build the connections among the first few terms. ASK "What is the next term after u_1?" [u_2] "What is the term right before u_5?" [u_4] "What is the next term after u_n?" [u_{n+1}] "What is the term before u_n?" [u_{n-1}] SMP 6, 8

ALERT Students might believe that the terms of a sequence must be in increasing or decreasing order. This is not necessarily true. Another misconception is that there is a simple pattern for each sequence.

ASK "Why is it necessary to include a restriction such as "where $n \geq 2$?" [The goal of the problem is to create a recursive rule that models the given sequence. Thus, in the context of this particular problem, it would not make sense to apply the rule when there are fewer than 2 tables.] SMP 1, 2, 6, 8

Modifying the Investigation

Whole Class Draw the table for Step 1 on the board. Then elicit student input and discussion to fill in the information. Discuss Steps 2 and 3. Note that to answer Step 2, you might continue the table to day 18, draw a graph, or use recursion in a spreadsheet or on a calculator. Elicit a variety of approaches.

Shortened Skip Step 2b.

One Step Have students read the investigation problem in the book or project the **One Step** found in your ebook. ASK "How many prints will have been delivered to the Little Print Shoppe when FineArt has received twice the number of prints that will still remain to be made?"

Example B

Consider projecting Example B from your ebook and having students work in groups of 2 – 4. As before, have students present their strategy as well as their solution, continuing to ask questions to emphasize the use of mathematically correct vocabulary. **ASK** "What are the differences between the terms *previous* and u_{n-1} (and, if students worked on calculators, the term *Ans*)?" [They all refer to the same thing.] Whether student responses are correct or incorrect, ask other students if they agree and why. **SMP 1, 3, 6**

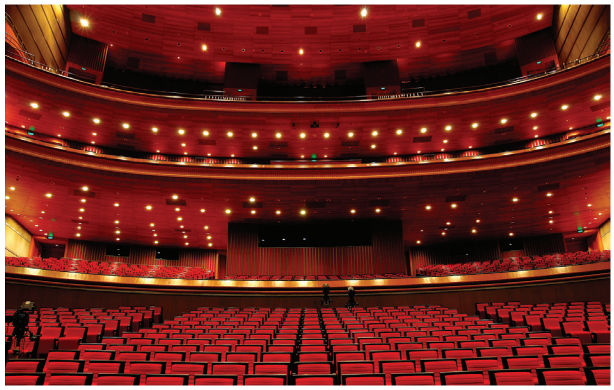
Another way to counter notation difficulties is to use home-screen recursion on graphing calculators. See Calculator Note: Recursion. Take some time to explain the connection between stating a starting value and the recursive rule on a graphing calculator— for example, [Ans + 2]. Relate u_{n-1} = (*previous answer*)(*Ans*) and u_1 = (*starting value*). If graphing calculators are new to your students, also give them copies of Calculator Notes: Getting Started, Reentry, and Making Spreadsheets Using the CellSheet App. **SMP 5, 6**

LANGUAGE Even though the term *arithmetic* in *arithmetic sequence* is spelled the same as the noun arithmetic, the adjective is pronounced "ar-ith-'me-tik."

ASK "Why aren't the data points connected with a line?" [You cannot have fractions of a seat.] **SMP 1, 2**

Before moving to the Investigation, ask students to write down their definition of an arithmetic sequence. After several students share their definitions, reach consensus as a class. Project the definition box. Discuss the differences in the definitions from the class and from the book. **SMP 3, 6**

EXAMPLE B A concert hall has 59 seats in Row 1, 63 seats in Row 2, 67 seats in Row 3, and so on. The concert hall has 35 rows of seats. Write a recursive formula to find the number of seats in each row. How many seats are in Row 4? Which row has 95 seats?



China National Grand Theater (Beijing)

Solution First, it helps to organize the information in a table.

Row	1	2	3	4	. . .
Seats	59	63	67		. . .

Every recursive formula requires a starting term. Here the starting term is 59, the number of seats in Row 1. That is, $u_1 = 59$.

This sequence also appears to have a common difference between successive terms: 63 is 4 more than 59, and 67 is 4 more than 63. Use this information to write the recursive rule for the *n*th term, $u_n = u_{n-1} + 4$.

Therefore, this recursive formula generates the sequence representing the number of seats in each row:

Row	1	2	3	4	. . .
Seats	59	63	67		

+4 +4 +4

$u_1 = 59$

$u_n = u_{n-1} + 4 \quad \text{where } n \geq 2$

You can use this recursive formula to calculate how many seats are in each row.

$u_1 = 59$ — The starting term is 59.

$u_2 = u_1 + 4 = 59 + 4 = 63$ — Substitute 59 for u_1.

$u_3 = u_2 + 4 = 63 + 4 = 67$ — Substitute 63 for u_2.

$u_4 = u_3 + 4 = 67 + 4 = 71$ — Continue using recursion.

$\vdots$

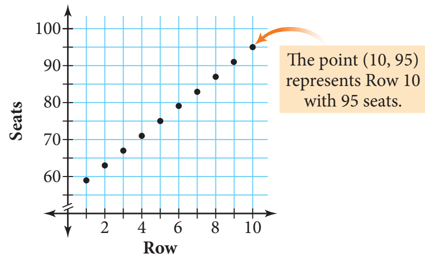
Because $u_4 = 71$, there are 71 seats in Row 4. If you continue the recursion process, you will find that $u_{10} = 95$, or that Row 10 has 95 seats.

Conceptual/Procedural

Conceptual The context of the examples and the investigation help students conceptualize both recursively defined sequences and recursive rules. A comparison of Examples A and C conceptually presents the difference between an arithmetic and a geometric sequence.

Procedural The solutions to the examples and the exercises help students become fluid in the procedure of solving for terms of a sequence.

You can graph the sequence from Example B by plotting (*row, seats*) or, more generally, (n, u_n).



In Examples A and B, a constant is added to each term of the sequence to generate the next term. This type of sequence is called an **arithmetic sequence.**

> **Arithmetic Sequence**
>
> **An arithmetic sequence is a sequence in which each term is equal to the previous term plus a constant. This constant is called the common difference. If d is the common difference, the recursive rule for the sequence has the form**
>
> $u_n = u_{n-1} + d$

The key to identifying an arithmetic sequence is recognizing the common difference. If you are given a few terms and need to write a recursive formula, first try subtracting consecutive terms. If $u_n - u_{n-1}$ is constant for each pair of terms, then you know your recursive rule must define an arithmetic sequence.

INVESTIGATION

Monitoring Inventory

Art Smith has been providing the prints of an engraving to FineArt Gallery. He plans to make just 2000 more prints. FineArt has already received 470 of Art's prints. The Little Print Shoppe also wishes to order prints. Art agrees to supply FineArt with 40 prints each month and Little Print Shoppe with 10 prints each month until he runs out.

Step 1 As a group, model what happens to the number of unmade prints, the number of prints delivered to FineArt, and the number delivered to Little Print Shoppe in a **spreadsheet** like the one below.

Month	Unmade prints	FineArt	Little Print Shoppe
1	2000	470	0
2	1950	510	10

Step 2 Use your table from Step 1 to answer these questions:

a. How many months will it be until FineArt has an equal number or a greater number of prints than the number of prints left unmade?

b. How many prints will have been delivered to the Little Print Shoppe when FineArt has received twice the number of prints that remain to be made?

Guiding the Investigation

There is a desmos simulation in your ebook. Students do not need any prior knowledge of recursion to do this investigation. To launch the investigation, ASK "What does it mean for a company to *monitor inventory?*"

Step 1 You may want to specify how many months students are to track.

Step 2 Encourage a variety of approaches to these problems.

For example, students might continue the table, use recursion, or draw a graph. Students might use statistics software or a spreadsheet to model the investigation. You could offer students the option of using this onscreen routine to generate the information for the table:

{1,2000,470,0}

{Ans[1] + 1, Ans[2] – 50, Ans[3] + 40, Ans[4] + 10}

See Calculator Note: Recursion for more information about home-screen recursion. SMP 1, 5

Be aware that desmos, which comes with your ebook, does not do recursion.

Choose a few groups to present a variety of approaches to Step 2 of the investigation. Whether student responses are correct or incorrect, ask other students if they agree and why. SMP 1, 3, 6

Step 1

Month	Unmade prints	FineArt	Little Print Shoppe
3	1900	550	20
…	…	…	…
n	$u_{n-1} - 50$	$v_{n-1} + 40$	$w_{n-1} + 10$

The three sequences have terms called u_n, v_n, and w_n.

Step 2a On Month 18, FineArt has 1150 prints, and there are 1150 unmade prints.

Step 2b On Month 27, FineArt has 1510 prints, there are 700 unmade, and Little Print Shoppe has 260. (On Month 26, FineArt had 1470, and there were 750 unmade.)

Example C

LANGUAGE This triangle is sometimes called the *Sierpiński gasket.* If extended for infinitely many steps, it's an example of a fractal, having fractional dimension. There is a desmos simulation for this example in your ebook.

Project the example and have students work in small groups. **ASK** "What would the initial figure, Stage 0, look like? Why?" [Stage 0 would be an equilateral triangle with no triangles inside because no change has occurred.] **SMP 1, 2, 7**

Many different sequences can be studied based on the Sierpiński triangle. **ASK** "If the area of the large triangle is 1 unit, what sequence represents the areas of the red triangles at successive stages?" $\left[\frac{3}{4}, \frac{9}{16}, \frac{27}{64}, \ldots\right]$ "What is the recursive formula for the sequence of areas?" $\left[u_1 = \frac{3}{4}, u_n = \frac{3}{4}u_{n-1} \text{ where } n \geq 2\right]$ "If the length of each side of the large triangle is 1, what sequence represents the perimeter (total distance around all red triangles at a particular stage)?" $\left[\frac{9}{2}, \frac{27}{4}, \frac{81}{8}, \ldots\right]$ "What is the recursive formula for the sequence of perimeters?" $\left[u_1 = \frac{9}{2}, u_n = \frac{3}{2}u_{n-1} \text{ where } n \geq 2\right]$

LANGUAGE *Common ratio* should be defined specifically here. Emphasize that it is the number, which can be a fraction, that each term is multiplied by to get the next. **ASK** "What is the common ratio for the sequence in Example C?" [3] "What would be the common ratio for the sequence if the terms in the series were reversed?" $\left[\frac{1}{3}\right]$

After Example C, ask students to write down their definition of a geometric sequence. After several students share their definitions, project the definition box. Discuss the differences in the definitions from the class and from the book. **SMP 3, 6**

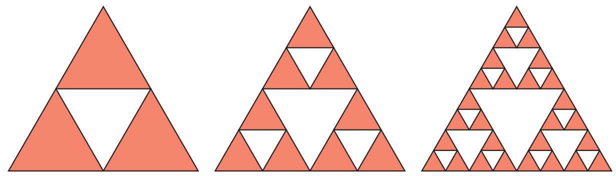
Step 3 Write a short summary of how you modeled the number of prints and how you found the answers to the questions in Step 2. Compare your methods with the methods of other groups. Answers will vary

***Career* CONNECTION**

Economics is the study of how goods and services are produced, distributed, and consumed. Economists in corporations, universities, and government agencies are concerned with the best way to meet human needs with limited resources. Professional economists use mathematics to study and model factors such as supply of resources, manufacturing costs, and selling price.

The sequences in Example A, Example B, and the investigation are arithmetic sequences. Example C introduces a different kind of sequence that is also defined recursively.

EXAMPLE C The geometric pattern below is created recursively. If you continue the pattern endlessly, you create a **fractal** called the Sierpiński triangle. How many red triangles are there at Stage 20?



Stage 1 Stage 2 Stage 3

***Mathematics* CONNECTION**

The Sierpiński triangle is named after the Polish mathematician Waclaw Sierpiński (1882–1969). He was most interested in number theory, set theory, and topology, three branches of mathematics that study the relations and properties of numbers, sets, and points, respectively. Sierpiński was highly involved in the development of mathematics in Poland between World War I and World War II. He published 724 papers and 50 books in his lifetime. He introduced his famous triangle pattern in a 1915 paper.

This stamp, part of Poland's 1982 "Mathematicians" series, portrays Waclaw Sierpiński.

Summarize

Have students present their solutions and explain their work in the Examples and the Investigation. Encourage a variety of approaches, especially making tables by hand or using a calculator or spreadsheet. Students may want to draw graphs or work with formulas. During the discussion, **ASK** "What does the situation in the Investigation have in common with those in the examples?" Bring out the idea of sequences, especially arithmetic versus geometric sequences. Emphasize the use of correct mathematical terminology. Whether student responses are correct or incorrect, ask other students if they agree and why. **SMP 1, 3, 5, 6**

Students may recognize the natural connection between arithmetic sequences and linear functions. The connection between recursive and explicit formulas will be addressed in a later lesson. Students may also see a connection between geometric sequences and exponential functions. If students are impatient with the recursive approach, mention that it is more useful than closed-form formulas for studying growth.

ASK "The arithmetic sequence 5, 2, −1, −4, and so on is decreasing. What's happening?" [We're adding a negative common difference.]

Solution

Count the number of red triangles at each stage and write a sequence.

3, 9, 27, . . .

The starting term, 3, represents the number of triangles at Stage 1. You can define the starting term as term one, or u_1. In this case, $u_1 = 3$.

Starting with the second term, each term of the sequence is 3 times the previous term, so that 9 is 3 times 3 and 27 is 3 times 9. Use this information to write the recursive rule and complete your recursive formula.

$u_1 = 3$

$u_n = 3 \cdot u_{n-1}$ where $n \geq 2$

Using the recursive rule 19 times, you find that $u_{20} = 3{,}486{,}784{,}401$. There are over 3 billion triangles at Stage 20!

fx | =B20*3

	A	B	
1	n	u_n	
2	1	3	
3	2	9	← B2·3
4	3	27	← B3·3
5	4	81	
6	5	243	
7	6	729	
8	7	2187	
9	8	6561	
10	9	19683	
11	10	59049	
12	11	177147	
13	12	531441	
14	13	1594323	
15	14	4782969	
16	15	14348907	
17	16	43046721	
18	17	129140163	
19	18	387420489	
20	19	1162261467	
21	20	3486784401	← B20·3

In Example C, each term is multiplied by a constant to generate the next term. This type of sequence is called a **geometric sequence**.

Geometric Sequence

A geometric sequence is a sequence in which each term is equal to the previous term multiplied by a constant. This constant is called the common ratio. If r is the common ratio, the recursive rule for the sequence has the form

$$u_n = r \cdot u_{n-1}$$

You identify a geometric sequence by dividing consecutive terms. If $\frac{u_n}{u_{n-1}}$ has the same value for each pair of terms, then you know the sequence is geometric.

Arithmetic and geometric sequences are the most basic sequences because their recursive rules use only one operation: addition in the case of arithmetic sequences, and multiplication in the case of geometric sequences. Recognizing these basic operations will help you easily identify sequences and write recursive formulas.

Resist the temptation to make a connection to the fact that u_n varies directly with n (with constant of variation u_1), because the emphasis of this lesson is on recursion, with the goal of a deep understanding of rate of change and slope.

CRITICAL QUESTION What are the parts of a recursive formula?

BIG IDEA A recursive formula consists of two parts: the initial value and a recursive rule that tells how to get each term from the previous term.

CRITICAL QUESTION How are arithmetic and geometric sequences and their recursive formulas alike and different?

BIG IDEA They're both sequences; arithmetic sequences are generated recursively by adding a constant, and geometric sequences by multiplying a constant.

CRITICAL QUESTION What generalizations can you make for geometric sequences?

BIG IDEA If the common ratio is between 0 and 1, the sequence is decreasing. If the common ratio is negative, the terms alternate between positive and negative.

CRITICAL QUESTION What is the difference between u_{n-1} and $u_n - 1$? Under what conditions would they be the same?

BIG IDEA u_{n-1} means the previous term, $u_n - 1$ means 1 less than the previous term. They would be the same if the common difference is 1.

Formative Assessment

Student group work and presentations of the Examples, Investigation and Exercises can provide evidence of meeting the objectives of the lesson. Answers and discussion of the Critical Questions also provide anecdotal evidence of understanding. Watch how students understand u_n notation and relate their recursive formulas to the situations and data. Encourage use of appropriate mathematical vocabulary to assess their understanding of the definition of arithmetic sequence (a starting value and a common difference) and a geometric sequence (a starting value and a common and a common ratio).

Apply

Extra Example

1. Write the first five terms of the sequence:
   $u_1 = -3$
   $u_n = u_{n-1} + 4$ where $n \geq 2$
   −3, 1, 5, 9, 13
2. Fill in the tables with the missing terms.

a.

1	2	3	4	…	10	u_n
8	2	−4	−10	…		

$u_{10} = -46$; $u_n = u_{n-1} - 6$

b.

0	1	2	3	…	10	u_n
−3	−1	$-\frac{1}{3}$	−10	…		

$u_{10} = \frac{1}{19,683}$; $u_n = \frac{1}{3}u_{n-1}$

c.

1	2	3	4	…	10	u_n
12	18	27		…		

$u_{10} = \frac{59,049}{128}$; $u_n = \frac{3}{2}u_{n-1}$

Closing Question

Consider the sequences from the Launch.

a. 4, 8, 12, 16, 20, …
b. 1, 0.1, 0.01, 0.001, 0.0001

Write a recursive formula to generate each sequence. Then find the eighth term of each. Which sequence is arithmetic and which is geometric? Explain.

a. $u_1 = 4$; $u_n = u_{n-1} + 4$; $u_8 = 32$

b. $u_1 = 1$; $u_n = u_{n-1} \cdot 0.1$; $u_8 = 0.0000001$

Part a is arithmetic, there is a common difference and part b is geometric, there is a common ratio.

Assigning Exercises

Suggested: 1 – 13
Additional Practice: 14 – 17

The exercises emphasize arithmetic sequences. Geometric sequences will receive more attention in later lessons.

2a. (iv) −18, −13.7, −9.4, −5.1; arithmetic; $d = 4.3$
2b. (ii) 47, 44, 41, 38; arithmetic; $d = -3$
2c. (i) 20, 26, 32, 38; arithmetic; $d = 6$
2d. (iii) 32, 48, 72, 108; geometric; $r = 1.5$

1.1 Exercises

Practice Your Skills

1. Match each description of a sequence to its recursive formula.
   a. The first term is −18. Keep adding 4.3. iv
   b. Start with 47. Keep subtracting 3. ii
   c. Start with 20. Keep adding 6. i
   d. The first term is 32. Keep multiplying by 1.5. iii

   i. $u_1 = 20$
   $u_n = u_{n-1} + 6$ where $n \geq 2$
   ii. $u_1 = 47$
   $u_n = u_{n-1} - 3$ where $n \geq 2$
   iii. $u_1 = 32$
   $u_n = 1.5 \cdot u_{n-1}$ where $n \geq 2$
   iv. $u_1 = -18$
   $u_n = u_{n-1} + 4.3$ where $n \geq 2$

2. For each sequence in Exercise 1, write the first 4 terms of the sequence and identify it as arithmetic or geometric. State the common difference or the common ratio for each sequence. @

3. Write a recursive formula and use it to find the missing table values. @

n	1	2	3	4	5	…	9
u_n	40	36.55	33.1	29.65	26.2	…	12.4

$u_1 = 40$ and $u_n = u_{n-1} - 3.45$ where $n \geq 2$

4. Write a recursive formula to generate an arithmetic sequence with a first term 6 and a common difference 3.2. Find the 10th term. $u_1 = 6$ and $u_n = u_{n-1} + 3.2$ where $n \geq 2$; $u_{10} = 34.8$

5. Write a recursive formula to generate each sequence. Then find the indicated term.
   a. 2, 6, 10, 14, . . . Find the 15th term. $u_1 = 2$ and $u_n = u_{n-1} + 4$ where $n \geq 2$; $u_{15} = 58$
   b. 0.4, 0.04, 0.004, 0.0004, . . . Find the 10th term. $u_1 = 0.4$ and $u_n = 0.1 \cdot u_{n-1}$ where $n \geq 2$; $u_{10} = 0.0000000004$
   c. −2, −8, −14, −20, −26, . . . Find the 30th term. $u_1 = -2$ and $u_n = u_{n-1} - 6$ where $n \geq 2$; $u_{30} = -176$
   d. −6.24, −4.03, −1.82, 0.39, . . . Find the 20th term. @ $u_1 = -6.24$ and $u_n = u_{n-1} + 2.21$ where $n \geq 2$; $u_{20} = 35.75$

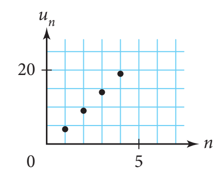
History CONNECTION

Hungarian mathematician Rózsa Péter (1905–1977) was the first person to propose the study of recursion in its own right. In an interview she described recursion in this way:

> The Latin technical term "recursion" refers to a certain kind of stepping backwards in the sequence of natural numbers, which necessarily ends after a finite number of steps. With the use of such recursions the values of even the most complicated functions used in number theory can be calculated in a finite number of steps.

In her book *Recursive Functions in Computer Theory,* Péter describes the important connections between recursion and computer languages.

6. Write a recursive formula for the sequence whose first four terms are graphed at right. Find the 46th term. @
   $u_1 = 4$ and $u_n = u_{n-1} + 5$ where $n \geq 2$; $u_{46} = 229$



Exercises 4, 5 Students may use either recursive sequences or home-screen recursion on their calculators to find the terms of these sequences. SMP 5

Exercise 6 Students might have difficulty making the transition from single numbers to ordered pairs (and points in a two-dimensional space). To help, ASK "What is the first coordinate in each pair?" [1, 2, 3, …; the term numbers] "The second coordinate?" [4, 9, 14, …; the terms of the sequence]

Exercise 11 Students might use the recursive rule $u_n = u_{n-1} - 0.25u_{n-1}$ or, equivalently, $u_n = 0.75u_{n-1}$.

12. David is correct. A common difference of −8 means add −8, or subtract 8, to get the next term. Start with 100, then keep subtracting 8 to get each subsequent term.

Exercise 13 ALERT Students may have difficulty finding the common difference, because they must divide the difference, 16, by 5 (not 4 or 6). As needed, remind students that the difference is *new* − *previous*.

13a. 3.2; $\frac{51-35}{5} = 3.2$

13b. 25.4, 28.6, 31.8, . . ., 38.2, 41.4, 44.6, 47.8, . . ., 54.2

ADVANCED This is an excellent extension to generating sequences given recursive routines. Students could describe their process for determining the common difference between terms and the starting value of the sequence.

Exercise 14 ELL Using a motion sensor and having students act out exercises like Exercise 14 will help develop a strong, conceptual understanding by modeling time-distance graphs.

14a.

Elapsed time (s)	Distance from motion sensor (m)
0.0	2.0
1.0	3.0
2.0	4.0
3.0	5.0
4.0	4.5
5.0	4.0
6.0	3.5
7.0	3.0

11. The week of February 14, the owner of Nickel's Appliances stocks hundreds of red, heart-shaped vacuum cleaners. The next week, he still has hundreds of red, heart-shaped vacuum cleaners. He tells the manager, "Discount the price 25 percent each week until they are gone."

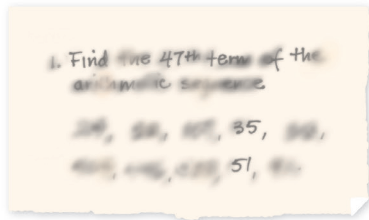
a. On February 14, the vacuums are priced at $80. What is the price of a vacuum during the second week? @ $60

b. What is the price during the fourth week? $33.75

c. When will a vacuum sell for less than $10? during the ninth week

12. Carl and David's teacher asked them to write the first four terms of a sequence that starts with 100 and has a common difference of −8. Carl says the first four terms are 100, 108, 116, 112. David says the first four terms are 100, 92, 86, 78. Who is correct? Write a clear explanation of how to use the common difference to build the sequence.

13. Taoufik picks up his homework paper from the puddle it fell in. Sadly he reads the first problem and finds that the arithmetic sequence is a blur except for two terms.



a. What is the common difference? How did you find it? (h)

b. What are the missing terms?

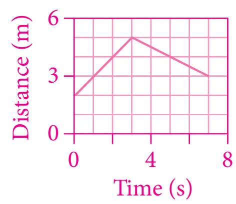
c. What is the answer to Taoufik's homework problem? 172.6

Review

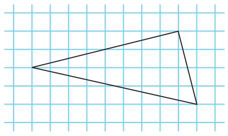
14. Ayaunna starts 2.0 m from a motion sensor. She walks away from the sensor at a rate of 1.0 m/s for 3.0 s and then walks toward the sensor at a rate of 0.5 m/s for 4.0 s.

a. Create a table of values for Ayaunna's distance from the motion sensor at 1-second intervals. @

b. Sketch a time-distance graph of Ayaunna's walk.



15. Find the area of this triangle using two different strategies. Describe your strategies.



17 square units. Sample answer: Enclose the triangle in a rectangle, and subtract the areas of the right triangles on the outside; or label the triangle ABC with $\angle B$ the largest, and notice that $AB \perp BC$, find their lengths using the distance formula, and use the formula $A = \frac{1}{2}(AB)(BC)$

16. **APPLICATION** Sherez is currently earning $390 per week as a store clerk and part-time manager. She is offered either a 7% increase or an additional $25 per week. Which offer should she accept? the 7% offer sat $417.30 per week

17. The cost of filling your car's gas tank varies directly with the number of gallons of gas you put in. It costs $41.88 to put 12 gallons in the gas tank.

a. How much does it cost to put 1 gallon of gas in the tank? This is the constant of variation for this relationship. $3.49

b. Write the equation for the direct variation between *c*, the cost of filling the tank, and *g*, the number of gallons put in the tank. $c = 3.49\,g$

c. Use your direct variation equation to find the cost of filling a tank with 22 gallons of gas. $76.78

IMPROVING YOUR Reasoning SKILLS

Fibonacci and the Rabbits

Suppose a pair of newborn rabbits, one male and one female, is put in a field. Assume that rabbits are able to mate at the age of one month, so at the end of its second month a female can produce another pair of rabbits. Suppose that each female who is old enough produces one new pair of rabbits (one male, one female) every month and that none of the rabbits die. Write the first few terms of a sequence that shows how many pairs there will be at the end of each month. Then write a recursive formula for the sequence.

In winter, the arctic hare, or polar rabbit, sports a brilliant white coat that provides excellent camouflage in the ice and snow of the North American tundra.

This sequence is called the **Fibonacci sequence** after Italian mathematician Leonardo Fibonacci (ca. 1170–1240), who included a similar problem in his book *Liber abaci* (1202). How is the Fibonacci sequence unique compared with the other sequences you have studied?

IMPROVING YOUR Reasoning SKILLS

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . .; $u_1 = 1$, $u_2 = 1$, and $u_n = u_{n-1} + u_{n-2}$ where $n \geq 3$. Thus the recursive rule for the Fibonacci sequence is defined by two preceding terms.

Rabbits do actually have a gestation period (from conception, or *breeding*, to birth, or *kindling*) of one month. However, in real life, rabbits usually have to be six to nine months old before breeding, and they usually are not able to breed again immediately after kindling. The Fibonacci sequence, which is related to the golden ratio, has many applications in biology. Fibonacci is credited with informing Europeans about the number system developed by the Hindus and Arabs.

LESSON 1.1 Recursively Defined Sequences 37