Illustrative Mathematics

IM 6 – 8 Math

Sample Materials

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About these materials

These materials were created by Illustrative Mathematics in collaboration with Open Up Resources. They were piloted and revised in the 2016–2017 school year.

The course contains nine units; each of the first eight are anchored by a few big ideas in grade 6 mathematics. Units contain between 15 and 20 lesson plans. Each unit has a diagnostic assessment for the beginning of the unit and an end-of-unit assessment. Longer units also have a mid-unit assessment. The last unit in the course is structured differently, and contains optional lessons that help students apply and tie together big ideas from the year.

The time estimates in these materials refer to instructional time. Each lesson plan is designed to fit within a 45–50 minute period. Some lessons contain optional activities that provide additional practice for teachers to use at their discretion.

There are two ways students can interact with these materials. Students can work solely with printed workbooks or pdfs. Alternatively, if all students have access to an appropriate device, students can look at the task statements on that device and write their responses in a notebook or the print companion for the digital materials. It is recommended that if students are to access the materials this way, they keep the notebook carefully organized so that they can go back to their work later.

Teachers can access the teacher materials either in print or in a browser. A classroom with a digital projector is recommended.

Many activities are written in a card sort, matching, or info gap format that requires teachers to provide students with a set of cards or slips of paper that have been photocopied and cut up ahead of time. Teachers might stock up on two sizes of resealable plastic bags: sandwich size and gallon size. For a given activity, one set of cards can go in each small bag, and then the small bags for one class can be placed in a large bag. If these are labeled and stored in an organized manner, it can facilitate preparing ahead of time and re-using card sets between classes. Additionally, if possible, it is often helpful to print the slips for different parts of an activity on different color paper. This helps facilitate quickly sorting the cards between classes.

Design Principles

Developing Conceptual Understanding and Procedural Fluency

Each unit begins with a pre-assessment that helps teachers gauge what students know about both prerequisite and upcoming concepts and skills, so that teachers can gauge where students are and make adjustments accordingly. The initial lesson in a unit is designed to activate prior knowledge and provide an easy entry to point to new concepts, so that students at different levels of both mathematical and English language proficiency can engage productively in the work. As the unit progresses, students are systematically introduced to representations, contexts, concepts, language and notation. As their learning progresses, they make connections between different representations and strategies, consolidating their conceptual understanding, and see and understand more efficient methods of solving problems, supporting the shift towards procedural fluency. The distributed practice problems give students ongoing practice, which also supports developing procedural proficiency.

Applying Mathematics

Students have opportunities to make connections to real-world contexts throughout the materials. Frequently, carefully-chosen anchor contexts are used to motivate new mathematical concepts, and students have many opportunities to make connections between contexts and the concepts they are learning. Additionally, most units include a real-world application lesson at the end. In some cases, students spend more time developing mathematical concepts before tackling more complex application problems, and the focus is on mathematical contexts. The first unit on geometry is an example of this.

The Five Practices

Selected activities are structured using Five Practices for Orchestrating Productive Mathematical Discussions (Smith & Stein, 2011), also described in Principles to Actions: Ensuring Mathematical Success for All (NCTM, 2014), and Intentional Talk: How to Structure and Lead Productive Mathematical Discussions (Kazemi & Hintz, 2014). These activities include a presentation of a task or problem (may be print or other media) where student approaches are anticipated ahead of time. Students first engage in independent think-time followed by partner or small-group work on the problem. The teacher circulates as students are working and notes groups using different approaches. Groups or individuals are selected in a specific, recommended sequence to share their approach with the class, and finally the teacher leads a whole-class discussion to make connections and highlight important ideas.

Task Purposes

- Provide experience with a new context. Activities that give all students experience with a new context ensure that students are ready to make sense of the concrete before encountering the abstract.

- Introduce a new concept and associated language. Activities that introduce a new concept and associated language build on what students already know and ask them to notice or put words to something new.

- Introduce a new representation. Activities that introduce a new representation often present the new representation of a familiar idea first and ask students to interpret it. Where appropriate, new representations are connected to familiar representations or extended from familiar representations. Students are then given clear instructions on how to create such a representation as a tool for understanding or for solving problems. For subsequent activities and lessons, students are given opportunities to practice using these representations and to choose which representation to use for a particular problem.

- Formalize a definition of a term for an idea previously encountered informally. Activities that formalize a definition take a concept that students have already encountered through examples, and give it a more general definition.

- Identify and resolve common mistakes and misconceptions that people make. Activities that give students a chance to identify and resolve common mistakes and misconceptions usually present some incorrect work and ask students to identify it as such and explain what is incorrect about it. Students deepen their understanding of key mathematical concepts as they analyze and critique the reasoning of others.

- Practice using mathematical language. Activities that provide an opportunity to practice using mathematical language are focused on that as the primary goal rather than having a primarily mathematical learning goal. They are intended to give students a reason to use mathematical language to communicate. These frequently use the Info Gap instructional routine.

- Work toward mastery of a concept or procedure. Activities where students work toward mastery are included for topics where experience shows students often need some additional time to work with the ideas. Often these activities are marked as optional because no new mathematics is covered, so if a teacher were to skip them, no new topics would be missed.

- Provide an opportunity to apply mathematics to a modeling or other application problem. Activities that provide an opportunity to apply mathematics to a modeling or other application problem are most often found toward the end of a unit. Their purpose is to give students experience using mathematics to reason about a problem or situation that one might encounter naturally outside of a mathematics classroom.

A note about standards alignments: There are three kinds of alignments to standards in these materials: building on, addressing, and building towards. Oftentimes a particular standard requires weeks, months, or years to achieve, in many cases building on work in prior grade-levels. When an activity reflects the work of prior grades but is being used to bridge to a grade-level standard, alignments are indicated as "building on." When an activity is laying the foundation for a grade-level standard but has not yet reached the level of the standard, the alignment is indicated as "building towards." When a task is focused on the grade-level work, the alignment is indicated as "addressing."

What is a Problem Based Curriculum?

What students should know and be able to do

Our ultimate purpose is to impact student learning and achievement. First, we define the attitudes and beliefs about mathematics and mathematics learning we want to cultivate in students, and what mathematics students should know and be able to do.

Attitudes and Beliefs We Want to Cultivate

Many people think that mathematical knowledge and skills exclusively belong to "math people." Yet research shows that students who believe that hard work is more important than innate talent learn more mathematics.[1] We want students to believe anyone can do mathematics and that persevering at mathematics will result in understanding and success. In the words of the NRC report Adding It Up, we want students to develop a "productive disposition—[the] habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy."[2]

[1] Uttal, D.H. (1997). Beliefs about genetic influences on mathematics achievement: a cross-cultural comparison. *Genetica*, 99(2-3), 165-172. doi.org/10.1023/A:1018318822120

[2] National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J.Kilpatrick, J. Swafford, and B.Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press. doi.org/10.17226/9822

Knowledge

Conceptual understanding: Students need to understand the why behind the how in mathematics. Concepts build on experience with concrete contexts. Students should access these concepts from a number of perspectives in order to see math as more than a set of disconnected procedures.

Procedural fluency: We view procedural fluency as solving problems expected by the standards with speed, accuracy, and flexibility.

Application: Application means applying mathematical or statistical concepts and skills to a novel mathematical or real-world context.

These three aspects of mathematical proficiency are interconnected: procedural fluency is supported by understanding, and deep understanding often requires procedural fluency. In order to be successful in applying mathematics, students must both understand and be able to do the mathematics.

Mathematical Practices

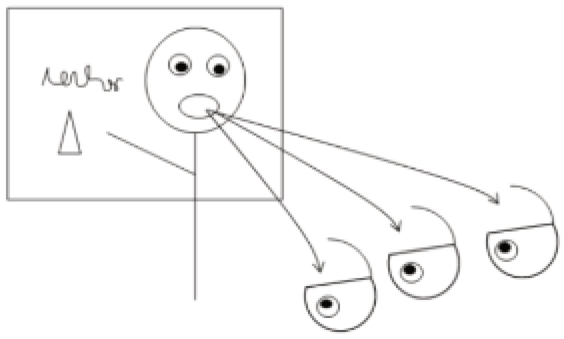
In a mathematics class, students should not just learn *about* mathematics, they should *do* mathematics. This can be defined as engaging in the mathematical practices: making sense of problems, reasoning abstractly and quantitatively, making arguments and critiquing the reasoning of others, modeling with mathematics, making appropriate use of tools, attending to precision in their use of language, looking for and making use of structure, and expressing regularity in repeated reasoning.

What teaching and learning should look like

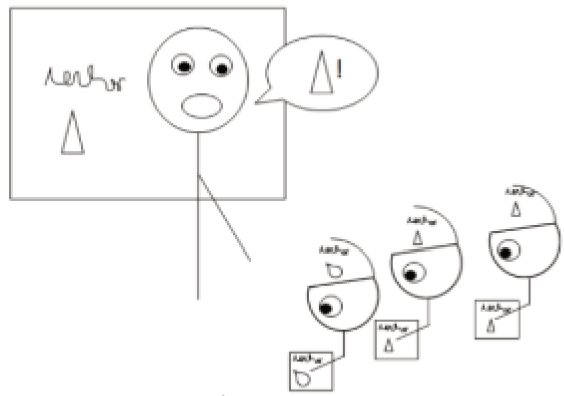
How teachers should teach depends on what we want students to learn. To understand what teachers need to know and be able to do, we need to understand how students develop the different (but intertwined) strands of mathematical proficiency, and what kind of instructional moves support that development.

Principles for Mathematics Teaching and Learning

Active learning is best: Students learn best and retain what they learn better by solving problems. Often, mathematics instruction is shaped by the belief that if teachers tell students how to solve problems and then students practice, students will learn how to do mathematics.



Teacher tells
Students listen



Students practice
Teacher corrects

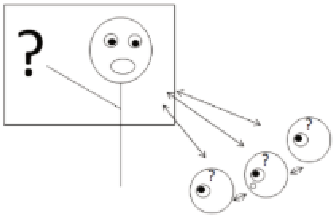
Decades of research tells us that the traditional model of instruction is flawed. Traditional instructional methods may get short-term results with procedural skills, but students tend to forget the procedural skills and do not develop problem solving skills, deep conceptual understanding, or a mental framework for how ideas fit together. They also don't develop strategies for tackling non-routine problems, including a propensity for engaging in productive struggle to make sense of problems and persevere in solving them.

In order to learn mathematics, students should spend time in math class *doing mathematics*.

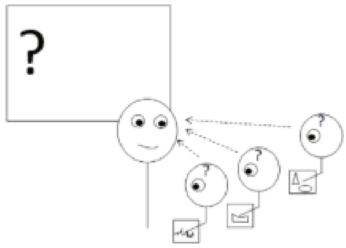
"Students learn mathematics as a result of solving problems. Mathematical ideas are the outcomes of the problem-solving experience rather than the elements that must be taught before problem solving."[3]

Students should take an active role, both individually and in groups, to see what they can figure out before having things explained to them or being told what to do. Teachers play a critical role in mediating student learning, but that role looks different than simply showing, telling, and correcting. The teacher's role is

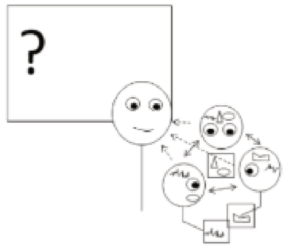
1. to ensure students understand the context and what is being asked,
2. ask questions to advance students' thinking in productive ways,
3. help students share their work and understand others' work through orchestrating productive discussions, and
4. synthesize the learning with students at the end of activities and lessons.



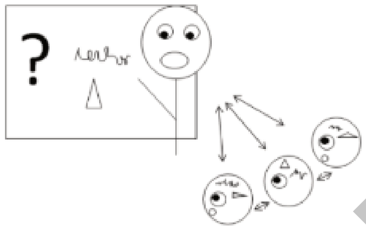
1. Teacher ensures students understand the question



2. Students work individually Teacher monitors, listens, questions



3. Students work in groups Teacher monitors, listens, and asks questions to understand students' thinking



4. Teacher helps students synthesize their learning

Teachers should build on what students know: New mathematical ideas are built on what students already know about mathematics and the world, and as they learn new ideas, students need to make connections between them.[4] In order to do this, teachers need to understand what knowledge students bring to the classroom and monitor what they do and do not understand as they are learning. Teachers must themselves know how the mathematical ideas connect in order to mediate students' learning.

Good instruction starts with explicit learning goals: Learning goals must be clear not only to teachers, but also to students, and they must influence the activities in which students participate. Without a clear understanding of what students should be learning, activities in the classroom, implemented haphazardly, have little impact on advancing students' understanding. Strategic negotiation of whole-class discussion on the part of the teacher during an activity synthesis is crucial to making the intended learning goals explicit. Teachers need to have a clear idea of the destination for the day, week, month, and year, and select and sequence instructional activities (or use well-sequenced materials) that will get the class to their destinations. If you are going to a party, you need to know the address and also plan a route to get there; driving around aimlessly will not get you where you need to go.

Different learning goals require different instructional moves: The kind of instruction that is appropriate at any given time depends on the learning goals of a particular lesson. Lessons and activities can:

- Introduce students to a new topic of study and invite them to the mathematics

- Study new concepts and procedures deeply
- Integrate and connect representations, concepts, and procedures
- Work towards mastery
- Apply mathematics

Lessons should be designed based on what the intended learning outcomes are. This means that teachers should have a toolbox of instructional moves that they can use as appropriate.

Each and every student should have access to the mathematical work: With proper structures, accommodations, and supports, all students can learn mathematics. Teachers' instructional toolboxes should include knowledge of and skill in implementing supports for different learners.

[3] Hiebert, J., et. al. (1996). Problem solving as a basis for reform in curriculum and instruction: the case of mathematics. *Educational Researcher* 25(4), 12-21. doi.org/10.3102/0013189X025004012

[4] National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J.Kilpatrick, J. Swafford, and B.Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press. doi.org/10.17226/9822

Critical Practices

Intentional planning: Because different learning goals require different instructional moves, teachers need to be able to plan their instruction appropriately. While a high-quality curriculum does reduce the burden for teachers to create or curate lessons and tasks, it does not reduce the need to spend deliberate time planning lessons and tasks. Instead, teachers' planning time can shift to high-leverage practices (practices that teachers without a high-quality curriculum often report wishing they had more time for): reading and understanding the high-quality curriculum materials; identifying connections to prior and upcoming work; diagnosing students' readiness to do the work; leveraging instructional routines to address different student needs and differentiate instruction; anticipating student responses that will be important to move the learning forward; planning questions and prompts that will help students attend to, make sense of, and learn from each other's work; planning supports and extensions to give as many students as possible access to the main mathematical goals; figuring out timing, pacing, and opportunities for practice;

preparing necessary supplies; and the never-ending task of giving feedback on student work.

Establishing norms: Norms around doing math together and sharing understandings play an important role in the success of a problem-based curriculum. For example, students must feel safe taking risks, listen to each other, disagree respectfully, and honor equal air time when working together in groups. Establishing norms helps teachers cultivate a community of learners where making thinking visible is both expected and valued.

Building a shared understanding of a small set of instructional routines: Instructional routines allow the students and teacher to become familiar with the classroom choreography and what they are expected to do. This means that they can pay less attention to what they are supposed to do and more attention to the mathematics to be learned. Routines can provide a structure that helps strengthen students' skills in communicating their mathematical ideas.

Using high quality curriculum: A growing body of evidence suggests that using a high-quality, coherent curriculum can have a significant impact on student learning.[5] Creating a coherent, effective instructional sequence from the ground up takes significant time, effort, and expertise. Teaching is already a full-time job, and adding curriculum development on top of that means teachers are overloaded or shortchanging their students.

Ongoing formative assessment: Teachers should know what mathematics their students come into the classroom already understanding, and use that information to plan their lessons. As students work on problems, teachers should ask questions to better understand students' thinking, and use expected student responses and potential misconceptions to build on students' mathematical understanding during the lesson. Teachers should monitor what their students have learned at the end of the lesson and use this information to provide feedback and plan further instruction.

[5] Steiner, D. (2017). Curriculum research: What we know and where we need to go. *Standards Work*. Retrieved from https://standardswork.org/wp-content/uploads/2017/03/sw-curriculum-research-report-fnl.pdf

A Typical Lesson

A note about optional activities: A relatively small number of activities throughout the course have been marked "optional." Some common reasons an activity might be optional include:

- The activity addresses a concept or skill that is below grade level, but we know that it is common for students to need a chance to focus on it before encountering grade-level material. If the pre-unit diagnostic assessment indicates that students don't need this review, an activity like this can be safely skipped.
- The activity addresses a concept or skill that goes beyond the requirements of a standard. The activity is nice to do if there is time, but students won't miss anything important if the activity is skipped.
- The activity provides an opportunity for additional practice on a concept or skill that we know many students (but not necessarily all students) need. Teachers should use their judgment about whether class time is needed for such an activity.

A typical lesson has four phases:

1. A warm-up
2. One or more instructional activities
3. The lesson synthesis
4. A cool-down

The Warm-up

The first event in every lesson is a warm-up. A warm-up either:

- helps students get ready for the day's lesson, or
- gives students an opportunity to strengthen their number sense or procedural fluency.

A warm-up that helps students get ready for today's lesson might serve to remind them of a context they have seen before, get them thinking about where the previous lesson left off, or preview a calculation that will happen in the lesson so that the calculation doesn't get in the way of learning new mathematics.

A warm-up that is meant to strengthen number sense or procedural fluency asks students to do mental arithmetic or reason numerically or algebraically. It gives them a chance to make deeper connections or become more flexible in their thinking.

Four instructional routines frequently used in warm-ups are Number Talks, Notice and Wonder, Which One Doesn't Belong, and True or False. In addition to the mathematical purposes, these routines serve the additional purpose of strengthening students' skills in listening and speaking about mathematics.

Once students and teachers become used to the routine, warm-ups should take 5–10 minutes. If warm-ups frequently take much longer than that, the teacher should work on concrete moves to more efficiently accomplish the goal of the warm-up.

At the beginning of the year, consider establishing a small, discreet hand signal students can display to indicate they have an answer they can support with reasoning. This signal could be a thumbs up, or students could show the number of fingers that indicate the number of responses they have for the problem. This is a quick way to see if students have had enough time to think about the problem and keeps them from being distracted or rushed by classmates' raised hands.

Classroom Activities

After the warm-up, lessons consist of a sequence of one to three classroom activities. The activities are the heart of the mathematical experience and make up the majority of the time spent in class.

An activity can serve one or more of many purposes.

- Provide experience with a new context.
- Introduce a new concept and associated language.
- Introduce a new representation.
- Formalize a definition of a term for an idea previously encountered informally.
- Identify and resolve common mistakes and misconceptions that people make.
- Practice using mathematical language.
- Work toward mastery of a concept or procedure.
- Provide an opportunity to apply mathematics to a modeling or other application problem.

The purpose of each activity is described in its Activity Narrative. Read more about how activities serve these different purposes in the section on Design Principles.

Lesson Synthesis

After the activities for the day are done, students should take time to synthesize what they have learned. This portion of class should take 5–10 minutes before students start working on the cool-down. Each lesson includes a Lesson Synthesis section that assists the teacher with ways to help students incorporate new insights gained during the activities into their big-picture understanding. Teachers can use this time in any number of ways, including posing questions verbally and calling on volunteers to respond, asking students to respond to prompts in a written journal, asking students to add on to a graphic organizer or concept map, or adding a new component to a persistent display like a word wall.

Cool-down

Each lesson includes a cool-down task to be given to students at the end of the lesson. Students are meant to work on the cool-down for about 5 minutes independently and turn it in. The cool-down serves as a brief formative assessment to determine whether students understood the lesson. Students' responses to the cool-down can be used to make adjustments to further instruction.

How to use these materials

Each lesson and Unit Tells a Story

This story each grade is told in nine units. Each unit has a narrative that describes the mathematical work that will unfold in that unit. Each lesson in the unit also has a narrative. Lesson Narratives explain:

- A description of the mathematical content of the lesson and its place in the learning sequence.
- The meaning of any new terms introduced in the lesson.
- How the mathematical practices come into play, as appropriate.

Activities within lessons also have a narrative, which explain:

- The mathematical purpose of the activity and its place in the learning sequence.
- What students are doing during the activity.
- What teacher needs to look for while students are working on an activity to orchestrate an effective synthesis.
- Connections to the mathematical practices when appropriate.

Launch - Work - Synthesize

Each classroom activity has three phases.

The Launch

During the launch, the teacher makes sure that students understand the context (if there is one) and *what the problem is asking them to do*. This is not the same as making sure the students know how to do the problem—part of the work that students should be doing for themselves is figuring out how to solve the problem.

Student Work Time

The launch for an activity frequently includes suggestions for grouping students. This gives students the opportunity to work individually, with a partner, or in small groups.

Activity Synthesis

During the activity synthesis, the teacher orchestrates some time for students to synthesize what they have learned. This time is used to ensure that all students have an opportunity to understand the mathematical punch line of the activity and situate the new learning within students' previous understanding.

Are You Ready For More?

Select classroom activities include an opportunity for differentiation for students ready for more of a challenge. We think of them as the "mathematical dessert" to follow the "mathematical entrée" of a classroom activity.

Every extension problem is made available to all students with the heading "Are You Ready for More?" These problems go deeper into grade-level mathematics and often make connections between the topic at hand and other concepts. Some of these problems extend the work of the associated activity, but some of them involve work from prior grades, prior units in the course, or reflect work that is related to the K-12 curriculum but a type of problem not required by the standards. They are not routine or procedural, and they are not just "the same thing again but with harder numbers."

They are intended to be used on an opt-in basis by students if they finish the main class activity early or want to do more mathematics on their own. It is not expected that an entire class engages in *Are You Ready for More?* problems, and it is not expected that any student works on all of them. *Are You Ready for More?* problems may also be good fodder for a Problem of the Week or similar structure.

Instructional Routines

Algebra Talk

What: One expression is displayed at a time. Students are given a few minutes to quietly think and give a signal when they have an answer and a strategy. The teacher selects students to share different strategies for each one, “Who thought about it a different way?” Their explanations are recorded for all to see. Students might be pressed to provide more details about why they decided to approach a problem a certain way. It may not be possible to share every possible strategy for the given limited time; the teacher may only gather two or three distinctive strategies per problem. Problems are purposefully chosen to elicit different approaches.

Where: Warm-ups

Why: Algebra Talks build algebraic thinking by encouraging students to think about the numbers and variables in an expression and rely on what they know about structure, patterns, and properties of operations to mentally solve a problem. Algebra Talks promote seeing structure in expressions and thinking about how changing one number affects others in an equation. While participating in these activities, students need to be precise in their word choice and use of language (MP6).

Anticipate, monitor, select, sequence, connect

What: These are the *5 Practices for Orchestrating Productive Mathematical Discussions* (Smith and Stein, 2011). In this curriculum, much of the work of anticipating, sequencing, and connecting is handled by the materials in the activity narrative, launch, and synthesis sections. Teachers need to prepare for and conduct whole-class discussions.

Where: Many classroom activities lend themselves to this structure.

Why: In a problem-based curriculum, many activities can be described as "do math and talk about it," but the 5 Practices lend more structure to these activities so that they more reliably result in students making connections and learning new mathematics.

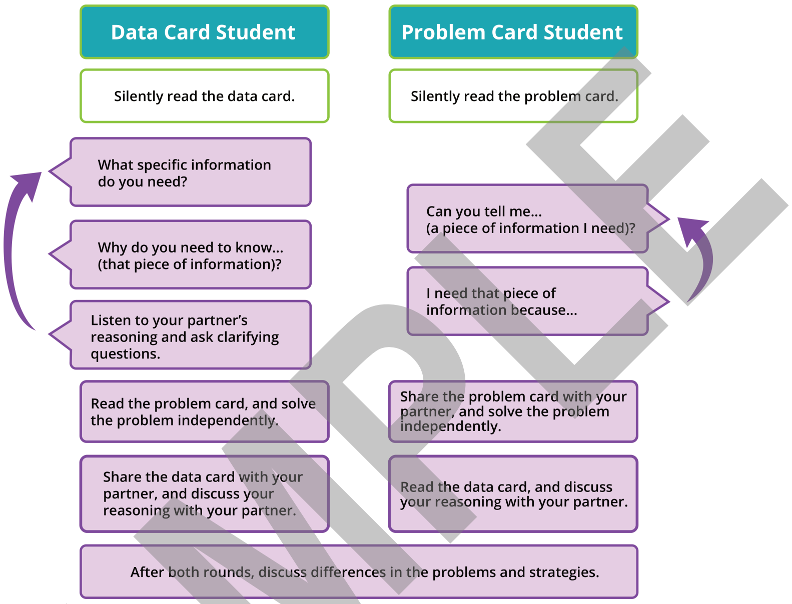
Group Presentations (6–8 ONLY)

Some activities instruct students to work in small groups to solve a problem with mathematical modeling, invent a new problem, design something, or organize and display data, and then create a visual display of their work. Teachers need to help groups organize their work so that others can follow it, and then facilitate different groups' presentation of work to the class. Teachers can develop specific questioning skills to help more students make connections and walk away from these experiences with desired mathematical learning. For example, instead of asking if anyone has any questions for the group, it is often more productive to ask a member of the class to restate their understanding of the group's findings in their own words.

Information Gap cards

What: Students conduct a dialog in a specific way. In an Info Gap, one partner gets a problem card with a math question that doesn't have enough given information, and the other partner gets a data card with information relevant to the problem card. Students ask each other questions like "What information do you need?" and are expected to explain what they will do with the information. The first few times students engage in these activities, the teacher should demonstrate, with a partner, how the discussion is expected to go. Once students are familiar with these structures, less set-up will be necessary.

Why: This activity structure is designed to strengthen the opportunities and supports for high-quality mathematical conversations. Mathematical language is learned by using mathematical language for real and engaging purposes. These activities were designed such that students *need* to communicate in order to bridge information gaps. During effective discussions, students should be supported to do the following: pose and answer questions, clarify what is asked and happening in a problem, build common understandings, and share experiences relevant to the topic.



MLR1: Stronger and Clearer Each Time

To provide a structured and interactive opportunity for students to revise and refine both their ideas and their verbal and written output. This routine provides a purpose for student conversation as well as fortifies output. The main idea is to have students think or write individually about a response, use a structured pairing strategy to have multiple opportunities to refine and clarify the response through conversation, and then finally revise their original written response. Throughout this process, students should be pressed for details, and encouraged to press each other for details.

MLR2: Collect and Display

To capture students' oral words and phrases into a stable, collective reference. The intent of this routine is to stabilize the fleeting language that students use during partner, small-group, or whole-class activities in order for student's own output to be used as a reference in developing their mathematical language. The teacher listens for, and scribes, the student output using written words, diagrams and pictures; this collected output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. This routine provides feedback for students in a way that increases accessibility while simultaneously supporting meta-awareness of language.

MLR3: Critique, Correct, and Clarify

To give students a piece of mathematical writing that is not their own to analyze, reflect on, and develop. The intent is to prompt student reflection with an incorrect, incomplete, or ambiguous written argument or explanation, and for students to improve upon the written work by correcting errors and clarifying meaning. This routine fortifies output and engages students in meta-awareness. Teachers can demonstrate with meta-think-alouds and press for details when necessary.

MLR5: Co-Craft Questions and Problems

To allow students to get inside of a context before feeling pressure to produce answers, and to create space for students to produce the language of mathematical questions themselves. Through this routine, students are able to use conversation skills as well as develop meta-awareness of the language used in mathematical questions and problems. Teachers should push for clarity and revoice oral responses as necessary.

MLR6: Three Reads

To ensure that students know what they are being asked to do, and to create an opportunity for students to reflect on the ways mathematical questions are presented. This routine supports reading comprehension of problems and meta-awareness of mathematical language. It also supports negotiating information in a text with a partner in mathematical conversation.

MLR7: Compare and Connect

To foster students' meta-awareness as they identify, compare, and contrast different mathematical approaches, representations, and language. Teachers should demonstrate thinking out loud (e.g., exploring why we one might do or say it this way, questioning an idea, wondering how an idea compares or connects to other ideas or language), and students should be prompted to reflect and respond. This routine supports meta-cognitive and meta-linguistic awareness, and also supports mathematical conversation.

MLR8: Discussion Supports

To support rich discussions about mathematical ideas, representations, contexts, and strategies. The examples provided can be combined and used together with any of the other routines. They include multi-modal strategies for helping students comprehend complex language and ideas, and can be used to make classroom communication accessible, to foster meta-awareness of language, and to demonstrate strategies students can use to enhance their own communication and construction of ideas.

Notice and wonder

What: This routine can appear as a warm-up or in the launch or synthesis of a classroom activity. Students are shown some media or a mathematical representation. The prompt to students is "What do you notice? What do you wonder?" Students are given a few minutes to write down things they notice and things they wonder. After students have had a chance to write down their responses, the teacher asks several students to share things they noticed and things they wondered; these are recorded by the teacher for all to see. Usually, the teacher steers the conversation to wondering about something mathematical that the class is about to focus on.

Where: Appears frequently in warm-ups but also appears in launches to classroom activities.

Why: The purpose is to make a mathematical task accessible to all students with these two approachable questions. By thinking about them and responding, students gain entry into the context and might get their curiosity piqued. Taking steps to become familiar with a context and the mathematics that might be involved is making sense of problems (MP1). Note: *Notice and Wonder* and *I Notice/I Wonder* are trademarks of NCTM and the Math Forum and used in these materials with permission.

Number Talk

What: One problem is displayed at a time. Students are given a few minutes to quietly think and give a signal when they have an answer and a strategy. The teacher selects students to share different strategies for each problem, "Who thought about it a different way?" Their explanations are recorded for all to see. Students might be pressed to provide more details about why they decided to approach a problem a certain way. It may not be possible to share every possible strategy for the given limited time; the teacher may only gather two or three distinctive strategies per problem. Problems are purposefully chosen to elicit different approaches, often in a way that builds from one problem to the next.

Where: Warm-ups

Why: Number talks build computational fluency by encouraging students to think about the numbers in a computation problem and rely on what they know about structure, patterns, and properties of operations to mentally solve a problem. Dot images are similar to number talks, except the image used is an arrangement of dots that students might count in different ways. While participating in these activities, students need to be precise in their word choice and use of language (MP6).

Poll the class

What: This routine is used to register an initial response or an estimate, most often in activity launches or to kick off a discussion. It can also be used when data needs to be collected from each student in class, for example, "What is the length of your ear in centimeters?" Every student in class reports a response to the prompt. Teachers need to develop a mechanism by which poll results are collected and displayed so that this frequent form of classroom interaction is seamless. Smaller classes might be able to conduct a roll call by voice. For larger classes, students might be given mini-whiteboards or a set of colored index cards to hold up. Free and paid commercial tools are also readily available.

Why: Collecting data from the class to use in an activity makes the outcome of the activity more interesting. In other cases, going on record with an estimate makes people want to know if they were right and increases investment in the outcome. If coming up with an estimate is too daunting, ask students for a guess that they are sure is too low or too high. Putting some boundaries on possible outcomes of a problem is an important skill for mathematical modeling (MP4).

Take turns

What: Students work with a partner or small group. They take turns in the work of the activity, whether it be spotting matches, explaining, justifying, agreeing or disagreeing, or asking clarifying questions. If they disagree, they are expected to support their case and listen to their partner's arguments. The first few times students engage in these activities, the teacher should demonstrate, with a partner, how the discussion is expected to go. Once students are familiar with these structures, less set-up will be necessary. While students are working, the teacher can ask students to restate their question more clearly or paraphrase what their partner said.

Why: Building in an expectation, through the routine, that students explain the rationale for their choices and listen to another's rationale deepens the understanding that can be achieved through these activities. Specifying that students take turns deciding, explaining, and listening limits the phenomenon where one student takes over and the other does not participate. Taking turns can also give students more opportunities to construct logical arguments and critique others' reasoning (MP3).

Think pair share

What: Students have quiet time to think about a problem and work on it individually, and then time to share their response or their progress with a partner. Once these partner conversations have taken place, some students are selected to share their thoughts with the class.

Why: This is a teaching routine useful in many contexts whose purpose is to give all students enough time to think about a prompt and form a response before they are expected to try to verbalize their thinking. First they have an opportunity to share their thinking in a low-stakes way with one partner, so that when they share with the class they can feel calm and confident, as well as say something meaningful that might advance everyone's understanding. Additionally, the teacher has an opportunity to eavesdrop on the partner conversations so that she can purposefully select students to share with the class.

True or False

What: One statement is displayed at a time. Students are given a few minutes to quietly think and give a signal when they have an answer. The teacher selects students to share different ways of reasoning for each statement. "Who thought about it a different way?" While students may evaluate each side of the equation to determine if it is true or false, encourage students to think about ways to reason that do not require complete computations. It may not be possible to share every possible reasoning approach for the given limited time; the teacher may only gather two or three distinctive strategies per problem. Statements are purposefully chosen to elicit different approaches, often in a way that builds from one statement to the next.

Where: Warm-ups

Why: Depending on the purpose of the task, the true or false structure encourages students to reason about numeric and algebraic expressions using base-ten structure, the meaning of fractions, meaning and properties of operations, and the meaning of comparison symbols. While the structure of a true or false is similar to that of a number talk, number talks are often focused on computational strategies, while true or false tasks are more likely to focus on more structural aspects of the expressions. Often students can determine whether an equation, an inequality, or a statement is true or false without doing any direct computation. While participating in these activities, students need to be precise in their word choice and use of language (MP6).

Which one doesn't belong?

What: Students are presented with four figures, diagrams, graphs, or expressions with the prompt "Which one doesn't belong?" Typically, each of the four options "doesn't belong" for a different reason, and the similarities and differences are mathematically significant.

Students are prompted to explain their rationale for deciding that one option doesn't belong and given opportunities to make their rationale more precise.

Where: Warm-ups

Why: Which One Doesn't Belong fosters a need to define terms carefully and use words precisely (MP6) in order to compare and contrast a group of geometric figures or other mathematical representations.

Supporting English Language Learners

Understanding Language/SCALE

These materials include embedded supports for English language learners (ELLs) tied to a framework developed by the team at Understanding Language/Stanford Center for Assessment, Learning, and Equity (UL/SCALE) at Stanford University. Some supports are built right into the curriculum because they help all learners. For example, task statements have been reviewed and modified to reduce unnecessary language complexity. Certain routines that are especially helpful for ELLs are included for all learners. For example, Info Gap activities appear regularly in the curriculum materials. There are also suggested supports embedded in the lesson plans themselves, included as annotations for supporting ELLs. Annotations often include suggestions for heavier or lighter supports, which are appropriate for students at different levels of language proficiency. Many of the annotations suggest using Mathematical Language Routines (MLRs).

The framework for supporting English language learners (ELLs) in this curriculum includes four design principles for promoting mathematical language use and development in curriculum and instruction. The design principles and related routines work to make language development an integral part of planning and delivering instruction while guiding teachers to amplify the most important language that students are expected to bring to bear on the central mathematical ideas of each unit. Read more in depth commentary about the design principles in the section about design principles.

Design Principles

1. Support sense making.

   Scaffold tasks and amplify language so students can make their own meaning.

   Students do not need to understand a language completely before they can start making sense of academic content and negotiate meaning in that language. Language learners of all levels can and should engage with grade-level content that is appropriately scaffolded. Students need multiple opportunities to talk about their mathematical thinking, negotiate meaning with others, and collaboratively solve problems with targeted guidance from the teacher. In addition, teachers can foster students' sense-making by amplifying rather than simplifying, or watering down, their own use of disciplinary language.

2. Optimize output.

   Strengthen the opportunities and supports for helping students to describe clearly their mathematical thinking to others, orally, visually, and in writing.

   Linguistic *output* is the language that students use to communicate their ideas to others. Output can come in various forms, such as oral, written, visual, etc. and refers to all forms of student linguistic expressions except those that include significant back-and-forth negotiation of ideas.

3. Cultivate conversation.

   Strengthen the opportunities and supports for constructive mathematical conversations (pairs, groups, and whole class).

   Conversations are back-and-forth interactions with multiple turns that build up ideas about math. Conversations act as scaffolds for students developing mathematical language because they provide opportunities to simultaneously make meaning and communicate that meaning. They also allow students to hear how other students express their understandings. When students have a reason or purpose to talk and listen to each other, interactive communication is more authentic.

4. Maximize meta-awareness

   Strengthen the "meta-" connections and distinctions between mathematical ideas, reasoning, and language.

   Language is a tool that not only allows students to communicate their math understanding to others, but also to organize their own experience, ideas, and learning for themselves. *Meta-awareness* is consciously thinking about one's own thought processes or language use. Meta-awareness develops when students and teachers engage in classroom activities or discussions that bring explicit attention to what students need to do to improve communication and reasoning about mathematical concepts.

Mathematical Language Routines

The mathematical language routines (MLRs) are designed to facilitate attention to student language in ways that support in-the-moment teacher-, peer-, and self- assessment for all learners and are especially helpful for English language learners. *Mathematical language routine* refers to a structured but adaptable format for amplifying, assessing, and developing students' language. These routines can be found throughout the curriculum.

MLR 1. Stronger and Clearer Each Time

The purpose of MLR 1 is to provide a structured and interactive opportunity for students to revise and refine both their ideas and their verbal and written output.

MLR 2. Collect and Display

The purpose of MLR 2 is to capture students' oral words and phrases into a stable, collective reference.

MLR 3. Critique, Correct, and Clarify

The purpose of MLR 3 is to give students a piece of mathematical writing that is not their own to analyze, reflect on, and develop

MLR 4. Information Gap

The purpose of MLR 4 is to create a need for students to communicate. This routine allows teachers to facilitate meaningful interactions by giving partners or team members different pieces of necessary information that must be used together to solve a problem or play a game.

MLR 5. Co-Craft Questions and Problems

The purpose of MLR 5 is to allow students to get inside of a context before feeling pressure to produce answers, and to create space for students to produce the language of mathematical questions themselves.

MLR 6. Three Reads

The purpose of MLR 6 is to ensure that students know what they are being asked to do, and to create an opportunity for students to reflect on the ways mathematical questions are presented.

MLR 7. Compare and Connect

The purpose of MLR 7 is to foster students’ meta-awareness as they identify, compare, and contrast different mathematical approaches, representations, and language.

MLR 8. Discussion Supports

The purpose of MLR 8 is to support rich discussions about mathematical ideas, representations, contexts, and strategies.

Many routines can support each of the design principles:

MLR 1	MLR 2	MLR 3	MLR 4	MLR 5	MLR 6	MLR 7	MLR 8
	x				x		x
x		x	x	x		x	
x		x	x	x		x	x
	x	x		x	x	x	x

Supporting Students with Disabilities

The philosophical stance that guided the creation of these materials is the belief that with proper structures, accommodations, and supports, all children can learn mathematics. Lessons are designed to maximize access for all students, and include additional suggested supports to meet the varying needs of individual students. While the suggested supports are designed for students with disabilities, they are also appropriate for many children who struggle to access rigorous, grade-level content. Teachers should use their professional judgment about which supports to use and when, based on their knowledge of the individual needs of students in their classroom.

Design Principles

These materials reflect three key design principles that support and engage all students in today’s diverse mathematics classrooms. The design principles and related supports work together to make each activity in each lesson accessible to all students.

Principle 1: Access for All

This foundational design principle draws from the Universal Design for Learning (UDL) framework, and shapes the instructional goals, recommended practices, lesson plans, and assessments to support a flexible approach to instruction, ensuring all students have an equitable opportunity to learn. For more information about Universal Design for Learning, visit http://www.udlcenter.org.

Principle 2: Presume Competence

All students are individuals who can learn, apply, and enjoy mathematics. The activities in these materials position students to capitalize on their existing abilities, and provide supports that eliminate potential barriers to learning when they arise. Each lesson is designed for wide range of ability, and all students are given access to grade-level problems. Student competence to engage with mathematical tasks should be assumed, with additional supports provided only when needed.

Principle 3: Strengths-based approach

All students, including students with disabilities, are resourceful and resilient members of the mathematics community. When the unique strengths and interests of students with disabilities are highlighted during class discussions, their contributions enhance the learning of all students in the classroom.

Design Elements for All Students

Each lesson is carefully designed to maximize engagement and accessibility for all students. Purposeful design elements that support all learners, but that are especially helpful for students with disabilities include:

Lesson Structures are Consistent

The structure of every lesson is the same: warm-up, activities, synthesis, cool-down. By keeping the components of each lesson similar from day to day, the flow of work in class becomes predictable for students. This reduces cognitive demand and enables students to focus on the mathematics at hand rather than the mechanics of the lesson.

Concepts Develop from Concrete to Abstract

Mathematical concepts are introduced simply, concretely, and repeatedly, with complexity and abstraction developing over time. Students begin with concrete examples, and transition to diagrams and tables before relying exclusively on symbols to represent the mathematics they encounter.

Individual to Pair or Small Group to Whole Class Progression

Providing students with time to think through a situation or question independently before engaging with others, allows students to carry the weight of learning, with supports arriving just in time from the community of learners. This progression allows students to first activate what they already know, and continue to build from this base with others.

Opportunities to Apply Mathematics to Real-World Contexts

Giving students opportunities to apply the mathematics they learn clarifies and deepens their understanding of core math concepts and skills and provides motivation and support. Mathematical modeling is a powerful activity for all students, but especially students with disabilities. Each unit has a culminating activity designed to explore, integrate, and apply all the big ideas of the unit. Centering instruction on these contextual situations can provide students with disabilities an anchor with which to base their mathematical understandings.

Supports for Students with Disabilities

The inclusion of additional supports for students with disabilities offers additional strategies for teachers to meet the individual needs of a diverse group of learners. Lesson and activity-level supports for students with disabilities are aligned to an area of cognitive functioning and are paired with a suggested strategy aimed to increase access and eliminate barriers. These lesson specific supports help students succeed with a specific activity without reducing the mathematical demand of the task. All of the supports can be used discreetly and are designed to be used as needed. Many of these supports that can be implemented throughout the academic year; for example, peer tutors can help build classroom culture, provide opportunities for teamwork, and build collaboration skills while also supporting those who struggle. Other supports should be faded out as students gain understanding and fluency with key ideas and procedures. Additional supports for students with disabilities are designed to address students' strengths and needs in the following areas of cognitive functioning, which are integral to learning mathematics (Addressing Accessibility project, Brodesky et al., 2002):

- Conceptual Processing includes perceptual reasoning, problem solving, and metacognition.
- Expressive & Receptive Language includes auditory and visual language processing and expression.
- Visual-Spatial Processing includes processing visual information and understanding relation in space (e.g., visual mathematical representations and geometric concepts).
- Executive Functioning includes organizational skills, attention, and focus.

- Memory includes working memory and short-term memory.
- Social-Emotional Functioning includes interpersonal skills and the cognitive comfort and safety required in order to take risks and make mistakes.
- Fine-motor Skills includes tasks that require small muscle movement and coordination such as manipulating objects (graphing, cutting, writing).

Suggestions for supports fall under the following categories:

Eliminate Barriers

Eliminate any barriers that students may encounter that prevent them from engaging with the important mathematical work of a lesson. This requires flexibility and attention to areas such as the physical environment of the classroom, access to tools, organization of lesson activities, and means of communication.

Processing Time

Increased time engaged in thinking and learning leads to mastery of grade level content for all students, including students with disabilities. Frequent switching between topics creates confusion and does not allow for content to deeply embed in the mind of the learner. Mathematical ideas and representations are carefully introduced in the materials in a gradual, purposeful way to establish a base of conceptual understanding. Some students may need additional time, which should be provided as required.

Peer Tutors

Develop peer tutors to help struggling students access content and solve problems. This support keeps all students engaged in the material by helping students who struggle and deepening the understanding of both the tutor and the tutee. For students with disabilities, peer tutor relationships with non-disabled peers can help them develop authentic, age-appropriate communication skills, and allow them to rely on a natural support while increasing independence.

Assistive Technology

Assistive technology can be a vital tool for students with learning disabilities, visual spatial needs, sensory integration, and students with autism. Assistive technology supports suggested in the materials are designed to either enhance or support learning, or to bypass unnecessary barriers. Physical manipulatives help students make connections between concrete ideas and abstract representations. Often, students with disabilities benefit from hands-on activities, which allow them to make sense of the problem at hand and communicate their own mathematical ideas and solutions.

Visual Aids

Visual aids such as images, diagrams, vocabulary anchor charts, color coding, or physical demonstrations, are suggested throughout the materials to support conceptual processing and language development. Many students with disabilities have working memory and processing challenges. Keeping visual aids visible on the board allows students to access them as needed, so that they can solve problems independently. Leaving visual aids on the board especially benefit students who struggle with working or short term memory issues.

Graphic Organizers

Word webs, Venn diagrams, tables, and other metacognitive visual supports provide structures that illustrate relationships between mathematical facts, concepts, words, or ideas. Graphic organizers can be used to support students with organizing thoughts and ideas, planning problem solving approaches, visualizing ideas, sequencing information, or comparing and contrasting ideas.

Brain Breaks

Brain breaks are short, structured, 2–3 minute movement breaks taken in between activities, or to break up a longer activity (approximately every 20–30 minutes during a class period). Brain breaks are a quick, effective way of refocusing and re-energizing the physical and mental state of students during a lesson. Brain breaks have also been shown to positively impact student concentration and stress levels, resulting in more time spent engaged in mathematical problem solving. This universal support is beneficial for all students, but especially those with ADHD.

Practice Problems

Each lesson includes an associated set of practice problems. Teachers may decide to assign practice problems for homework or for extra practice in class; they may decide to collect and score it or to provide students with answers ahead of time for self-assessment. It is up to teachers to decide which problems to assign (including assigning none at all).

The practice problem set associated with each lesson includes a few questions about the contents of that lesson, plus additional problems that review material from earlier in the unit and previous units. Distributed practice (revisiting the same content over time) is more effective than massed practice (a large amount of practice on one topic, but all at once).

Assessments

Learning Goals and Learning Targets

Learning Goals

Teacher-facing learning goals appear at the top of lesson plans. They describe, for a teacher audience, the mathematical and pedagogical goals of the lesson.

Student-facing learning goals appear in student materials at the beginning of each lesson and start with the word "Let's." They are intended to invite students into the work of that day without giving away too much and spoiling the problem-based instruction. They are suitable for writing on the board before class begins.

Learning Targets

These appear in student materials at the end of each unit. They describe, for a student audience, the mathematical goals of each lesson.

We do not recommend writing learning targets on the board before class begins, because doing so might spoil the problem-based instruction. (The student-facing learning goals (that start with "Let's") are more appropriate for this purpose.)

Teachers and students might use learning targets in a number of ways. Some examples include:

- targets for standards-based grading
- prompts for a written reflection as part of a lesson synthesis
- a study aid for self-assessment, review, or catching up after an absence from school

How to Assess Progress

The materials contain many opportunities and tools for both formative and summative assessment. Some things are purely formative, but the tools that can be used for summative assessment can also be used formatively.

- Each unit begins with a diagnostic assessment of concepts and skills that are prerequisite to the unit as well as a few items that assess what students already know of the key contexts and concepts that will be addressed by the unit.

- Each instructional task is accompanied by commentary about expected student responses and potential misconceptions so that teachers can adjust their instruction depending on what students are doing in response to the task. Often there are suggested questions to help teachers better understand students' thinking.

- Each lesson includes a cool-down (analogous to an exit slip or exit ticket) to assess whether students understood the work of that day's lesson. Teachers may use this as a formative assessment to provide feedback or to plan further instruction.

- A set of cumulative practice problems is provided for each lesson that can be used for homework or in-class practice. The teacher can choose to collect and grade these or simply provide feedback to students.

- Each unit includes an end-of-unit written assessment that is intended for students to complete individually to assess what they have learned at the conclusion of the unit. Longer units also include a mid-unit assessment. The mid-unit assessment states which lesson in the middle of the unit it is designed to follow.

Pre-Unit-Diagnostic Assessments

At the start of each unit is a *pre-unit diagnostic assessment*. These assessments vary in length. Most of the problems in the pre-unit diagnostic assessment address prerequisite concepts and skills for the unit. Teachers can use these problems to identify students with particular below-grade needs, or topics to carefully address during the unit. The pre-unit diagnostic assessment also may include problems that assess what students already know of the upcoming unit's key ideas, which teachers can use to pace or tune instruction; in rare cases, this may signal the opportunity to move more quickly through a topic to optimize instructional time.

What if a large number of students can't do the same pre-unit assessment problem? Teachers are encouraged to address below-grade skills while continuing to work through the on-grade tasks and concepts of each unit, instead of abandoning the current work in favor of material that only addresses below-grade skills. Look for opportunities within the upcoming unit where the target skill could be addressed in context. For example, an upcoming activity might require solving an equation in one variable. Some strategies might include:

- ask a student who can do the skill to present their method

- add additional questions to the warm-up with the purpose of revisiting the skill

- add to the activity launch a few related equations to solve, before students need to solve an equation while working on the activity

- pause the class while working on the activity to focus on the portion that requires solving an equation

Then, attend carefully to students as they work through the activity. If difficulty persists, add more opportunities to practice the skill, by adapting tasks or practice problems.

What if all students do really well on the pre-unit diagnostic assessment? That means they are ready for the work ahead, and special attention doesn't likely need to be paid to below-grade skills.

Cool-downs

Each lesson includes a cool-down (also known as an exit slip or exit ticket) to be given to students at the end of the lesson. This activity serves as a brief checkpoint to determine whether students understood the main concepts of that lesson. Teachers can use this as a formative assessment to plan further instruction.

What if the feedback from a cool-down suggests students haven't understood a key concept? Choose one or more of these strategies:

- Look at the next few lessons to see if students have more opportunities to engage with the same topic. If so, plan to focus on the topic in the context of the new activities.
- During the next lesson, display the work of a few students on that cool-down. Anonymize their names, but show some correct and incorrect work. Ask the class to observe some things each student did well and could have done better.
- Give each student brief, written feedback on their cool-down that asks a question that nudges them to re-examine their work. Ask students to revise and resubmit.
- Look for practice problems that are similar to, or involve the same topic as the cool-down, then assign those problems over the next few lessons.

Here is an example. For a lesson in grade 6, unit 2, the learning goals are

- Understand that doubling, tripling, or halving a recipe yields something that tastes the same.
- Understand that “doubling, tripling, or halving the recipe” means “doubling, tripling, or halving each ingredient.”

The cool-down reads:

> Usually when Elena makes bird food, she mixes 9 cups of seeds with 6 tablespoons of maple syrup. However, today she is short on ingredients. Think of a recipe that would yield a smaller batch of bird food but still taste the same. Explain or show your reasoning.

A number of students responded with 8 cups of seeds and 5 tablespoons of maple syrup, and did not provide an explanation or show their reasoning. Here are some possible strategies:

- Notice that this lesson is the first of several that familiarize students with contexts where equivalent ratios carry physical meaning, for example, the taste of a recipe or the result of mixing paint colors. Over the next several lessons, there are more opportunities to reason about multiple batches of a recipe. When launching these activities, pause to assist students to interpret this correctly. Highlight the strategies of any students who use a discrete diagram or other representation to correctly represent multiple batches.
- Select the work of one student who answered correctly and one student whose work had the common error. In the next class, display these together for all to see (hide the students' names). Ask each student to decide which interpretation is correct, and defend their choice to their partner. Select students to share their reasoning with the class who have different ways of representing that $9 : 6$ is equivalent to $3 : 2$, $6 : 4$, or $4\frac{1}{2} : 3$.
- Write feedback for each student along the lines of "If this recipe is 3 batches, how could you make 1 batch?" Allow students to revise and resubmit their work.
- Look for practice problems in upcoming lessons that require students to generate examples of different numbers of batches equivalent to a given ratio, and be sure to assign those problems.

End-of-Unit Assessments

At the end of each unit is the *end-of-unit assessment*. These assessments have a specific length and breadth, with problem types that are intended to gauge students' understanding of the key concepts of the unit while also preparing students for new-generation standardized exams. Problem types include multiple-choice, multiple response, short answer, restricted constructed response, and extended response. Problems vary in difficulty and depth of knowledge.

Teachers may choose to grade these assessments in a standardized fashion, but may also choose to grade more formatively by asking students to show and explain their work on all problems. Teachers may also decide to make changes to the provided assessments to better suit their needs. If making changes, teachers are encouraged to keep the format of problem types provided, which helps students know what to expect and ensures each assessment will take approximately the same amount of time.

In longer units, a *mid-unit assessment* is also available. This assessment has the same form and structure as an end-of-unit assessment. In longer units, the end-of-unit assessment will include the breadth of all content for the full unit, with emphasis on the content from the second half of the unit.

All summative assessment problems include a complete solution and standard alignment. Multiple-choice and multiple response problems often include a reason for each potential error a student might make. Restricted constructed response and extended response items include a rubric.

Unlike formative assessments, problems on summative assessments generally do not prescribe a method of solution.

Design Principles for Summative Assessments

Students should get the correct answer on assessment problems for the right reasons, and get incorrect answers for the right reasons. To help with this, our assessment problems are targeted and short, use consistent, positive wording, and have clear, undebatable correct responses.

In multiple choice problems, distractors are common errors and misconceptions directly relating to what is being assessed, since problems are intended to test whether the student has proficiency on a specific skill. The distractors serve as a diagnostic, giving teachers the chance to quickly see which of the most common errors are being made. There are no “trick” questions, and the phrases “all of the above” and “none of the above” are never used, since they do not give useful information about the methods a student used.

Multiple response prompts always include the phrase “select **all**” to clearly indicate their type. Each part of a multiple response problem addresses a different piece of the same overall skill, again serving as a diagnostic for teachers to understand which common errors students are making.

Short answer, restricted constructed response, and extended response problems are careful to avoid the “double whammy” effect, where a part of the problem asks for students to use correct work from a previous part. This choice is made to ensure that students have all possible opportunities to show proficiency on assessments.

When possible, extended response problems provide multiple ways for students to demonstrate understanding of the content being assessed, through some combination of arithmetic or algebra, use of representations (tables, graphs, diagrams, expressions, and equations) and explanation.

Rubrics for Evaluating Student Answers

Restricted constructed response and extended response items have rubrics that can be used to evaluate the level of student responses.

Restricted Constructed Response

- *Tier 1 response*: Work is complete and correct.
- *Tier 2 response*: Work shows general conceptual understanding and mastery, with some errors.
- *Tier 3 response*: Significant errors in work demonstrate lack of conceptual understanding or mastery. Two or more error types from Tier 2 responsecan be given as the reason for a Tier 3 response instead of listing combinations.

Extended Response

- *Tier 1 response*: Work is complete and correct, with complete explanation or justification.
- *Tier 2 response*: Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.
- *Tier 3 response*: Work shows a developing but incomplete conceptual understanding, with significant errors.
- *Tier 4 response*: Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.

Typically, sample errors are included. Acceptable errors can be listed at any Tier (as an additional bullet point), notably Tier 1, to specify exclusions.

Mathematical Modeling Prompts

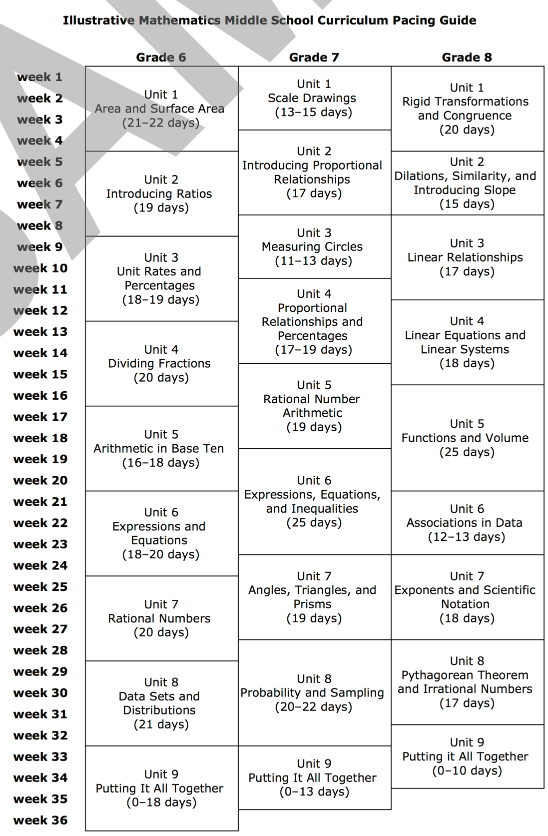
Each unit has a culminating lesson where student have an opportunity to show off their problem-solving skills or apply the mathematics they have learned to a real-world problem. The end unit assessments, combined with students' work on the culminating lessons, will show a multi-faceted view of students' learning over the course of the unit.

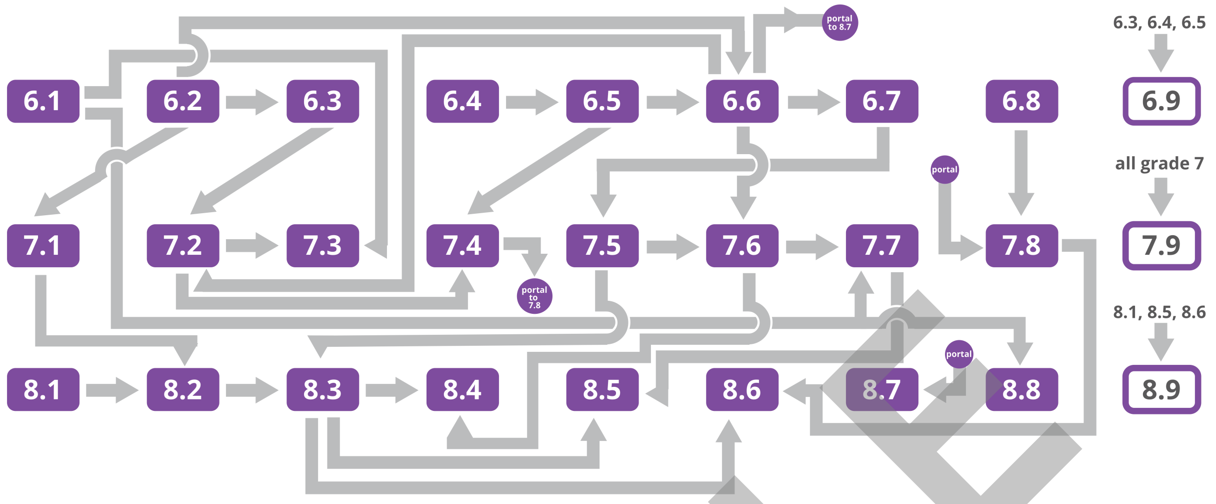
Scope and sequence - Grade 6

Course Overview

Grade 6 begins with a unit on reasoning about area and understanding and applying concepts of surface area. It is common to begin the year by reviewing the arithmetic learned in previous grades, but starting instead with a mathematical idea that students haven't seen before sets up opportunities for students to surprise the teacher and themselves with the connections they make. Instead of front-loading review and practice from prior grades, these materials incorporate opportunities to practice elementary arithmetic concepts and skills through warm-ups, in the context of instructional tasks, and in practice problems as they are reinforcing the concepts they are learning in the unit.

One of the design principles of these materials is that students should encounter plenty of examples of a mathematical or statistical idea in various contexts before that idea is named and studied as an object in its own right. For example, in the first unit, students will generalize arithmetic by writing simple expressions like $\frac{1}{2}bh$ and $6s^2$ before they study algebraic expressions as a class of objects in the sixth unit. Sometimes this principle is put into play several units before a concept is developed more fully, and sometimes in the first several lessons of a unit, where students have a chance to explore ideas informally and concretely, building toward a more formal and abstract understanding later in the unit.

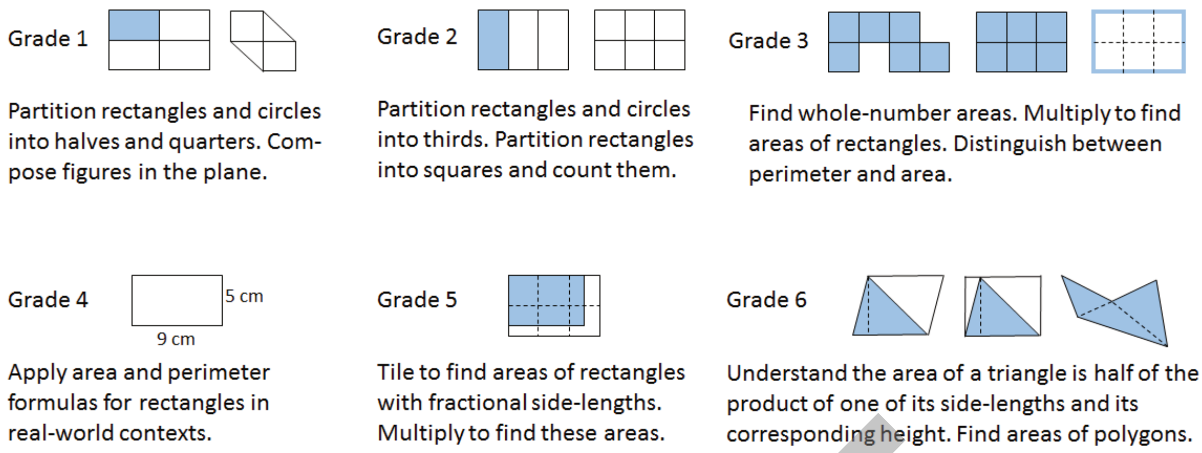




In the unit dependency chart, an arrow indicates that a particular unit is designed for students who already know the material in a previous unit. Reversing the order would have a negative effect on mathematical or pedagogical coherence. For example, there is an arrow from 6.2 to 6.6, because students are expected to use their knowledge of contexts involving ratios (from 6.2) to write and solve equations representing such contexts (in 6.6).

Area and Surface Area

Work with area in grade 6 draws on earlier work with geometry and geometric measurement. Students began to learn about two- and three-dimensional shapes in kindergarten, and continued this work in grades 1 and 2, composing, decomposing, and identifying shapes. Students' work with geometric measurement began with length and continued with area. Students learned to "structure two-dimensional space," that is, to see a rectangle with whole-number side lengths as composed of an array of unit squares or composed of iterated rows or iterated columns of unit squares. In grade 3, students distinguished between perimeter and area. They connected rectangle area with multiplication, understanding why (for whole-number side lengths) multiplying the side lengths of a rectangle yields the number of unit squares that tile the rectangle. They used area diagrams to represent instances of the distributive property. In grade 4, students applied area and perimeter formulas for rectangles to solve real-world and mathematical problems, and learned to use protractors. In grade 5, students extended the formula for the area of rectangles to rectangles with fractional side lengths.



In grade 6, students extend their reasoning about area to include shapes that are not composed of rectangles. Doing this draws on abilities developed in earlier grades to compose and decompose shapes, for example, to see a rectangle as composed of two congruent right triangles. Through activities designed and sequenced to allow students to make sense of problems and persevere in solving them (MP1), students build on these abilities and their knowledge of areas of rectangles to find the areas of polygons by decomposing and rearranging them to make figures whose areas they can determine (MP7). They learn strategies for finding areas of parallelograms and triangles, and use regularity in repeated reasoning (MP8) to develop formulas for these areas, using geometric properties to justify the correctness of these formulas. They use these formulas to solve problems. They understand that any polygon can be decomposed into triangles, and use this knowledge to find areas of polygons. Students find the surface areas of polyhedra with triangular and rectangular surfaces. They study, assemble, and draw nets for polyhedra and use nets to determine surface areas. Throughout, they discuss their mathematical ideas and respond to the ideas of others (MP3, MP6).

Because grade 6 students will be writing algebraic expressions and equations involving the letter x and x is easily confused with ×, these materials use the “dot” notation, e.g., $2 \cdot 3$, for multiplication instead of the “cross” notation, e.g., 2×3. The dot notation will be new for many students, and they will need explicit guidance in using it.

Many of the lessons in this unit ask students to work on geometric figures that are not set in a real-world context. This design choice respects the significant intellectual work of reasoning about area. Tasks set in real-world contexts that involve areas of polygons are often contrived and hinder rather than help understanding. Moreover, mathematical contexts are legitimate contexts that are worthy of study. Students do have an opportunity at the end of the unit to tackle a real-world application (MP2, MP4).

In grade 6, students are likely to need physical tools in order to check that one figure is an identical copy of another where “identical copy” is defined as follows:

> One figure is an *identical copy* of another if one can be placed on top of the other so that they match up exactly.

In grade 8, students will understand “identical copy of” as “congruent to” and understand congruence in terms of rigid motions, that is, motions such as reflection, rotation, and translation. In grade 6, students do not have any way to check for congruence except by inspection, but it is not practical to cut out and stack every pair of figures one sees. Tracing paper is an excellent tool for verifying that figures “match up exactly,” and students should have access to this and other tools at all times in this unit. Thus, each lesson plan suggests that each student should have access to a *geometry toolkit*, which contains tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles. Providing students with these toolkits gives opportunities for students to develop abilities to select appropriate tools and use them strategically to solve problems (MP5). Note that even students in a digitally enhanced classroom should have access to such tools; apps and simulations should be considered additions to their toolkits, not replacements for physical tools. In this grade, all figures are drawn and labeled so that figures that look congruent actually are congruent; in later grades when students have the tools to reason about geometric figures more precisely, they will need to learn that visual inspection is not sufficient for determining congruence. Also note that all arguments laid out in this unit can (and should) be made more precise in later grades, as students' geometric understanding deepens.

Introducing Ratios

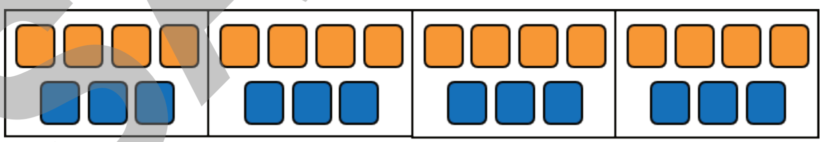
Work with ratios in grade 6 draws on earlier work with numbers and operations. In elementary school, students worked to understand, represent, and solve arithmetic problems involving quantities with the same units. In grade 4, students began to use two-column tables, e.g., to record conversions between measurements in inches and yards. In grade 5, they began to plot points on the coordinate plane, building on their work with length and area. These early experiences were a brief introduction to two key representations used to study relationships between quantities, a major focus of work that begins in grade 6 with the study of ratios.

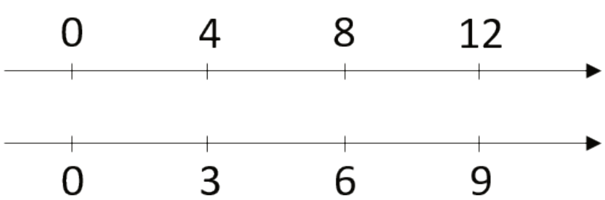
Starting in grade 3, students worked with relationships that can be expressed in terms of ratios and rates (e.g., conversions between measurements in inches and in yards), however, they did not use these terms. In grade 4, students studied multiplicative comparison. In grade 5, they began to interpret multiplication as scaling, preparing them to think about simultaneously scaling two quantities by the same factor. They learned what it means to divide one whole number by another, so they are well equipped to consider the quotients $\frac{a}{b}$ and $\frac{b}{a}$ associated with a ratio $a : b$ for non-zero whole numbers a and b.

In this unit, students learn that a ratio is an association between two quantities, e.g., “1 teaspoon of drink mix to 2 cups of water.” Students analyze contexts that are often expressed in terms of ratios, such as recipes, mixtures of different paint colors, constant speed (an association of time measurements with distance measurements), and uniform pricing (an association of item amounts with prices).

One of the principles that guided the development of these materials is that students should encounter examples of a mathematical concept in various contexts before the concept is named and studied as an object in its own right. The development of ratios, equivalent ratios, and unit rates in this unit and the next unit is in accordance with that principle. In this unit, equivalent ratios are first encountered in terms of multiple batches of a recipe and “equivalent” is first used to describe a perceivable sameness of two ratios, for example, two mixtures of drink mix and water taste the same or two mixtures of red and blue paint are the same shade of purple. Building on these experiences, students analyze situations involving both discrete and continuous quantities, and involving ratios of quantities with units that are the same and that are different. Several lessons later, *equivalent* acquires a more precise meaning (MP6): All ratios that are equivalent to $a : b$ can be made by multiplying both a and b by the same non-zero number (note that students are not yet considering negative numbers).

This unit introduces *discrete diagrams* and *double number line diagrams*, representations that students use to support thinking about equivalent ratios before their work with tables of equivalent ratios.





Initially, discrete diagrams are used because they are similar to the kinds of diagrams students might have used to represent multiplication in earlier grades. Next come double number line diagrams. These can be drawn more quickly than discrete diagrams, but are more similar to tables while allowing reasoning based on the lengths of intervals on the number lines. After some work with double number line diagrams, students use tables to represent equivalent ratios. Because equivalent pairs of ratios can be written in any order in a table and there is no need to attend to the distance between values, tables are the most flexible and concise of the three representations for equivalent ratios, but they are also the most abstract. Use of tables to represent equivalent ratios is an important stepping stone toward use of tables to represent linear and other functional relationships in grade 8 and beyond. Because of this, students should learn to use tables to solve all kinds of ratio problems, but they should always have the option of using discrete diagrams and double number line diagrams to support their thinking.

When a ratio involves two quantities with the same units, we can ask and answer questions about ratios of each quantity and the total of the two. Such ratios are sometimes called "part-part-whole" ratios and are often used to introduce ratio work. However, students often struggle with them so, in this unit, the study of part-part-whole ratios occurs at the end. (Note that tape diagrams are reserved for ratios in which all quantities have the same units.) The major use of part-part-whole ratios occurs with certain kinds of percentage problems, which comes in the next unit.

On using the terms ratio, rate, and proportion. In these materials, a *quantity* is a measurement that is or can be specified by a number and a unit, e.g., 4 oranges, 4 centimeters, "my height in feet," or "my height" (with the understanding that a unit of measurement will need to be chosen). The term *ratio* is used to mean an association between two or more quantities and the fractions $\frac{a}{b}$ and $\frac{b}{a}$ are never called ratios. Ratios of the form $1 : \frac{b}{a}$ or $\frac{a}{b} : 1$ (which are equivalent to $a : b$) are highlighted as useful but $\frac{a}{b}$ and $\frac{b}{a}$ are not identified as *unit rates* for the ratio $a : b$ until the next unit. However, the meanings of these fractions in contexts is very carefully developed. The word "per" is used with students in interpreting a unit rate in context, as in "$3 per ounce," and "at the same rate" is used to signify a situation characterized by equivalent ratios.

In the next unit, students learn the term "unit rate" and that if two ratios $a : b$ and $c : d$ are equivalent, then the unit rates $\frac{a}{b}$ and $\frac{c}{d}$ are equal.

The terms *proportion* and *proportional relationship* are not used anywhere in the grade 6 materials. A proportional relationship is a collection of equivalent ratios, and such collections are objects of study in grade 7. In high school—after their study of ratios, rates, and proportional relationships—students discard the term "unit rate," referring to a to b, $a : b$, and $\frac{a}{b}$ as "ratios."

Unit Rates and Percentages

In the previous unit, students began to develop an understanding of ratios and rates. They started to describe situations using terms such as "ratio," "rate," "equivalent ratios," "per," "constant speed," and "constant rate" (MP6). They understood specific instances of the idea that $a : b$ is equivalent to every other ratio of the form $sa : sb$, where s is a positive number. They learned that "at this rate" or "at the same rate" signals a situation that is characterized by equivalent ratios. Although the usefulness of ratios of the form $\frac{a}{b} : 1$ and $1 : \frac{b}{a}$ was highlighted, the term "unit rate" was not introduced.

In this unit, students find the two values $\frac{a}{b}$ and $\frac{b}{a}$ that are associated with the ratio $a : b$, and interpret them as rates per 1. For example, if a person walks 13 meters in 10 seconds at a constant rate, that means they walked at a speed of $\frac{13}{10}$ meters per 1 second and a pace of $\frac{10}{13}$ seconds per 1 meter.

Students learn that one of the two values ($\frac{a}{b}$ or $\frac{b}{a}$) may be more useful than the other in reasoning about a given situation. They find and use rates per 1 to solve problems set in contexts (MP2), attending to units and specifying units in their answers. For example, given item amounts and their costs, which is the better deal? Or given distances and times, which object is moving faster? Measurement conversions provide other opportunities to use rates.

Students observe that if two ratios $a : b$ and $c : d$ are equivalent, then $\frac{a}{b} = \frac{c}{d}$. The values $\frac{a}{b}$ and $\frac{c}{d}$ are called *unit rates* because they can be interpreted in the context from which they arose as rates per unit. Students note that in a table of equivalent ratios, the entries in one column are produced by multiplying a unit rate by the corresponding entries in the other column. Students learn that "percent" means "per 100" and indicates a rate. Just as a unit rate can be interpreted in context as a rate per 1, a percentage can be interpreted in the context from which it arose as a rate per 100. For example, suppose a beverage is made by mixing 1 cup of juice with 9 cups of water. The *percentage* of juice in 20 cups of the beverage is 2 cups and 10 *percent* of the beverage is juice. Interpreting the 10 as a rate: "there are 10 cups of juice per 100 cups of beverage" or, more generally, "there are 10 units of juice per 100 units of beverage." The percentage—and the rate—indicate equivalent ratios of juice to beverage, e.g., 2 cups to 20 cups and 10 cups to 100 cups.

In this unit, tables and double number line diagrams are intended to help students connect percentages with equivalent ratios, and reinforce an understanding of percentages as rates per 100. Students should internalize the meaning of important benchmark percentages, for example, they should connect "75% of a number" with "$\frac{3}{4}$

times a number” and “0.75 times a number.” Note that 75% (“seventy-five per hundred”) does not represent a fraction or decimal (which are numbers), but that “75% of a number” is calculated as a *fraction of* or a *decimal times* the number.

Work done in grades 4 and 5 supports learning about the concept of a percentage. In grade 5, students understand why multiplying a given number by a fraction less than 1 results in a product that is less than the original number, and why multiplying a given number by a fraction greater than 1 results in a product that is greater than the original number. This understanding of multiplication as scaling comes into play as students interpret, for example,

- 35% of 2 cups of juice as $\frac{35}{100} \cdot 2$ cups of juice.
- 250% of 2 cups of juice as $\frac{250}{100} \cdot 2$ cups of juice.

Dividing Fractions

Work with fractions in grade 6 draws on earlier work in operations and algebraic thinking, particularly the knowledge of multiplicative situations developed in grades 3 to 5, and making use of the relationship between multiplication and division. Multiplicative situations include three types: equal groups; comparisons of two quantities; dimensions of arrays or rectangles. In the equal groups and comparison situations, there are two subtypes, sometimes called the partitive and the quotitive (or measurement) interpretations of division. Students are not expected to identify the three types of situations or use the terms “partitive” or “quotitive.” However, they should recognize the associated interpretations of division in specific contexts (MP7).

For example, in an equal groups situation when the group size is unknown, division can be used to answer the question, “How many in each group?” If the number of groups is unknown, division answers the question, “How many groups?” For example, if 12 pounds of almonds are equally shared among several bags:

> There are 2 bags. How many pounds in each bag? (partitive)
>
> There are 6 pounds in each bag. How many bags? (quotitive)

In a comparison situation that involves division, the size of one object may be unknown or the relative sizes of two objects may be unknown. For example, when comparing two ropes:

> A rope is 12 feet long. It is twice as long as another rope. How long is the second rope? (partitive)

> One rope is 12 feet long. One rope is 6 feet long. How many times longer than the second rope is the first rope? (quotitive)

In situations that involve arrays or rectangles, division can be used to find an unknown factor. In an array situation, the unknown is the number of entries in a row or a column; in a rectangle, the unknown is a length or a width measurement. For example, "The area of a rectangle is 12 square feet. One side is 6 feet long. How long is the other side?" If the rectangle is viewed as tiled by an array of 12 unit squares with 6 tiles in each row, this question can seen as asking for the number of entries in each column.

At beginning of the unit, students consider how the relative sizes of numerator and denominator affect the size of their quotient. Students first compute quotients of whole numbers, then—without computing—consider the relative magnitudes of quotients that include divisors which are whole numbers, fractions, or decimals, e.g., "Is $800 \div \frac{1}{10}$ larger than or smaller than $800 \div 2.5$?"

The second section of the unit focuses on equal groups and comparison situations. It begins with partitive and quotitive situations that involve whole numbers, represented by tape diagrams and equations. Students interpret the numbers in the two situations (MP2) and consider analogous situations that involve one or more fractions, again accompanied by tape diagrams and equations. Students learn to interpret, represent, and describe these situations, using terminology such as "What fraction of 6 is 2?," "How many 3s are in 12?," "How many fourths are in 3?," "is one-third as long as," "is two-thirds as much as," and "is one-and-one-half times the size of."

The third section concerns computing quotients of fractions. Students build on their work from the previous section by considering quotients related to products of numbers and unit fractions, e.g., "How many 3s in 12?" and "What is $\frac{1}{3}$ of 12?," to establish that dividing by a unit fraction $\frac{1}{b}$ is the same as multiplying by its reciprocal b. Building on this and their understanding that $\frac{a}{b} = a \cdot \frac{1}{b}$ (from grade 4), students understand that dividing by a fraction $\frac{a}{b}$ is the same as multiplying by its reciprocal $\frac{b}{a}$.

The fourth section returns to interpretations of division in situations that involve fractions. This time, the focus is on using division to find an unknown area or volume measurement. In grade 3, students connected areas of rectangles with multiplication, viewing a rectangle as tiled by an array of unit squares and understanding that, for whole-number side lengths, multiplying the side lengths yields the number of unit squares that tile the rectangle. In grade 5, students extended the formula for the area of rectangles with whole-number side lengths to rectangles with fractional side lengths. For example, they viewed a $\frac{2}{3}$-by-$\frac{5}{7}$ rectangle as tiled by 10 $\frac{1}{3}$-by-$\frac{1}{7}$ rectangles, reasoning that 21 such rectangles compose 1 square unit, so the area of one such rectangle is $\frac{1}{21}$, thus the area of a shape composed of 10 such rectangles is $\frac{10}{21}$. In a previous grade 6 unit, students used their familiarity with this formula to develop formulas for areas of triangles and parallelograms. In this unit, they return to this formula, using their understanding of it to extend the formula for the volume of a right rectangular prism (developed in grade 5) to right rectangular prisms with fractional side lengths.

The unit ends with two lessons in which students use what they have learned about working with fractions (including the volume formula) to solve problems set in real-world contexts, including a multi-step problem about calculating shipping costs. These require students to formulate appropriate equations that use the four operations or draw diagrams, and to interpret results of calculations in the contexts from which they arose (MP2).

Arithmetic in Base Ten

By the end of grade 5, students learn to use efficient algorithms to fluently calculate sums, differences, and products of multi-digit whole numbers. They calculate quotients of multi-digit whole numbers with up to four-digit dividends and two-digit divisors. These calculations use strategies based on place value, the properties of operations, and the relationship between multiplication and division. Grade 5 students illustrate and explain these calculations with equations, rectangular arrays, and area diagrams.

In grade 5, students also calculate sums, differences, products, and quotients of decimals to hundredths, using concrete representations or drawings, and strategies based on place value, properties of operations, and the relationship between addition and subtraction. They connect their strategies to written methods and explain their reasoning.

In this unit, students learn an efficient algorithm for division and extend their use of other base-ten algorithms to decimals of arbitrary length. Because these algorithms rely on the structure of the base-ten system, students build on the understanding of place value and the properties of operations developed during earlier grades (MP7).

The unit begins with a lesson that revisits sums and differences of decimals to hundredths, and products of a decimal and whole number. The tasks are set in the context of shopping and budgeting, allowing students to be reminded of appropriate magnitudes for results of calculations with decimals.

The next section focuses on extending algorithms for addition, subtraction, and multiplication, which students used with whole numbers in earlier grades, to decimals of arbitrary length.

Students begin by using “base-ten diagrams,” diagrams analogous to base-ten blocks for ones, tens, and hundreds. These diagrams show, for example, ones as large squares, tenths as rectangles, hundredths as medium squares, thousandths as small rectangles, and ten-thousandths as small squares. These are designed so that the area of a figure that represents a base-ten unit is one tenth of the area of the figure that represents the base-ten unit of next highest value. Thus, a group of 10 figures that represent 10 like base-ten units can be replaced by a figure whose area is the sum of the areas of the 10 figures.

Students first calculate sums of two decimals by representing each number as a base-ten diagram, combining representations of like base-ten units and replacing representations of 10 like units by a representation of the unit of next highest value, e.g., 10 rectangles compose 1 large square. Next, they examine “vertical calculations,” representations of calculations with symbols that show one summand above the other, with the sum written below. They check each vertical calculation by representing it with base-ten diagrams. This is followed by a similar lesson on subtraction of decimals. The section concludes with a lesson designed to illustrate efficient algorithms and their advantages, and to promote their use.

The third section, multiplication of decimals, begins by asking students to estimate products of a whole number and a decimal, allowing students to be reminded of appropriate magnitudes for results of calculations with decimals. In this section, students extend their use of efficient algorithms for multiplication from whole numbers to decimals. They begin by writing products of decimals as products of fractions, calculating the product of the fractions, then writing the product as a decimal. They discuss the effect of multiplying by powers of 0.1, noting that multiplying by 0.1 has the same effect as dividing by 10. Students use area diagrams to represent products of decimals. The efficient multiplication algorithms are introduced and students use them, initially supported by area diagrams.

In the fourth section, students learn long division. They begin with quotients of whole numbers, first representing these quotients with base-ten diagrams, then proceeding to efficient algorithms, initially supporting their use with base-ten diagrams. Students then tackle quotients of whole numbers that result in decimals, quotients of decimals and whole numbers, and finally quotients of decimals.

The unit ends with two lessons in which students use calculations with decimals to solve problems set in real-world contexts. These require students to interpret diagrams, and to interpret results of calculations in the contexts from which they arose (MP2). The second lesson draws on work with geometry and ratios from previous units. Students fold papers of different sizes to make origami boxes of different dimensions, then compare the lengths, widths, heights, and surface areas of the boxes.

Expressions and Equations

Students begin the unit by working with linear equations that have single occurrences of one variable, e.g., $x + 1 = 5$ and $4x = 2$. They represent relationships with tape diagrams and with linear equations, explaining correspondences between these representations. They examine values that make a given linear equation true or false, and what it means for a number to be a solution to an equation. Solving equations of the form $px = q$ where p and q are rational numbers can produce complex fractions (i.e., quotients of fractions), so students extend their understanding of fractions to include those with numerators and denominators that are not whole numbers.

The second section introduces balanced and unbalanced “hanger diagrams” as a way to reason about solving the linear equations of the first section. Students write linear equations to represent situations, including situations with percentages, solve the equations, and interpret the solutions in the original contexts (MP2), specifying units of measurement when appropriate (MP6). They represent linear expressions with tape diagrams and use the diagrams to identify values of variables for which two linear expressions are equal. Students write linear expressions such as $6w - 24$ and $6(w - 4)$ and represent them with area diagrams, noting the connection with the distributive property (MP7). They use the distributive property to write equivalent expressions.

In the third section of the unit, students write expressions with whole-number exponents and whole-number, fraction, or variable bases. They evaluate such expressions, using properties of exponents strategically (MP5). They understand that a solution to an equation in one variable is a number that makes the equation true when the number is substituted for all instances of the variable. They represent algebraic expressions and equations in order to solve problems. They determine whether pairs of numerical exponential expressions are equivalent and explain their reasoning (MP3). By examining a list of values, they find solutions for simple exponential equations of the form $a = b^x$, e.g., $2^x = 32$, and simple quadratic and cubic equations, e.g., $64 = x^3$.

In the last section of the unit, students represent collections of equivalent ratios as equations. They use and make connections between tables, graphs, and linear equations that represent the same relationships (MP1).

Rational Numbers

In this unit, students are introduced to signed numbers and plot points in all four quadrants of the coordinate plane for the first time. They work with simple inequalities in one variable and learn to understand and use “common factor,” “greatest common factor,” “common multiple,” and “least common multiple.”

The first section of the unit introduces signed numbers. Students begin by considering examples of positive and negative temperatures, plotting each temperature on a vertical number line on which 0 is the only label. Next, they consider examples of positive and negative numbers used to denote height relative to sea level. In the second lesson, they plot positive and negative numbers on horizontal number lines, including “opposites”—pairs of numbers that are the same distance from zero. They use “less than,” “greater than,” and the corresponding symbols to describe the relationship of two signed numbers, noticing correspondences between the relative positions of two numbers on the number line and statements that use these symbols, e.g., $0.8 > \text{-}1.3$ means that 0.8 is to the right of -1.3 on the number line. Students learn that the sign of a number indicates whether the number is positive or negative, and that zero has no sign. They learn that the absolute value of a number is its distance from zero, how to use absolute value notation, and that opposites have the same absolute value because they have the same distance from zero.

Previously, when students worked only with non-negative numbers, magnitude and order were indistinguishable: if one number was greater than another, then on the number line it was always to the right of the other number *and* always farther from zero. In comparing two signed numbers, students distinguish between magnitude (the absolute value of a number) and order (relative position on the number line), distinguishing between “greater than” and “greater absolute value,” and “less than” and “smaller absolute value.”

Students examine opposites of numbers, noticing that the opposite of a negative number is positive.

The second section of the unit concerns inequalities. Students graph simple inequalities in one variable on the number line, using a circle or disk to indicate when a given point is, respectively, excluded or included. In these materials, inequality symbols in grade 6 are limited to < and > rather than $\leq$ and $\geq$. However, in this unit students encounter situations when they need to represent statements such as $2 < x$ or $2 = x$.

Students represent situations that involve inequalities, symbolically and with the number line, understanding that there may be infinitely many solutions for an inequality. They interpret and graph solutions in contexts (MP2), understanding that some results do not make sense in some contexts, and thus the graph of a solution might be different from the graph of the related symbolic inequality. For example, the graph describing the situation "A fishing boat can hold fewer than 9 people" omits values other than the whole numbers from 0 to 8, but the graph of $x < 8$ includes all numbers less than 8. Students encounter situations that require more than one inequality statement to describe, e.g., "It rained for more than 10 minutes but less than 30 minutes" ($t > 10$ and $t < 30$, where t is the amount of time that it rained in minutes) but which can be described by one number line graph.

The third section of the unit focuses on the coordinate plane. In grade 5, students learned to plot points in the coordinate plane, but they worked only with non-negative numbers, thus plotted points only in the first quadrant. In a previous unit, students again worked in the first quadrant of the coordinate plane, plotting points to represent ratio and other relationships between two quantities with positive values. In this unit, students work in all four quadrants of the coordinate plane, plotting pairs of signed number coordinates in the plane. They understand that for a given data set, there are more and less strategic choices for the scale and extent of a set of axes. They understand the correspondence between the signs of a pair of coordinates and the quadrant of the corresponding point. They interpret the meanings of plotted points in given contexts (MP2), and use coordinates to calculate horizontal and vertical distances between two points.

The last section of the unit returns to consideration of whole numbers. In the first lesson, students are introduced to "common factor" and "greatest common factor," and solve problems that illustrate how the greatest common factor of two numbers can be used in real-world situations, e.g., determining the largest rectangular tile with whole-number dimensions that can tile a given rectangle with whole-number dimensions. The second lesson introduces "common multiple" and "least common multiple," and students solve problems that involve listing common multiples or identifying common multiples of two or more numbers. In the third and last lesson, students solve problems that revisit situations similar to those in the first two lessons and identify which of the new concepts is involved in each problem. This lesson includes two optional classroom activities.

Data Sets and Distributions

In this unit, students learn about populations and study variables associated with a population. They understand and use the terms "numerical data," "categorical data," "survey" (as noun and verb), "statistical question," "variability," "distribution," and "frequency." They make and interpret histograms, bar graphs, tables of frequencies, and box plots. They describe distributions (shown on graphical displays) using terms such as "symmetrical," "peaks," "gaps," and "clusters." They work with measures of center—understanding and using the terms "mean," "average," and "median." They work with measures of variability—understanding and using the terms "range," "mean absolute deviation” or MAD, “quartile,” and “interquartile range” or IQR. They interpret measurements of center and variability in contexts.

Putting it All Together

This optional unit consists of six lessons. Each of the first three lessons is independent of the others, requiring only the mathematics of the previous units. The last three lessons build on each other.

The first lesson concerns Fermi problems—problems that require making rough estimates for quantities that are difficult or impossible to measure directly (MP4). The three problems in this lesson involve measurement conversion and calculation of volumes and surface areas of three-dimensional figures or the relationship of distance, rate, and time.

The second lesson involves finding approximately equivalent ratios for groups from two populations, one very large (the population of the world) and one comparatively small (a 30-student class). Students work with percent rates that describe subgroups of the world population, e.g., about 59% of the world population lives in Asia. Using these rates, which include numbers expressed in the form $\frac{a}{b}$ or as decimals, they determine, for example, the number of students who would live in Asia—“if our class were the world” (MP2). Because students choose their own methods to determine these numbers, possibly making strategic use of benchmark percentages or spreadsheets (MP5), there is an opportunity for them to see correspondences between approaches (MP1). Because the size of the world population and its subgroups are estimates, and because pairs of values in ratios may both be whole numbers, considerations of accuracy may arise (MP6).

The third lesson is an exploration of the relationship between the greatest common factor of two numbers, continued fractions, and decomposition of rectangles with whole-number side lengths, providing students an opportunity to perceive this relationship through repeated reasoning (MP8) and to see correspondences between two kinds of numerical relationships, and between numerical and geometric relationships (MP1).

The remaining three lessons explore the mathematics of voting (MP2, MP4). In some activities, students chose how to assign votes and justify their choices (MP3). The first of these lessons focuses on proportions of voters and votes cast in elections in which there are two choices. It requires only the mathematics of the previous units, in particular, equivalent ratios, part–part ratios, percentages, unit rates, and, in the final activity, the concept of area. The second of these lessons focuses on methods for voting when there are more than two choices: plurality, runoff, and instant runoff. They compute percentages, finding that different voting methods have different outcomes. The third of these lessons focuses on representation in the case when voters have two choices. It’s not always possible to have the same number of constituents per representative. How can we fairly share a small number of representatives? Students again compute percentages to find outcomes.

Scope and sequence - Grade 7

Course Overview

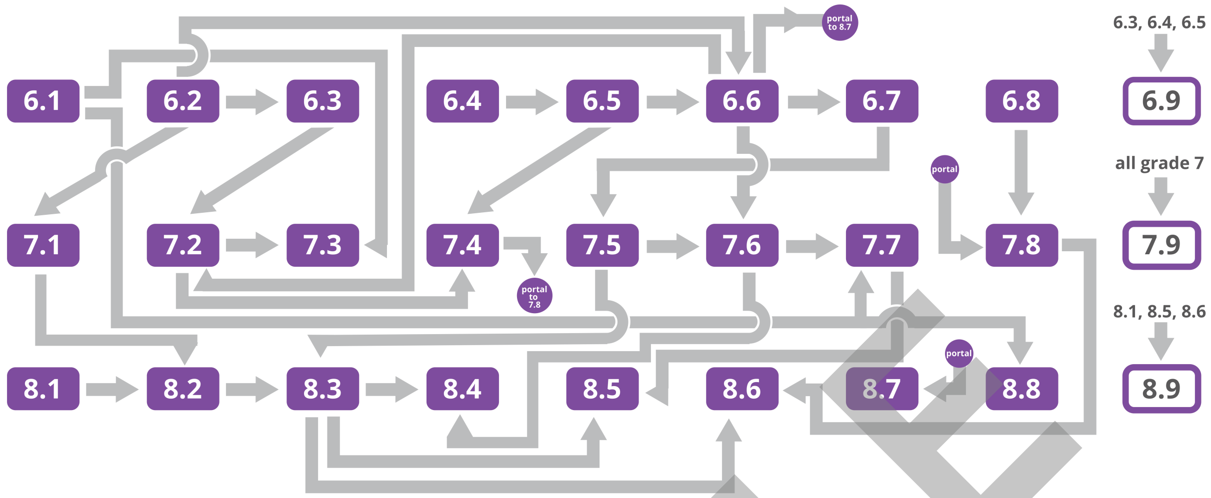
As in grade 6, students start grade 7 by studying scale drawings, an engaging geometric topic that supports the subsequent work on proportional relationships in the second and fourth units. It also makes use of grade 6 arithmetic understanding and skill, without arithmetic becoming the major focus of attention at this point. Geometry and proportional relationships are also interwoven in the third unit on circles, where the important proportional relationship between a circle's circumference and its diameter is studied. By the time students reach the fifth unit on operations with rational numbers, both positive and negative, students have had time to brush up on and solidify their understanding and skill in grade 6 arithmetic. The work on operations on rational numbers, with its emphasis on the role of the properties of operations in determining the rules for operating with negative numbers, is a natural lead-in to the work on expressions and equations in the next unit. Students then put their arithmetical and algebraic skills to work in the last two units, on angles, triangles, and prisms, and on probability and sampling.

Illustrative Mathematics Middle School Curriculum Pacing Guide

week 1, week 2, week 3, week 4, week 5, week 6, week 7, week 8, week 9, week 10, week 11, week 12, week 13, week 14, week 15, week 16, week 17, week 18, week 19, week 20, week 21, week 22, week 23, week 24, week 25, week 26, week 27, week 28, week 29, week 30, week 31, week 32, week 33, week 34, week 35, week 36

Grade 6	Grade 7	Grade 8
Unit 1 Area and Surface Area (21–22 days)	Unit 1 Scale Drawings (13–15 days)	Unit 1 Rigid Transformations and Congruence (20 days)
Unit 2 Introducing Ratios (19 days)	Unit 2 Introducing Proportional Relationships (17 days)	Unit 2 Dilations, Similarity, and Introducing Slope (15 days)
Unit 3 Unit Rates and Percentages (18–19 days)	Unit 3 Measuring Circles (11–13 days)	Unit 3 Linear Relationships (17 days)
Unit 4 Dividing Fractions (20 days)	Unit 4 Proportional Relationships and Percentages (17–19 days)	Unit 4 Linear Equations and Linear Systems (18 days)
Unit 5 Arithmetic in Base Ten (16–18 days)	Unit 5 Rational Number Arithmetic (19 days)	Unit 5 Functions and Volume (25 days)
Unit 6 Expressions and Equations (18–20 days)	Unit 6 Expressions, Equations, and Inequalities (25 days)	Unit 6 Associations in Data (12–13 days)
Unit 7 Rational Numbers (20 days)	Unit 7 Angles, Triangles, and Prisms (19 days)	Unit 7 Exponents and Scientific Notation (18 days)
Unit 8 Data Sets and Distributions (21 days)	Unit 8 Probability and Sampling (20–22 days)	Unit 8 Pythagorean Theorem and Irrational Numbers (17 days)
Unit 9 Putting It All Together (0–18 days)	Unit 9 Putting It All Together (0–13 days)	Unit 9 Putting it All Together (0–10 days)

Unit Dependency Chart



In the unit dependency chart, an arrow indicates that a particular unit is designed for students who already know the material in a previous unit. Reversing the order would have a negative effect on mathematical or pedagogical coherence. For example, there is an arrow from 7.4 to 7.8, because students are expected to use their skills in representing percentages (from 7.4) when solving problems about probability (in 7.8).

Scale Drawings

Work with scale drawings in grade 7 draws on earlier work with geometry and geometric measurement. Students began to learn about two- and three-dimensional shapes in kindergarten, and continued this work in grades 1 and 2, composing, decomposing, and identifying shapes. Students' work with geometric measurement began with length and continued with area. Students learned to "structure two-dimensional space," that is, to see a rectangle with whole-number side lengths as an array of unit squares, or rows or columns of unit squares. In grade 3, students distinguished between perimeter and area. They connected rectangle area with multiplication, understanding why (for whole-number side lengths) multiplying the side lengths of a rectangle yields the number of unit squares that tile the rectangle. They used area diagrams to represent instances of the distributive property. In grade 4, students applied area and perimeter formulas for rectangles to solve real-world and mathematical problems, and learned to use protractors. In grade 5, students extended the formula for the area of a rectangle to include rectangles with fractional side lengths. In grade 6, students built on their knowledge of geometry and geometric measurement to produce formulas for the areas of parallelograms and triangles, using these formulas to find surface areas of polyhedra.

In this unit, students study scaled copies of pictures and plane figures, then apply what they have learned to scale drawings, e.g., maps and floor plans. This provides geometric preparation for grade 7 work on proportional relationships as well as grade 8 work on dilations and similarity.

Students begin by looking at copies of a picture, some of which are to scale, and some of which are not. They use their own words to describe what differentiates scaled and non-scaled copies of a picture. As the unit progresses, students learn that all lengths in a scaled copy are multiplied by a scale factor and all angles stay the same. They draw scaled copies of figures. They learn that if the scale factor is greater than 1, the copy will be larger, and if the scale factor is less than 1, the copy will be smaller. They study how area changes in scaled copies of an image.

Next, students study scale drawings. They see that the principles and strategies that they used to reason about scaled copies of figures can be used with scale drawings. They interpret and draw maps and floor plans. They work with scales that involve units (e.g., "1 cm represents 10 km"), and scales that do not include units (e.g., "the scale is 1 to 100"). They learn to express scales with units as scales without units, and vice versa. They understand that actual lengths are products of a scale factor and corresponding lengths in the scale drawing, thus lengths in the drawing are the product of the actual lengths and the reciprocal of that scale factor. They study the relationship between regions and lengths in scale drawings. Throughout the unit, they discuss their mathematical ideas and respond to the ideas of others (MP3, MP6). In the culminating lesson of this unit, students make a floor plan of their classroom or some other room or space at their school. This is an opportunity for them to apply what they have learned in the unit to everyday life (MP4).

In the unit, several lesson plans suggest that each student have access to a *geometry toolkit*. Each toolkit contains tracing paper, graph paper, colored pencils, scissors, centimeter ruler, protractor (clear protractors with no holes that show radial lines are recommended), and an index card to use as a straightedge or to mark right angles. Providing students with these toolkits gives opportunities for students to develop abilities to select appropriate tools and use them strategically to solve problems (MP5). Note that even students in a digitally enhanced classroom should have access to such tools; apps and simulations should be considered additions to their toolkits, not replacements for physical tools.

Note that the study of scaled copies is limited to pairs of figures that have the same rotation and mirror orientation (i.e. that are not rotations or reflections of each other), because the unit focuses on scaling, scale factors, and scale drawings. In grade 8, students will extend their knowledge of scaled copies when they study translations, rotations, reflections, and dilations.

Introducing Proportional Relationships

In this unit, students develop the idea of a proportional relationship out of the grade 6 idea of equivalent ratios. Proportional relationships prepare the way for the study of linear functions in grade 8.

In grade 6, students learned two ways of looking at equivalent ratios. First, if you multiply both values in a ratio $a : b$ by the same positive number s (called the scale factor) you get an equivalent ratio $sa : sb$. Second, two ratios are equivalent if they have the same *unit rate*. A unit rate is the "amount per 1" in a ratio; the ratio $a : b$ is equivalent to $\frac{a}{b} : 1$, and $\frac{a}{b}$ is a unit rate giving the amount of the first quantity per unit of the second quantity. You could also talk about the amount of the second quantity per unit of the first quantity, which is the unit rate $\frac{b}{a}$, coming from the equivalent ratio $1 : \frac{b}{a}$.

In a table of equivalent ratios, a multiplicative relationship between the pair of rows is given by a scale factor. By contrast, the multiplicative relationship between the columns is given by a unit rate. Every number in the second column is obtained by multiplying the corresponding number in the first column by one of the unit rates, and every number in the first column is obtained by multiplying the number in the second column by the other unit rate. The relationship between pairs of values in the two columns is called a *proportional relationship*, the unit rate that describes this relationship is called a *constant of proportionality*, and the quantity represented by the right column is said to be *proportional to* the quantity represented by the left. (Although a proportional relationship between two quantities represented by a and b is associated with *two* constants of proportionality, $\frac{a}{b}$ and $\frac{b}{a}$, throughout the unit, the convention is if a and b are, respectively, in the left and right columns of a table, then $\frac{b}{a}$ is the constant of proportionality for the relationship represented by the table.)

For example, if a person runs at a constant speed and travels 12 miles in 2 hours, then the distance traveled is proportional to the time elapsed, with constant of proportionality 6, because

$$\text{distance} = 6 \cdot \text{time}.$$

The time elapsed is proportional to distance traveled with constant of proportionality $\frac{1}{6}$, because

$$\text{time} = \frac{1}{6} \cdot \text{distance}.$$

Students learn that any proportional relationship can be represented by an equation of the form $y = kx$ where k is the constant of proportionality, that its graph lies on a line through the origin that passes through Quadrant I, and that the constant of proportionality indicates the steepness of the line. By the end of the unit, students should be able to easily work with common contexts associated with proportional relationships (such as constant speed, unit pricing, and measurement conversions) and be able to determine whether a relationship is proportional or not.

Because this unit focuses on understanding what a proportional relationship is, how it is represented, and what types of contexts give rise to proportional relationships, the contexts have been carefully chosen. The first tasks in the unit employ contexts such as servings of food, recipes, constant speed, and measurement conversion, that should be familiar to students from the grade 6 course. These contexts are revisited throughout the unit as new aspects of proportional relationships are introduced.

Associated with the contexts from the grade 6 course are derived units: miles per hour; meters per second; dollars per pound; or cents per minute. In this unit, students build on their grade 6 experiences in working with a wider variety of derived units, such as cups of flour per tablespoon of honey, hot dogs eaten per minute, and centimeters per millimeter. The tasks in this unit avoid discussion of measurement error and statistical variability, which will be addressed in later units.

On using the terms quantity, ratio, proportional relationship, unit rate, and fraction. In these materials, a *quantity* is a measurement that is or can be specified by a number and a unit, e.g., 4 oranges, 4 centimeters, “my height in feet,” or “my height” (with the understanding that a unit of measurement will need to be chosen, MP6). The term *ratio* is used to mean a type of association between two or more quantities. A *proportional relationship* is a collection of equivalent ratios.

A *unit rate* is the numerical part of a rate per 1 unit, e.g., the 6 in 6 miles per hour. The fractions $\frac{a}{b}$ and $\frac{b}{a}$ are never called ratios. The fractions $\frac{a}{b}$ and $\frac{b}{a}$ are identified as “unit rates” for the ratio $a : b$. In high school—after their study of ratios, rates, and proportional relationships—students discard the term “unit rate,” referring to a to b, $a : b$, and $\frac{a}{b}$ as “ratios.”

In grades 6–8, students write rates without abbreviated units, for example as “3 miles per hour” or “3 miles in every 1 hour.” Use of notation for derived units such as $\frac{\text{mi}}{\text{hr}}$ waits for high school—except for the special cases of area and volume. Students have worked with area since grade 3 and volume since grade 5. Before grade 6, they have learned the meanings of such things as sq cm and cu cm. After students learn exponent notation in grade 6, they also use cm^2 and cm^3.

A *fraction* is a point on the number line that can be located by partitioning the segment between 0 and 1 into equal parts, then finding a point that is a whole number of those parts away from 0. A fraction can be written in the form $\frac{a}{b}$ or as a decimal.

Measuring Circles

In this unit, students extend their knowledge of circles and geometric measurement, applying their knowledge of proportional relationships to the study of circles. They extend their grade 6 work with perimeters of polygons to circumferences of circles, and recognize that the circumference of a circle is proportional to its diameter, with constant of proportionality π. They encounter informal derivations of the relationship between area, circumference, and radius.

The unit begins with activities designed to help students come to a more precise understanding of the characteristics of a circle (MP6): a “circle” is the set of points that are equally distant from a point called the “center”; the diameter of a circle is a line segment that passes through its center with endpoints on the circle; the radius is a line segment with one endpoint on the circle and one endpoint at the center. Students identify these characteristics in a variety of contexts (MP2). They use compasses to draw circles with given diameters or radii, and to copy designs that involve circles. Using their newly gained familiarity with circumference and diameter, students measure circular objects, investigating the relationship between measurements of circumference and diameter by making tables and graphs.

The second section involves area. Students encounter two informal derivations of the fact that the area of a circle is equal to π times the square of its radius. The first involves dissecting a disk into sectors and rearranging them to form a shape that approximates a parallelogram of height r and width $2\pi r$. A second argument involves considering a disk as formed of concentric rings, “cutting” the rings with a radius, and “opening” the rings to form a shape that approximates an isosceles triangle of height r and base $2\pi \cdot 2r$.

In the third and last section, students select and use formulas for the area and circumference of a circle to solve abstract and real-world problems that involve calculating lengths and areas. They express measurements in terms of π or using appropriate approximations of π to express them numerically. In grade 8, they will use and extend their knowledge of circles and radii at the beginning of a unit on dilations and similarity.

On using the term circle. Strictly speaking, a circle is one-dimensional—the boundary of a two-dimensional region rather than the region itself. Because students are not yet expected to make this distinction, these materials refer to both circular regions (i.e., disks) and boundaries of disks as “circles,” using illustrations to eliminate ambiguity.

Proportional Relationships and Percentages

Students began their work with ratios, rates, and unit rates in grade 6, representing them with expressions, tape diagrams, double number line diagrams, and tables. They used these to reason about situations involving color mixtures, recipes, unit price, discounts, constant speed, and measurement conversions. They extended their understanding of rates to include percentages as rates per 100, reasoning about situations involving whole-number percentages. They did not use the terms “proportion” and “proportional relationship” in grade 6.

A proportional relationship is a collection of equivalent ratios, and such collections are objects of study in grade 7. In previous grade 7 units, students worked with scale factors and scale drawings, and with proportional relationships and constants of proportionality. Although students have learned how to compute quotients of fractions in grade 6, these first units on scaling and proportional relationships do not require such calculations, allowing the new concept (scaling or proportional relationship) to be the main focus.

In this unit, students deepen their understanding of ratios, scale factors, unit rates (also called constants of proportionality), and proportional relationships, using them to solve multi-step problems that are set in a wide variety of contexts that involve fractions and percentages.

In the first section of the unit, students extend their use of ratios and rates to problems that involve computing quotients of fractions, and interpreting these quotients in contexts such as scaling a picture or running at constant speed (MP2). They use long division to write fractions presented in the form $\frac{a}{b}$ as decimals, e.g., $\frac{11}{30} = 0.3\overline{6}$.

The section begins by revisiting scale factors and proportional relationships, each of which has been the focus of a previous unit. Both of these concepts can be used to solve problems that involve equivalent ratios. However, it is often more efficient to view equivalent ratios as pairs that are in the same proportional relationship rather than seeing one pair as obtained by multiplying both entries of the other by a scale factor. From the scaling perspective, to see that one ratio is equivalent to another or to generate a ratio equivalent to a given ratio, a scale factor is needed—which may be different for each pair of ratios in the proportional relationship. From the proportional relationship perspective, all that is needed is the constant of proportionality—which is the same for every ratio in the proportional relationship.

The second section of the unit is about percent increase and decrease. Students consider situations for which percentages can be used to describe a change relative to an initial amount, e.g., prices before and after a 25% increase. They begin by considering situations with unspecified amounts, e.g., matching tape diagrams with statements such as "Compared with last year's strawberry harvest, this year's strawberry harvest increased by 25%". They next consider situations with a specified amount and percent change, or with initial and final amounts, using double number line diagrams to find the unknown amount or percent change. Next, they use equations to represent such situations, using the distributive property to show that different expressions for the same amount are equivalent, e.g., $x - 0.25x = 0.75x$. So far, percent change in this section has focused on whole-number rates per 100, e.g., 75%. The last lesson asks students to compute fractional percentages of given amounts.

In the third section of the unit, students begin by using their abilities to find percentages and percent rates to solve problems that involve sales tax, tip, discount, markup, markdown, and commission (MP2). The remaining lessons of the section continue the focus on situations that can be described in terms of percentages, but the situations involve error rather than change—describing an incorrect value as a percentage of the correct value rather than describing an initial amount as a percentage of a final amount (or vice versa).

The last section of the unit consists of a lesson in which students analyze news items that involve percent increase or decrease. In small groups, students identify important quantities in a situation described in a news item, use diagrams to map the relationship of of the quantities, and reason mathematically to draw conclusions (MP4). This is an opportunity to choose an appropriate type of diagram (MP5), to state the meanings of symbols used in the diagram, to specify units of measurement, and to label the diagram accurately (MP6). Each group creates a display to communicate its reasoning and critiques the reasoning shown in displays from other groups (MP3).

These materials follow specific conventions for the use of language around ratios, rates, and proportional relationships. Please see the unit narrative for the second unit to read about those conventions.

Rational Number Arithmetic

In grade 6, students learned that the rational numbers comprise positive and negative fractions. They plotted rational numbers on the number line and plotted pairs of rational numbers in the coordinate plane. In this unit, students extend the operations of addition, subtraction, multiplication, and division from fractions to all rational numbers, written as decimals or in the form $\frac{a}{b}$.

The unit begins by revisiting ideas familiar from grade 6: how signed numbers are used to represent quantities such as measurements of temperature and elevation, opposites (pairs of numbers on the number line that are the same distance from zero), and absolute value.

In the second section of the unit, students extend addition and subtraction from fractions to all rational numbers. They begin by considering how changes in temperature and elevation can be represented—first with tables and number line diagrams, then with addition and subtraction expressions and equations. Initially, physical contexts provide meanings for sums and differences that include negative numbers. Students work with numerical addition and subtraction expressions and equations, becoming more fluent in computing sums and differences of signed numbers. Using the meanings that they have developed for addition and subtraction of signed numbers, they write equivalent numerical addition and subtraction expressions, e.g., $-8 + -3$ and $-8 - 3$; and they write different equations that express the same relationship.

The third section of the unit focuses on multiplication and division. It begins with problems about position, direction, constant speed, and constant velocity in which students represent quantities with number line diagrams and tables of numerical expressions with signed numbers. This allows products of signed numbers to be interpreted in terms of position and direction, using the understanding that numbers that are additive inverses are located at the same distance but opposite sides of the starting point. These examples illustrate how multiplication of how multiplication of fractions extends to rational numbers. The third lesson of this section focuses on computing products of signed numbers and is optional. In the fourth lesson, students use the relationship between multiplication and division to understand how division extends to rational numbers. In the process of solving problems set in contexts (MP4), they write and solve multiplication and division equations.

In the fourth section of the unit, students work with expressions that use the four operations on rational numbers, making use of structure (MP7), e.g., to see without calculating that the product of two factors is positive because the values of the factors are both negative. They extend their use of the “next to” notation (which they used in expressions such as $5x$ and $6(3 + 2)$ in grade 6) to include negative numbers and products of numbers, e.g., writing $-5x$ and $(-5)(-10)$ rather than $(-5) \cdot (x)$ and $(-5) \cdot (-10)$. They extend their use of the fraction bar to include variables as well as numbers, writing $-8.5 \div x$ as well as $\frac{-8.5}{x}$. They solve problems that involve interpreting negative numbers in context, for instance, when a negative number represents a rate at which water is flowing (MP2).

In the fifth section of the unit, students begin working with linear equations in one variable that have rational number coefficients. The focus of this section is representing situations with equations (MP4) and what it means for a number to be a solution for an equation, rather than methods for solving equations. Such methods are the focus of a later unit.

The last section of the unit is a lesson in which students use rational numbers in the context of stock-market situations, finding values of quantities such as the value of a portfolio or changes due to interest and depreciation (MP4).

Note. In these materials, an *expression* is built from numbers, variables, operation symbols ($+$, $-$, $\cdot$, $\div$), parentheses, and exponents. (Exponents—in particular, negative exponents—are not a focus of this unit. Students work with integer exponents in grade 8 and non-integer exponents in high school.) An *equation* is a statement that two expressions are equal, thus always has an equal sign. *Signed numbers* include all rational numbers, written as decimals or in the form $\frac{a}{b}$.

Expressions, Equations, and Inequalities

In this unit, students solve equations of the forms $px + q = r$ and $p(x + q) = r$, and solve related inequalities, e.g., those of the form $px + q > r$ and $px + q \geq r$, where p, q, and r are rational numbers.

In the first section of the unit, students represent relationships of two quantities with tape diagrams and with equations, and explain correspondences between the two types of representations (MP1). They begin by examining correspondences between descriptions of situations and tape diagrams, then draw tape diagrams to represent situations in which the variable representing the unknown is specified. Next, they examine correspondences between equations and tape diagrams, then draw tape diagrams to represent equations, noticing that one tape diagram can be described by different (but related) equations. At the end of the section, they draw tape diagrams to represent situations in which the variable representing the unknown is not specified, then match the diagrams with equations. The section concludes with an example of the two main types of situations examined, characterized in terms of whether or not they involve equal parts of an amount or equal *and* unequal parts of an amount, and as represented by equations of different forms, e.g., $6(x + 8) = 72$ and $6x + 8 = 72$. This initiates a focus on seeing two types of structure in the situations, diagrams, and equations of the unit (MP7).

In the second section of the unit, students solve equations of the forms $px + q = r$ and $p(x + q) = r$, then solve problems that can be represented by such equations (MP2). They begin by considering balanced and unbalanced “hanger diagrams,” matching hanger diagrams with equations, and using the diagrams to understand two algebraic steps in solving equations of the form $px + q = r$: subtract the same number from both sides, then divide both sides by the same number. Like a tape diagram, the same balanced hanger diagram can be described by different (but related) equations, e.g., $3x + 6 = 18$ and $3(x + 2) = 18$. The second form suggests using the same two algebraic steps to solve the equation, but in reverse order: divide both sides by the same number, then subtract the same number from both sides. Each of these algebraic steps and the associated structure of the equation is illustrated by hanger diagrams (MP1, MP7).

So far, the situations in the section have been described by equations in which p is a whole number, and q and r are positive (and frequently whole numbers). In the remainder of the section, students use the algebraic methods that they have learned to solve equations of the forms $px + q = r$ and $p(x + q) = r$ in which p, q, and r are rational numbers. They use the distributive property to transform an equation of one form into the other (MP7) and note how such transformations can be used strategically in solving an equation (MP5). They write equations in order to solve problems involving percent increase and decrease (MP2).

In the third section of the unit, students work with inequalities. They begin by examining values that make an inequality true or false, and using the number line to represent values that make an inequality true. They solve equations, examine values to the left and right of a solution, and use those values in considering the solution of a related inequality. In the last two lessons of the section, students solve inequalities that represent real-world situations (MP2).

In the last section of the unit, students work with equivalent linear expressions, using properties of operations to explain equivalence (MP3). They represent expressions with area diagrams, and use the distributive property to justify factoring or expanding an expression.

Angles, Triangles, and Prisms

In this unit, students investigate whether sets of angle and side length measurements determine unique triangles or multiple triangles, or fail to determine triangles. Students also study and apply angle relationships, learning to understand and use the terms “complementary,” “supplementary,” “vertical angles,” and “unique” (MP6). The work gives them practice working with rational numbers and equations for angle relationships. Students analyze and describe cross-sections of prisms, pyramids, and polyhedra. They understand and use the formula for the volume of a right rectangular prism, and solve problems involving area, surface area, and volume (MP1, MP4). Students should have access to their geometry toolkits so that they have an opportunity to select and use appropriate tools strategically (MP5).

Note: It is not expected that students memorize which conditions result in a unique triangle, are impossible to create a triangle, or multiple possible triangles. Understanding that, for example, SSS information results in zero or exactly one triangle will be explored in high school geometry. At this level, students should attempt to draw triangles with the given information and notice that there is only one way to do it (or that it is impossible to do).

Probability and Sampling

In this unit, students understand and use the terms “event,” “sample space,” “outcome,” “chance experiment,” “probability,” “simulation,” “random,” “sample,” “random sample,” “representative sample,” “overrepresented,” “underrepresented,” “population,” and “proportion.” They design and use simulations to estimate probabilities of outcomes of chance experiments and understand the probability of an outcome as its long-run relative frequency. They represent sample spaces (that is, all possible outcomes of a chance experiment) in tables and tree diagrams and as lists. They calculate the number of outcomes in a given sample space to find the probability of a given event. They consider the strengths and weaknesses of different methods for obtaining a representative sample from a given population. They generate samples from a given population, e.g., by drawing numbered papers from a bag and recording the numbers, and examine the distributions of the samples, comparing these to the distribution of the population. They compare two populations by comparing samples from each population.

Putting it All Together

In this optional unit, students use concepts and skills from previous units to solve three groups of problems. In calculating or estimating quantities associated with running a restaurant, e.g., number of calories in one serving of a recipe, expected number of customers served per day, or floor space, they use their knowledge of proportional relationships, interpreting survey findings, and scale drawings. In estimating quantities such as age in hours and minutes or number of times their hearts have beaten, they use measurement conversions and consider accuracy of their estimates. Estimation of area and volume measurements from length measurements introduces considerations of measurement error. In designing a five-kilometer race course for their school, students use their knowledge of measurement and scale drawing. They select appropriate tools and methods for measuring their school campus, build a trundle wheel and use it to make measurements, make a scale drawing of the course on a map or a satellite image of the school grounds, and describe the number of laps, start, and finish of the race.

Scope and sequence - Grade 8

Course Overview

Students begin grade 8 with transformational geometry. They study rigid transformations and congruence, then dilations and similarity (this provides background for understanding the slope of a line in the coordinate plane). Next, they build on their understanding of proportional relationships from grade 7 to study linear relationships. They express linear relationships using equations, tables, and graphs, and make connections across these representations. They expand their ability to work with linear equations in one and two variables. Building on their understanding of a solution to an equation in one or two variables, they understand what is meant by a solution to a system of equations in two variables. They learn that linear relationships are an example of a special kind of relationship called a function. They apply their understanding of linear relationships and functions to contexts involving data with variability. They extend the definition of exponents to include all integers, and in the process codify the properties of exponents. They learn about orders of magnitude and scientific notation in order to represent and compute with very large and very small quantities. They encounter irrational numbers for the first time and informally extend the rational number system to the real number system, motivated by their work with the Pythagorean Theorem.

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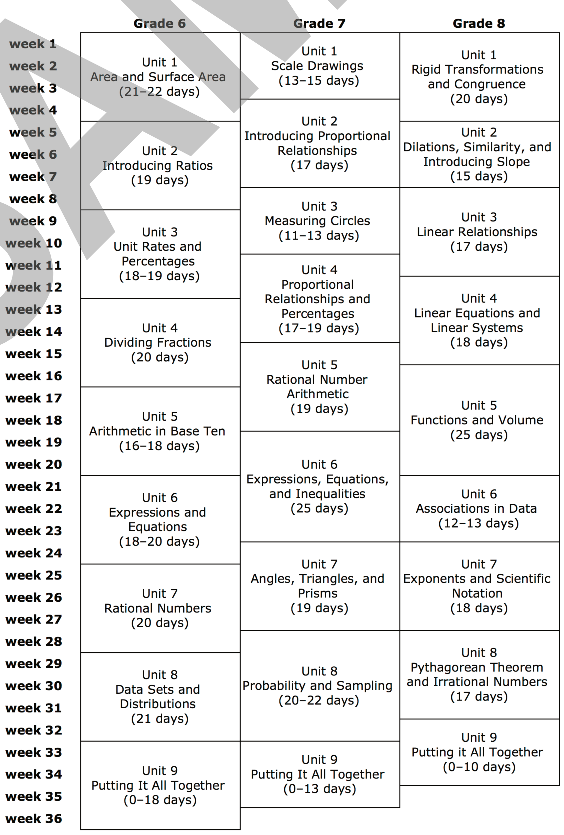
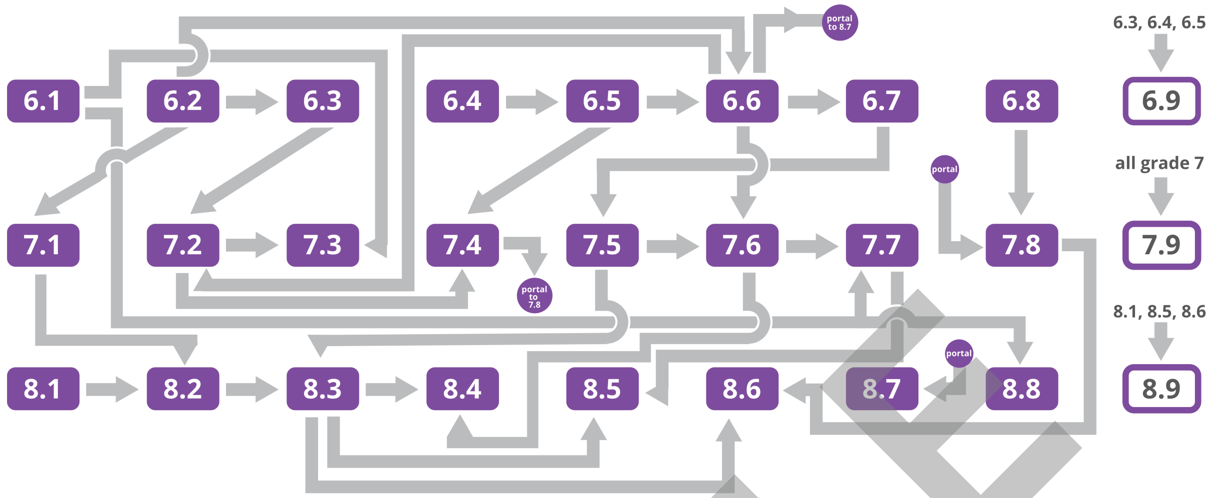




In the unit dependency chart, an arrow indicates that a particular unit is designed for students who already know the material in a previous unit. Reversing the order would have a negative effect on mathematical or pedagogical coherence. For example, there is an arrow from 8.3 to 8.6, because students are expected to use their skills in writing and interpreting an equation that represents a line (from 8.3) to interpret the parameters in an equation that represents a line that fits a scatter plot (in 8.6).

Rigid Transformations and Congruence

Work with transformations of plane figures in grade 8 draws on earlier work with geometry and geometric measurement. Students began to learn about two- and three-dimensional shapes in kindergarten, and continued this work in grades 1 and 2, composing, decomposing, and identifying shapes. Students' work with geometric measurement began with length and continued with area. Students learned to "structure two-dimensional space," that is, to see a rectangle with whole-number side lengths as composed of an array of unit squares or composed of iterated rows or iterated columns of unit squares. In grade 3, students distinguished between perimeter and area. They connected rectangle area with multiplication, understanding why (for whole-number side lengths) multiplying the side lengths of a rectangle yields the number of unit squares that tile the rectangle. They used area diagrams to represent instances of the distributive property. In grade 4, students applied area and perimeter formulas for rectangles to solve real-world and mathematical problems, and learned to use protractors. In grade 5, students extended the formula for the area of rectangles to rectangles with fractional side lengths. In grade 6, students combined their knowledge of geometry and geometric measurement to produce formulas for the areas of parallelograms and triangles, using these formulas to find surface areas of polyhedra. In grade 7, students worked with scaled copies and scale drawings, learning that angle measures are preserved in scaled copies, but areas increase or decrease proportionally to the square of the scale factor. Their study of scaled copies was limited to pairs of figures with the same rotation and mirror orientation. Viewed from the perspective of grade 8, a scaled copy is a dilation and translation, not a rotation or reflection, of another figure.

In grade 8, students extend their reasoning to plane figures with different rotation and mirror orientations.

Through activities designed and sequenced to allow students to make sense of problems and persevere in solving them (MP1), students use and extend their knowledge of geometry and geometric measurement. They begin the unit by looking at pairs of cartoons, each of which illustrates a translation, rotation, or reflection. Students describe in their own words how to move one cartoon figure onto another. As the unit progresses, they solidify their understanding of these transformations, increase the precision of their descriptions (MP6), and begin to use associated terminology, recognizing what determines each type of transformation, e.g., two points determine a translation. They identify and describe translations, rotations, and reflections, and sequences of these. In describing images of figures under rigid transformations on and off square grids and the coordinate plane, students use the terms "corresponding points," "corresponding sides," and "image." Students learn that angles and distances are preserved by any sequence of translations, rotations, and reflections, and that such a sequence is called a "rigid transformation." They learn the definition of "congruent": two figures are said to be congruent if there is a rigid transformation that takes one figure to the other. Students experimentally verify the properties of translations, rotations, and reflections, and use these properties to reason about plane figures, understanding informal arguments showing that the alternate interior angles cut by a transversal have the same measure and that the sum of the angles in a triangle is 180°. The latter will be used in a subsequent grade 8 unit on similarity and dilations. Throughout the unit, students discuss their mathematical ideas and respond to the ideas of others (MP3, MP6).

Many of the lessons in this unit ask students to work on geometric figures that are not set in a real-world context. This design choice respects the significant intellectual work of reasoning about area. Tasks set in real-world contexts are sometimes contrived and hinder rather than help understanding. Moreover, mathematical contexts are legitimate contexts that are worthy of study. Students do have opportunities in the unit to tackle real-world applications. In the culminating activity of the unit, students examine and create different patterns formed by plane figures. This is an opportunity for them to apply what they have learned in the unit (MP4).

In this unit, several lesson plans suggest that each student have access to a *geometry toolkit*. These contain tracing paper, graph paper, colored pencils, scissors, ruler, protractor, and an index card to use as a straightedge or to mark right angles, giving students opportunities to develop their abilities to select appropriate tools and use them strategically to solve problems (MP5). Note that even students in a digitally enhanced classroom should have access to such tools; apps and simulations should be considered additions to their toolkits, not replacements for physical tools.

Dilations, Similarity, and Introducing Slope

Work with transformations of plane figures in grade 8 builds on earlier work with geometry and geometric measurement, using students' familiarity with geometric figures, their knowledge of formulas for the areas of rectangles, parallelograms, and triangles, and their abilities to use rulers and protractors. Grade 7 work with scaled copies is especially relevant. This work was limited to pairs of figures with the same rotation and mirror orientations (i.e. that are not rotations or reflections of each other). In grade 8, students study pairs of scaled copies that have different rotation or mirror orientations, examining how one member of the pair can be transformed into the other, and describing these transformations. Initially, they view transformations as moving one figure in the plane onto another figure in the plane. As the unit progresses, they come to view transformations as moving the entire plane.

Through activities designed and sequenced to allow students to make sense of problems and persevere in solving them (MP1), students use and extend their knowledge of geometry and geometric measurement. Students begin the first lesson of the unit by looking at cut-out figures, first comparing them visually to determine if they are scaled copies of each other, then representing the figures in a diagram, and finally representing them on a circular grid with radial lines. They encounter the term "scale factor" (familiar from grade 7) and the new terms "dilation" and "center of dilation." In the next lesson, students again use a circular grid with radial lines to understand that under a dilation the image of a circle is a circle and the image of a line is a line parallel to the original. During the rest of the unit, students draw images of figures under dilations on and off square grids and the coordinate plane. In describing correspondences between a figure and its dilation, they use the terms "corresponding points," "corresponding sides," and "image." Students learn that angle measures are preserved under a dilation, but lengths in the image are multiplied by the scale factor. They learn the definition of "similar": two figures are said to be similar if there is a sequence of translations, rotations, reflections, and dilations that takes one figure to the other. They use the definition of "similar" and properties of similar figures to justify claims of similarity or non-similarity and to reason about similar figures (MP3). Using these properties, students conclude that if two triangles have two angles in common, then the triangles must be similar. Students also conclude that the quotient of a pair of side lengths in a triangle is equal to the quotient of the corresponding side lengths in a similar triangle. This conclusion is used in the lesson that follows: students learn the terms “slope” and “slope triangle,” and use the similarity of slope triangles on the same line to understand that any two distinct points on a line determine the same slope (MP7). In the following lesson, students use their knowledge of slope to find an equation for a line. They will build on this initial work with slope in a subsequent grade 8 unit on linear relationships. Throughout the unit, students discuss their mathematical ideas and respond to the ideas of others (MP3, MP6).

Many of the lessons in this unit ask students to work on geometric figures that are not set in a real-world context. This design choice respects the significant intellectual work of reasoning about area. Tasks set in real-world contexts are sometimes contrived and hinder rather than help understanding. Moreover, mathematical contexts are legitimate contexts that are worthy of study. Students do have opportunities in the unit to tackle real-world applications. In the culminating activity of the unit, students examine shadows cast by objects in the sun. This is an opportunity for them to apply what they have learned about similar triangles (MP4).

In this unit, several lesson plans suggest that each student have access to a *geometry toolkit*. Each toolkit contains tracing paper, graph paper, colored pencils, scissors, ruler, protractor, and an index card to use as a straightedge or to mark right angles, giving students opportunities to develop their abilities to select appropriate tools and use them strategically to solve problems (MP5). Note that even students in a digitally enhanced classroom should have access to such tools; apps and simulations should be considered additions to their toolkits, not replacements for physical tools.

Linear Relationships

Work with linear relationships in grade 8 builds on earlier work with rates and proportional relationships in grade 7, and grade 8 work with geometry. At the end of the previous unit on dilations, students learned the terms “slope” and “slope triangle,” used the similarity of slope triangles on the same line to understand that any two distinct points on a line determine the same slope, and found an equation for a line with a positive slope and vertical intercept. In this unit, students gain experience with linear relationships and their representations as graphs, tables, and equations through activities designed and sequenced to allow them to make sense of problems and persevere in solving them (MP1). Because of this dependency, this unit and the previous one should be done in order.

The unit begins by revisiting different representations of proportional relationships (graphs, tables, and equations), and the role of the constant of proportionality in each representation and how it may be interpreted in context (MP2).

Next, students analyze the relationship between number of cups in a given stack of cups and the height of the stack—a relationship that is linear but not proportional—in order to answer the question "How many cups are needed to get to a height of 50 cm?" They are not asked to solve this problem in a specific way, giving them an opportunity to choose and use strategically (MP5) representations that appeared earlier in this unit (table, equation, graph) or in the previous unit (equation, graph). Students are introduced to "rate of change" as a way to describe the rate per 1 in a linear relationship and note that its numerical value is the same as that of the slope of the line that represents the relationship. Students analyze another linear relationship (height of water in a cylinder vs number of cubes in the cylinder) and establish a way to compute the slope of a line from any two distinct points on the line via repeated reasoning (MP8). They learn a third way to obtain an equation for a linear relationship by viewing the graph of a line in the coordinate plane as the vertical translation of a proportional relationship (MP7).

So far, the unit has involved only lines with positive slopes and y-intercepts. Students next consider the graph of a line with a negative y-intercept and equations that might represent it. They consider situations represented by linear relationships with negative rates of change, graph these (MP4), and interpret coordinates of points on the graphs in context (MP2).

The unit concludes with two lessons that involve graphing two equations in two unknowns and finding and interpreting their solutions (MP2). Doing this involves considering correspondences among different representations (MP1), in particular, what it means for a pair of values to be a solution for an equation and the correspondence between coordinates of points on a graph and solutions of an equation.

In this unit, several lesson plans suggest that each student have access to a *geometry toolkit*. Each toolkit contains tracing paper, graph paper, colored pencils, scissors, ruler, protractor, and an index card to use as a straightedge or to mark right angles, giving students opportunities to select appropriate tools and use them strategically to solve problems (MP5). Note that even students in a digitally enhanced classroom should have access to such tools; apps and simulations should be considered additions to their toolkits, not replacements for physical tools.

On using the terms ratio, rate, and proportion. In these materials, a *quantity* is a measurement that is or can be specified by a number and a unit, e.g., 4 oranges, 4 centimeters, "my height in feet," or "my height" (with the understanding that a unit of measurement will need to be chosen). The term *ratio* is used to mean an association between two or more quantities and the fractions $\frac{a}{b}$ and $\frac{b}{a}$ are never called ratios. The fractions $\frac{a}{b}$ and $\frac{b}{a}$ are identified as "unit rates" for the ratio $a : b$. The word "per" is used with students in interpreting a unit rate in context, as in “$3 per ounce,” and “at the same rate” is used to signify a situation characterized by equivalent ratios.

In grades 6–8, students write rates without abbreviated units, for example as “3 miles per hour” or “3 miles in every 1 hour.” Use of notation for derived units such as $\frac{\text{mi}}{\text{hr}}$ waits for high school—except for the special cases of area and volume. Students have worked with area since grade 3 and volume since grade 5. Before grade 6, they have learned the meanings of such things as sq cm and cu cm. After students learn exponent notation in grade 6, they also use cm^2 and cm^3.

A *proportional relationship* is a collection of equivalent ratios. In high school—after their study of ratios, rates, and proportional relationships—students discard the term “unit rate,” referring to a to b, $a : b$, and $\frac{a}{b}$ as “ratios.”

A proportional relationship between two quantities represented by a and b is associated with *two* constants of proportionality: $\frac{a}{b}$ and $\frac{b}{a}$. Throughout the unit, the convention is if a and b are represented by columns in a table and the column for a is to the left of the column for b, then $\frac{b}{a}$.is the constant of proportionality for the relationship represented by the table.

Linear Equations and Linear Systems

In this unit, students build on their grades 6 and 7 work with equivalent expressions and equations with one occurrence of one variable, learning algebraic methods to solve linear equations with multiple occurrences of one variable. Students learn to use algebraic methods to solve systems of linear equations in two variables, building on their grades 7 and 8 work with graphs and equations of linear relationships. Understanding of linear relationships is, in turn, built on the understanding of proportional relationships developed in grade 7 that connected ratios and rates with lines and triangles.

The unit begins with a lesson on “number puzzles” in which students are shown a number line diagram that displays numerical changes (e.g., as in grade 7 work with signed numbers) and asked to write descriptions of situations and equations that the diagram could represent. Students are then given descriptions of situations in which an unknown quantity is linearly related to a combination of known quantities and asked to determine the unknown quantities in any way they can, e.g., using diagrams or writing equations.

In the second and third sections of the unit, students write and solve equations, abstracting from contexts (MP2) to represent a problem situation, stating the meanings of symbols that represent unknowns (MP6), identifying assumptions such as constant rate (MP4), selecting methods and representations to use in obtaining a solution (MP5), reasoning to obtain a solution (MP1), interpreting solutions in the contexts from which they arose (MP2) and writing them with appropriate units (MP6), communicating their reasoning to others (MP3), and identifying correspondences between verbal descriptions, tables, diagrams, equations, and graphs, and between different solution approaches (MP1).

The second section focuses on linear equations in one variable. Students analyze “hanger diagrams” that depict two collections of shapes that balance each other. Assuming that identical shapes have the same weight, they decide which actions of adding or removing weights preserve that balance. Given a hanger diagram that shows one type of shape with unknown weight, they use the diagram and their understanding of balance to find the unknown weight. Abstracting actions of adding or removing weights that preserve balance (MP7), students formulate the analogous actions for equations, using these along with their understanding of equivalent expressions to develop algebraic methods for solving linear equations in one variable. They analyze groups of linear equations in one unknown, noting that they fall into three categories: no solution, exactly one solution, and infinitely many solutions. They learn that any one such equation is false, true for one value of the variable, or (using properties of operations) true for all values of the variable. Given descriptions of real-world situations, students write and solve linear equations in one variable, interpreting solutions in the contexts from which the equations arose.

The third section focuses on systems of linear equations in two variables. It begins with activities intended to remind students that a point lies on the graph of a linear equation if and only if its coordinates make the equation true. Given descriptions of two linear relationships students interpret points on their graphs, including points on both graphs. Students categorize pairs of linear equations graphed on the same axes, noting that there are three categories: no intersection (lines distinct and parallel, no solution), exactly one intersection (lines not parallel, exactly one solution), and same line (infinitely many solutions).

Functions and Volume

In this unit, students are introduced to the concept of a function as a relationship between “inputs” and “outputs” in which each allowable input determines exactly one output. In the first three sections of the unit, students work with relationships that are familiar from previous grades or units (perimeter formulas, proportional relationships, linear relationships), expressing them as functions. In the remaining three sections of the unit, students build on their knowledge of the formula for the volume of a right rectangular prism from grade 7, learning formulas for volumes of cylinders, cones, and spheres. Students express functional relationships described by these formulas as equations. They use these relationships to reason about how the volume of a figure changes as another of its measurements changes, transforming algebraic expressions to get the information they need (MP1).

The first section begins with examples of “input–output rules” such as “divide by 3” or “if even, then . . . ; if odd, then . . . ” In these examples, the inputs are (implicitly) numbers, but students note that some inputs are not allowable for some rules, e.g., $\frac{1}{2}$ is not even or odd. Next, students work with tables that list pairs of inputs and outputs for rules specified by “input–output diagrams,” noting that a finite list of pairs does not necessarily determine a unique input–output rule (MP6). Students are then introduced to the term “function” as describing a relationship that assigns exactly one output to each allowable input.

In the second section, students connect the terms “independent variable” and “dependent variable” (which they learned in grade 6) with the inputs and outputs of a function. They use equations to express a dependent variable as a function of an independent variable, viewing formulas from earlier grades (e.g., $C = 2\pi r$), as determining functions. They work with tables, graphs, and equations for functions, learning the convention that the independent variable is generally shown on the horizontal axis. They work with verbal descriptions of a function arising from a real-world situation, identifying tables, equations, and graphs that represent the function (MP1), and interpreting information from these representations in terms of the real-world situation (MP2).

The third section of the unit focuses on linear and piecewise linear functions. Students use linear and piecewise linear functions to model relationships between quantities in real-world situations (MP4), interpreting information from graphs and other representations in terms of the situations (MP2). The lessons on linear functions provide an opportunity for students to coordinate and synthesize their understanding of new and old terms that describe aspects of linear and piecewise functions. In working with proportional relationships in grade 7, students learned the term “constant of proportionality,” and that any proportional relationship can be represented by an equation of the form $y = kx$ where k is the constant of proportionality, that its graph lies on a line through the origin that passes through Quadrant I, and that the constant of proportionality indicates the steepness of the line. In an earlier grade 8 unit, students were introduced to “rate of change” as a way to describe the rate per 1 in a linear relationship and noted that its numerical value is the same as that of the slope of the line that represents the relationship. In this section, students connect their understanding of “increasing” and “decreasing” from the previous section with their understanding of linear functions, noting, for example, that if a linear function is increasing, then its graph has positive slope, and that its rate of change is positive. Similarly, they connect their understanding of y-intercept (learned in an earlier unit) with the new term “initial value,” noting, for example, when the numerical part of an initial value of a function is given by the y-intercept of its graph (MP1).

In the remaining three sections of the unit, students work with volume, using abilities developed in earlier work with geometry and geometric measurement.

Students' work with geometry began in kindergarten, where they identified and described two- and three-dimensional shapes that included cones, cylinders, and spheres. They continued this work in grades 1 and 2, composing, decomposing, and identifying two- and three-dimensional shapes.

Students' work with geometric measurement began with length and continued with area and volume. Students learned to "structure two-dimensional space," that is, to see a rectangle with whole-number side lengths as composed of an array of unit squares or composed of iterated rows or iterated columns of unit squares. In grade 3, students connected rectangle area with multiplication, understanding why (for whole-number side lengths) multiplying the side lengths of a rectangle yields the number of unit squares that tile the rectangle. They used area models to represent instances of the distributive property. In grade 4, students used area and perimeter formulas for rectangles to solve real-world and mathematical problems. In grade 5, students extended the formula for the area of rectangles to rectangles with fractional side lengths. They found volumes of right rectangular prisms by viewing them as layers of arrays of cubes and used formulas to calculate these volumes as products of edge lengths or as products of base area and height. In grade 6, students extended the formula for the volume of a right rectangular prism to right rectangular prisms with fractional side lengths and used it to solve problems. They extended their reasoning about area to include shapes not composed of rectangles and combined their knowledge of geometry and geometric measurement to produce formulas for the areas of parallelograms and triangles, using these formulas to find surface areas of polyhedra. In grade 7, students analyzed and described cross-sections of prisms (including prisms with polygonal but non-rectangular bases), pyramids, and polyhedra, and used the formula for the volume of a right rectangular prism (volume is area of base times height of prism) to solve problems involving area, surface area, and volume.

In this grade 8 unit, students extend their understanding of volume from right rectangular prisms to right cylinders, right cones, and spheres. They begin by investigating the volume of water in a graduated cylinder as a function of the height of the water, and vice versa. They examine depictions of of a cylinder, prism, sphere, and cone, in order to develop their abilities to identify radii, bases, and heights of these objects. They estimate volumes of prisms, cylinders, cones, and spheres, in order to reinforce the idea that a measurement of volume indicates the amount of space within an object. Students use their abilities to identify radii, bases, and heights, together with the geometric abilities developed in earlier grades, to perceive similar structure (MP7) in formulas for the volume of a rectangular prism and the volume of a cylinder—both are the product of base and height. After gaining familiarity with a formula for the volume of a cylinder by using it to solve problems, students perceive similar structure (MP7) in a formula for the volume of a cone.

The fifth section of the unit begins with an examination of functional relationships between two quantities that are illustrated by changes in scale for three-dimensional figures. For example, if the radius of a cylinder triples, its volume becomes nine times larger. This work combines grade 7 work on scale and proportional relationships. In grade 7, students studied scaled copies of two-dimensional figures, recognizing lengths are scaled by a scale factor and areas by the square of the scale factor, and applied their knowledge to scale drawings, e.g., maps and floor plans. In their study of proportional relationships, grade 7 students solved problems set in contexts commonly associated with proportional relationships such as constant speed, unit pricing, and measurement conversions, and learned that any proportional relationship can be represented by an equation of the form $y = kx$ where k is the constant of proportionality. In this section, students use their knowledge of scale, proportional relationships, and volume to reason about how the volume of a prism, cone, or cylinder changes as another measurement changes.

In the last section of the unit, students reason about how the volume of a sphere changes as its radius changes. They consider a situation in which water flows into a cylinder, cone, and sphere at the same constant rate. Information about the height of the water in each container is shown in an equation, graph, or table, allowing students to use it strategically (MP5) to compare water heights and capacities for the containers.

Associations in Data

In this unit, students analyze bivariate data—using scatter plots and fitted lines to analyze numerical data, and using two-way tables, bar graphs, and segmented bar graphs to analyze categorical data.

The unit begins with an investigation of a table of data. Measurements of a leg and perimeter of an isosceles right triangle are shown in each row, but column entries are not in order, making it hard to discern a pattern. Students manipulate the data to look for patterns in the table (MP7), then examine a scatter plot of the same data. This motivates the need to use different representations of the same data to find and analyze any patterns.

The second section begins with investigation of two questions: “Are older students always taller?” and “Do taller students tend to have bigger hands?” Students collect data (measurements of each student’s arm span, hand span, and height) and record each student’s measurements together with the student’s age in months. They make a scatter plot for height vs. hand span and select their own methods to display the height data (MP5).

The second section focuses on using scatter plots and fitted lines to analyze numerical data. Students make and examine scatter plots, interpreting points in terms of the quantities represented (MP2) and identifying scatter plots that could represent verbal descriptions of associations between two numerical variables (MP1). They see examples of how a line can be used to model an association between measurements displayed in a scatter plot and they compare values predicted by a linear model with the actual values given in the scatter plot (MP4). They draw lines to fit data displayed in scatter plots and informally assess how well the line fits by judging the closeness of the data points to the lines (MP4). Students compare scatter plots that show different types of associations (MP7) and learn to identify these types, making connections between the overall shape of a cloud of points and trends in the data represented, e.g., a scatter plot of used car price vs. mileage shows a cloud of points that descends from left to right and prices of used cars decrease with increased mileage (MP2). They make connections between the overall shape of a cloud of points, the slope of a fitted line, and trends in the data, e.g., "a line fit to the data has a negative slope and the scatter plot shows a negative association between price of a used car and its mileage." Outliers are informally identified based on their relative distance from other points in a scatter plot. Students examine scatter plots that show linear and non-linear associations as well as some sets of data that show clustering, describing their differences (MP7). They return to the data on height and arm span gathered at the beginning of the unit, describe the association between the two, and fit a line to the data (MP4).

The third section focuses on using two-way tables to analyze categorical data (MP4). Students use a two-way frequency table to create a relative frequency table to examine the percentages represented in each intersection of categories to look for any associations between the categories. Students also examine and create bar and segmented bar graphs to visualize any associations.

The unit ends with a lesson in which students collect and analyze numerical data using a scatter plot, then categorize the data based on a threshold and analyze the categories based on a two-way table (MP4).

Exponents and Scientific Notation

Students were introduced to exponent notation in grade 6. They worked with expressions that included parentheses and positive whole-number exponents with whole-number, fraction, decimal, or variable bases, using properties of exponents strategically, but did not formulate rules for use of exponents.

In this unit, students build on their grade 6 work. The first section of the unit begins with a lesson that reviews exponential expressions, including work with exponential expressions with bases 2 and $\frac{1}{2}$. In the next two lessons, students examine powers of 10, formulating the rules $10^n \cdot 10^m = 10^{n+m}$, $10^{n^m} = 10^{nm}$, and, for $n > m$, $10^n \cdot 10^m = 10^{n-m}$ where n and m are positive integers. After working with these powers of 10, they consider what the value of 10^0 should be and define 10^0 to be 1. In the next lesson, students consider what happens when the exponent rules are used on exponential expressions with base 10 and negative integer exponents and define 10^{-n} to be $\frac{1}{10^n}$. In the next two lessons, they expand their work to numerical bases other than 10, using exponent rules with products of exponentials with the same base and contrasting it with products of exponentials with different bases. They note numerical instances of $a^n \cdot b^n = (a \cdot b)^n$.

The second section of the unit returns to powers of 10 as a prelude to the introduction of scientific notation. Students consider differences in magnitude of powers of 10 and use powers of 10 and multiples of powers of 10 to describe magnitudes of quantities, e.g., the distance from the Earth to the Sun or the population of Russia. Initially, they work with large quantities, locating powers of 10 and positive-integer multiples of powers of 10 on the number line. Most of these multiples are products of single-digit numbers and powers of 10. The remainder are products of two- or three-digit numbers and powers of 10, allowing students to notice that these numbers may be expressed in different ways, e.g., $75 \cdot 10^5$ can be written $7.5 \cdot 10^6$, and that some forms may be more helpful in finding locations on the number line. In the next lesson, students do similar work with small quantities.

In the remaining five lessons, students write estimates of quantities in terms of integer or non-integer multiples of powers of 10 and use their knowledge of exponential expressions to solve problems, e.g., How many meter sticks does it take to equal the mass of the Moon? They are introduced to the term "scientific notation," practice distinguishing scientific from other notation, and use scientific notation (with no more than three significant figures) in order to make additive and multiplicative comparisons of pairs of quantities. They compute sums, differences, products, and quotients of numbers written in scientific notation (some with as many as four significant figures), using such calculations to estimate quantities. They make measurement conversions that involve powers of ten, e.g., converting bytes to kilobytes or gigabytes, choose appropriate units for measurements and express them in scientific notation.

Pythagorean Theorem and Irrational Numbers

Work in this unit is designed to build on and connect students' understanding of geometry and numerical expressions. The unit begins by foreshadowing algebraic and geometric aspects of the Pythagorean Theorem and strategies for proving it. Students are shown three squares and asked to compare the area of the largest square with the sum of the areas of the other two squares. The comparison can be done by counting grid squares and comparing the counts—when the three squares are on a grid with their sides on grid lines and vertices on intersections of grid lines—using the understanding of area measurement established in grade 3. The comparison can also be done by showing that there is a shape that can be decomposed and rearranged to form the largest square or the two smallest squares. Students are provided with opportunities to use and discuss both strategies.

In the second section, students work with figures shown on grids, using the grids to estimate lengths and areas in terms of grid units, e.g., estimating the side lengths of a square, squaring their estimates, and comparing them with estimates made by counting grid squares. The term "square root" is introduced as a way to describe the relationship between the side length and area of a square (measured in units and square units, respectively), along with the notation $\sqrt{}$. Students continue to work with side lengths and areas of squares. They learn and use definitions for "rational number" and "irrational number." They plot rational numbers and square roots on the number line. They use the meaning of "square root," understanding that if a given number p is the square root of n, then $x^2 = n$. Students learn (without proof) that $\sqrt{2}$ is irrational. They understand two proofs of the Pythagorean Theorem—an algebraic proof that involves manipulation of two expressions for the same area and a geometric proof that involves decomposing and rearranging two squares. They use the Pythagorean Theorem in two and three dimensions, e.g., to determine lengths of diagonals of rectangles and right rectangular prisms and to estimate distances between points in the coordinate plane.

In the third section, students work with edge lengths and volumes of cubes and other rectangular prisms. (In this grade, all prisms are right prisms.) They are introduced to the term "cube root" and the notation $\sqrt[3]{}$. They plot square and cube roots on the number line, using the meanings of "square root" and "cube root," e.g., understanding that if a given number x is the square root of n and n is between m and p, then x^2 is between m and p and that x is between $\sqrt{m}$ and $\sqrt{p}$.

In the fourth and last section, students work with decimal representations of rational numbers and decimal approximations of irrational numbers. In grade 7, they used long division in order to write fractions as decimals and learned that such decimals either repeat or terminate. They build on their understanding of decimals to make successive decimal approximations of $\sqrt{2}$ and π which they plot on number lines.

Putting it All Together

This section consists of two optional lessons in which students solve complex problems. In the first, they investigate relationships of temperature and latitude, climate, season, cloud cover, or time of day. In particular, they use scatter plots and lines of best fit to investigate the question of modeling temperature as a function of latitude. In the second, they consider tessellations of the plane, understanding and using the terms “tessellation” and “regular tessellation” in their work, and using properties of shapes (e.g., the sum of the interior angles of a quadrilateral is 360 degrees) to make inferences about regular tessellations.

Grade 6, Unit 2, Lesson 4: Color Mixtures

Goals

- Understand and communicate that doubling, tripling, or halving a recipe for colored water yields the same resulting color.
- Understand that “doubling the recipe” means “doubling each ingredient,” and more generally, multiple batches of a recipe result from multiplying the amounts of each ingredient by the same number.
- Understand that equivalent ratios represent mixtures that are comprised of multiple batches of the same recipe.

Learning Targets

- I can explain the meaning of equivalent ratios using a color mixture as an example.
- I can use a diagram to represent a single batch, a double batch, and a triple batch of a color mixture.
- I know what it means to double or triple a color mixture.

Lesson Narrative

This is the second of two lessons that help students make sense of equivalent ratios through physical experiences. In this lesson, students mix different numbers of batches of a recipe for green water by combining blue and yellow water (created ahead of time with food coloring) to see if they produce the same shade of green. They also change the ratio of blue and yellow water to see if it changes the result. The activities here reinforce the idea that scaling a recipe up (or down) requires scaling the amount of each ingredient by the same factor (MP7). Students continue to use discrete diagrams as a tool to represent a situation.

For students who do not see color, the lesson can be adapted by having students make batches of dough with flour and water. 1 cup of flour to 5 tablespoons of water makes a very stiff dough, and 1 cup of flour to 6 tablespoons of water makes a soft (but not sticky) dough. In this case, doubling a recipe yields dough with the same tactile properties, just as doubling a colored-water recipe yields a mixture with the same color. The invariant property is stiffness rather than color. The principle that equivalent ratios yield products that are identical in some important way applies to both types of experiments.

Alignments

Building On

- 4.NBT.B.5: Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Addressing

- 6.RP.A.1: Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. For example, “The ratio of wings to beaks in the bird house at the zoo was $2 : 1$, because for every 2 wings there was 1 beak.” “For every vote candidate A received, candidate C received nearly three votes.”

Required Materials

beakers

food coloring

graduated cylinders

markers

paper cups

Required Preparation

Mix blue water and yellow water; each group of 2 students will need 1 cup of each. To make colored water, add 1 teaspoon of food coloring to 1 cup of water. It would be best to give each mixture to students in a beaker or another container with a pour spout. If possible, conduct this lesson in a room with a sink.

Note that a digital version of this activity is available at this link: https://ggbm.at/Hcz2rDHc. It is embedded in the digital version of the student materials, but if classrooms using the print version of materials have access to enough student devices, it could be used in place of mixing actual colored water.

Student Learning Goals

Let’s see what color-mixing has to do with ratios.

4.1 Number Talk: Adjusting a Factor

Warm Up: 10 minutes

This number talk encourages students to use the structure of base ten numbers and the properties of operations to find the product of two whole numbers (MP7).

While many strategies may emerge, the focus of this string of problems is for students to see how adjusting a factor impacts the product, and how this insight can be used to reason about other problems. Four problems are given, however, it may not be possible to share every possible strategy. Consider gathering only two or three different strategies per problem. Each problem was chosen to elicit a slightly different reasoning, so as students explain their strategies, ask how the factors impacted how they approached the problem.

Building On

- 4.NBT.B.5

Instructional Routines

- **Number Talk**
  What: One problem is displayed at a time. Students are given a few minutes to quietly think and give a signal when they have an answer and a strategy. The teacher selects students to share different strategies for each problem, "Who thought about it a different way?" Their explanations are recorded for all to see. Students might be pressed to provide more details about why they decided to approach a problem a certain way. It may not be possible to share every possible strategy for the given limited time; the teacher may only gather two or three distinctive strategies per problem. Problems are purposefully chosen to elicit different approaches, often in a way that builds from one problem to the next.

  Where: Warm-ups

  Why: Number talks build computational fluency by encouraging students to think about the numbers in a computation problem and rely on what they know about structure, patterns, and properties of operations to mentally solve a problem. Dot images are similar to number talks, except the image used is an arrangement of dots that students might count in different ways. While participating in these activities, students need to be precise in their word choice and use of language (MP6).

Launch

Display one problem at a time. Give students 1 minute of quiet think time per problem and ask them to give a signal when they have an answer and a strategy. Follow with a whole-class discussion.

Support for Students with Disabilities

Memory: Processing Time. Provide sticky notes or mini whiteboards to aid students with working memory challenges.

Student Task Statement

Find the value of each product mentally.

$6 \cdot 15$

$12 \cdot 15$

$6 \cdot 45$

$13 \cdot 45$

Student Response

1. 90. Possible strategy: $(6 \cdot 10) + (6 \cdot 5) = 90$
2. 180. Possible strategy: Since the 6 from the first question doubled to 12, and the 15 stayed the same, the product doubles to 180. This is because there are twice as many groups of 15 than in the first question.
3. 270. Possible strategy: Since the 6 is the same as the in the first question, and the 15 tripled to 45, the product triples to 270. This is because the number of groups stayed the same, but the amount in each group got three times as large.
4. 585. Possible strategy: Since the 45 is the same as the previous question, we can double the 6 and the product to get 540. We need one more group of 45, and $540 + 45 = 585$.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. Ask students if or how the factors in the problem impacted the strategy choice. To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Did anyone solve the problem the same way but would explain it differently?"
- "Did anyone solve the problem in a different way?"
- "Does anyone want to add on to _____'s strategy?"
- "Do you agree or disagree? Why?"

4.2 Turning Green

35 minutes

In this activity, students mix different numbers of batches of a color recipe to obtain a certain shade of green. They observe how multiple batches of the same recipe produce the same shade of green as a single batch, which suggests that the ratios of blue to yellow for the two situations are equivalent. They also come up with a ratio that is not equivalent to produce a mixture that is a different shade of green.

As students make the mixtures, ensure that they measure accurately so they will get accurate outcomes. As students work, note the different diagrams students use to represent their recipes. Select a few examples that could be highlighted in discussion later.

Addressing

- 6.RP.A.1

Launch

Arrange students in groups of 2–4. (Smaller groups are better, but group size might depend on available equipment.) Each group needs a beaker of blue water and one of yellow water, one graduated cylinder, a permanent marker, a craft stick, and 3 opaque white cups (either styrofoam, white paper, or with a white plastic interior).

For classes using the print-only version: Show students the blue and yellow water. Demonstrate how to pour from the beakers to the graduated cylinder to measure and mix 5 ml of blue water with 15 ml of yellow water. Demonstrate how to get an accurate reading on the graduated cylinder by working on a level surface and by reading the measurement at eye level. Tell students they will experiment with different mixtures of green water and observe the resulting shades.

For classes using the digital version: Display the dynamic color mixing cylinders for all to see. Tell students, “The computer mixes colors when you add colored water to each cylinder. You can add increments of 1 or 5. You can’t remove water (once it’s mixed, it’s mixed), but you can start over. The computer mixes yellow and blue.” Ask students a few familiarization questions before they start working on the activity:

- What happens when you mix yellow and blue? (A shade of green is formed.)
- What happens if you add more blue than yellow? (Darker green, blue green, etc.)

If necessary, demonstrate how it works by adding some yellows and blues to both the left and the right cylinder. Show how the “Reset” button lets you start over.

Support for English Language Learners

Heavier Support. Assign one member from each group to stay behind to answer questions from students visiting from other groups. Provide visitors with question prompts such as, "These look like the same shade, but how can we be sure they are equivalent?", "If you want a smaller amount with the same shade, what can you do?" or "What is the ratio for your new mixture? And if I want a triple batch, what do I need to do?"

Support for Students with Disabilities

Strengths-based Approach:

- This activity leverages many natural strengths of students with ADHD, LD, and other concrete learners in terms of its multi-sensory and hands-on nature.
- This may be an opportunity for the teacher to highlight this strength in class and allow an individual with disability to lead peer interactions/discussions, increasing buy-in.

Anticipated Misconceptions

If any students come up with an incorrect recipe, consider letting this play out. A different shade of green shows that the ratio of blue to yellow in their mixture is not equivalent to the ratio of blue to yellow in the other mixtures. Rescuing the incorrect mixture to display during discussion may lead to meaningful conversations about what equivalent ratios mean.

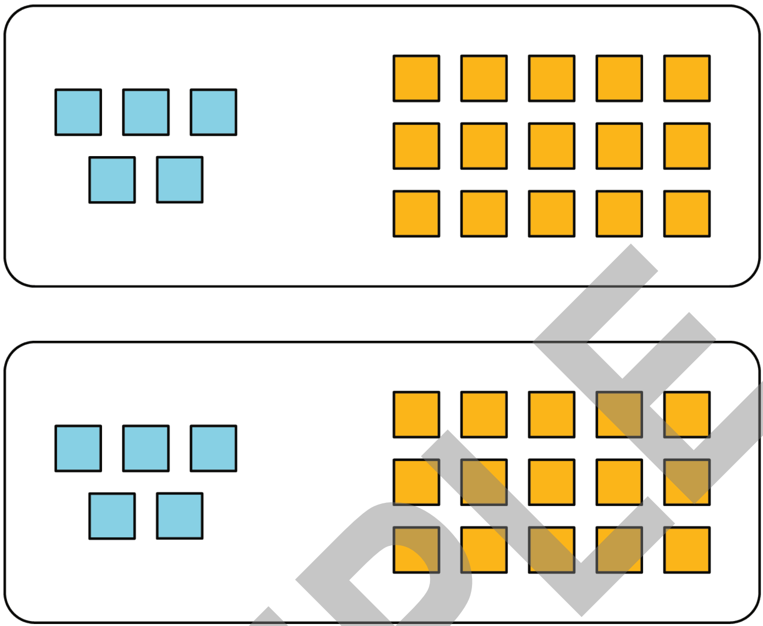
Student Task Statement

Your teacher mixed milliliters of blue water and milliliters of yellow water in the ratio $5 : 15$.

1. Doubling the original recipe:
    a. Draw a diagram to represent the amount of each color that you will combine to double your teacher's recipe.
    b. Use a marker to label an empty cup with the ratio of blue water to yellow water in this double batch.
    c. Predict whether these amounts of blue and yellow will make the same shade of green as your teacher's mixture. Next, check your prediction by measuring those amounts and mixing them in the cup.
    d. Is the ratio in your mixture equivalent to the ratio in your teacher's mixture? Explain your reasoning.
2. Tripling the original recipe:
    a. Draw a diagram to represent triple your teacher's recipe.
    b. Label an empty cup with the ratio of blue water to yellow water.
    c. Predict whether these amounts will make the same shade of green. Next, check your prediction by mixing those amounts.
    d. Is the ratio in your new mixture equivalent to the ratio in your teacher's mixture? Explain your reasoning.
3. Next, invent your own recipe for a *bluer* shade of green water.
    a. Draw a diagram to represent the amount of each color you will combine.
    b. Label the final empty cup with the ratio of blue water to yellow water in this recipe.
    c. Test your recipe by mixing a batch in the cup. Does the mixture yield a bluer shade of green?
    d. Is the ratio you used in this recipe equivalent to the ratio in your teacher's mixture? Explain your reasoning.

Student Response

1. Doubling the recipe.
    a. Here is one example of a diagram. Students may arrange the groups differently or use different symbols to represent 1 ml of water.



    b. A cup is labeled $10 : 30$ or "10 to 30."

    c. If the recipe is correct, the shade of green is identical to the teacher's.

    d. $10 : 30$ is equivalent to $5 : 15$ because it is 2 batches of the same recipe. It creates an identical shade of green.

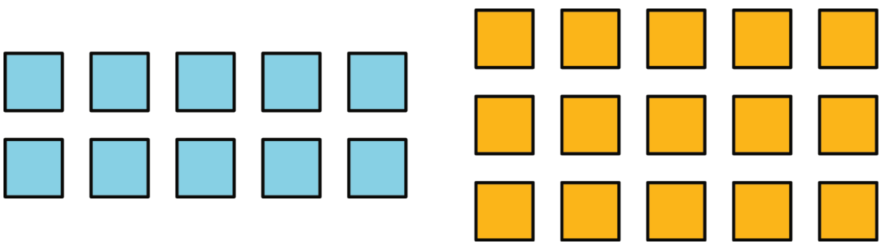
2. Tripling the recipe.
    a. Like the previous diagram, except showing 3 batches.

    b. A cup is labeled $15 : 45$ or "15 to 45."

    c. If the recipe is correct, the shade of green is identical to the teacher's.

    d. $15 : 45$ is equivalent to $5 : 15$ because it is 3 batches of the same recipe. It creates an identical shade of green.

3. A bluer shade of green.
    a. Answers vary. You might use more blue for the same amount of yellow, or less yellow for the same amount of blue. Sample response:



b. Answers vary. Sample responses: $10 : 15$ (more blue for the same amount of yellow) or $5 : 10$ (less yellow for the same amount of blue).

c. If a correct ratio is used, the mixture should be a bluer shade of green than the other mixtures.

d. No, it was not the same shade of green. The first and second parts were not, respectively, obtained by multiplying 5 and 15 by the same number.

Are You Ready for More?

Someone has made a shade of green by using 17 ml of blue and 13 ml of yellow. They are sure it cannot be turned into the original shade of green by adding more blue or yellow. Either explain how more can be added to create the original green shade, or explain why this is impossible.

Student Response

You could add 3 ml of blue to get 20 ml of blue, and 47 ml of yellow to get 60 ml of yellow. The blue to yellow ratio of $20 : 60$ will make the same shade of green as $5 : 15$. It's a quadruple batch.

Activity Synthesis

After each group has completed the task, have the students rotate through each group's workspace to observe the mixtures and diagrams. As they circulate, pose some guiding questions. (For students using the digital version, these questions refer to the mixtures on their computers.)

- Are each group's results for the first two mixtures the same shade of green?
- Are the ratios representing the double batch, the triple batch, and your new mixture all equivalent to each other? How do you know?
- What are some different ways groups drew diagrams to represent the ratios?

Highlight the idea that a ratio is equivalent to another if the two ratios describe different numbers of batches of the same recipe.

4.3 Perfect Purple Water

Optional: 10 minutes

Students revisit color mixing—this time by producing purple-colored water—to further understand equivalent ratios. They recall that doubling, tripling, or halving a recipe for colored water yields the same resulting color, and that equivalent ratios can represent different numbers of batches of the same recipe.

As students work, monitor for students who use different representations to answer both questions, as well as students who come up with different ratios for the second question.

Addressing

- 6.RP.A.1

Instructional Routines

- **Anticipate, monitor, select, sequence, connect**
  What: These are the *5 Practices for Orchestrating Productive Mathematical Discussions* (Smith and Stein, 2011). In this curriculum, much of the work of anticipating, sequencing, and connecting is handled by the materials in the activity narrative, launch, and synthesis sections. Teachers need to prepare for and conduct whole-class discussions.

  Where: Many classroom activities lend themselves to this structure.

  Why: In a problem-based curriculum, many activities can be described as "do math and talk about it," but the 5 Practices lend more structure to these activities so that they more reliably result in students making connections and learning new mathematics.

- **MLR2: Collect and Display**
  To capture students' oral words and phrases into a stable, collective reference. The intent of this routine is to stabilize the fleeting language that students use during partner, small-group, or whole-class activities in order for student's own output to be used as a reference in developing their mathematical language. The teacher listens for, and scribes, the student output using written words, diagrams and pictures; this collected output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. This routine provides feedback for students in a way that increases accessibility while simultaneously supporting meta-awareness of language.

- **MLR3: Critique, Correct, and Clarify**
  To give students a piece of mathematical writing that is not their own to analyze, reflect on, and develop. The intent is to prompt student reflection with an incorrect, incomplete, or ambiguous written argument or explanation, and for students to improve upon the written work by correcting errors and clarifying meaning. This routine fortifies output and engages students in meta-awareness. Teachers can demonstrate with meta-think-alouds and press for details when necessary.

- **Think pair share**
  What: Students have quiet time to think about a problem and work on it individually, and then time to share their response or their progress with a partner. Once these partner conversations have taken place, some students are selected to share their thoughts with the class.

  Why: This is a teaching routine useful in many contexts whose purpose is to give all students enough time to think about a prompt and form a response before they are expected to try to verbalize their thinking. First they have an opportunity to share their thinking in a low-stakes way with one partner, so that when they share with the class they can feel calm and confident, as well as say something meaningful that might advance everyone's understanding.

Additionally, the teacher has an opportunity to eavesdrop on the partner conversations so that she can purposefully select students to share with the class.

Launch

Arrange students in groups of 2. Remind students of the previous “Turning Green” activity. Ask students to discuss the following questions with a partner. Then, discuss responses together as a whole class:

- Why did the different mixtures of blue and yellow water result in the same shade of green? (If mixed correctly, the amount of the ingredients were all doubled or all tripled. The ratio of blue water to yellow water was equivalent within each recipe.)
- How were you able to get a darker shade of green? (We changed the ratio of ingredients, so there was more blue for the same amount of yellow.)

Explain to students that the task involves producing purple-colored water, but they won’t actually be mixing colored water. Ask students to use the ideas just discussed from the previous activity to predict the outcomes of mixing blue and red water.

Ensure students understand the abbreviation for milliliters is ml.

Support for English Language Learners

Heavier Support: MLR 2 (Collect and Display) with Oral to Written Math Explanation. Use this routine as students discuss and share their responses from the above questions about the "Turning Green" activity.

Support for Students with Disabilities

Conceptual Processing: Manipulatives. Provide manipulatives (i.e., snap cubes) to aid students who benefit from hands-on activities.

Anticipated Misconceptions

At a quick glance, students may think that since Andre is mixing a multiple of 8 with a multiple of 3, it will also result in Perfect Purple Water. If this happens, ask them to take a closer look at how the values are related or draw a diagram showing batches.

Student Task Statement

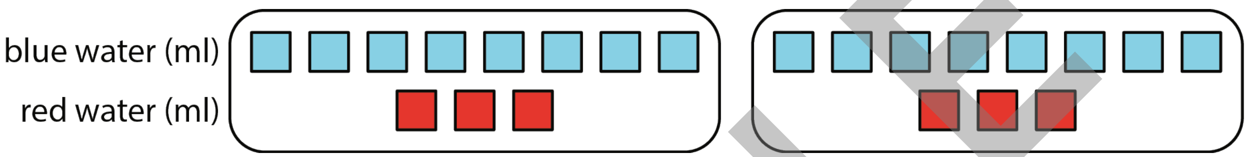
The recipe for Perfect Purple Water says, “Mix 8 ml of blue water with 3 ml of red water.”

Jada mixes 24 ml of blue water with 9 ml of red water. Andre mixes 16 ml of blue water with 9 ml of red water.

1. Which person will get a color mixture that is the same shade as Perfect Purple Water? Explain or show your reasoning.
2. Find another combination of blue water and red water that will also result in the same shade as Perfect Purple Water. Explain or show your reasoning.

Student Response

1. Jada's mixture will result in the same shade of purple, because both ingredients were tripled. $8 \cdot 3 = 24$ and $3 \cdot 3 = 9$. Andre's mixture will not result in the same shade of purple, because the amount of red water is doubled, but the amount of blue water was tripled.
2. Answers vary. One possible answer is $16 : 6$ (each ingredient is doubled or multiplied by 2. $8 \cdot 2 = 16$, there are 16 ml blue. $3 \cdot 2 = 6$, there are 6 ml red.)



Activity Synthesis

Select students to share their answers to the questions.

- For the first question, emphasize that not only did Jada triple each amount of red and blue, but this means that amount of each color is being *multiplied by the same value*, in this case, 3.
- For the second question, list all the different ratios students brought up for all to see. Discuss how each ratio differed from that for the original mixture. Point out that as long as both terms are multiplied by the same quantity, the resulting ratio will be *equivalent* and will yield the same shade of purple.

Support for English Language Learners

Lighter Support: MLR 3 (Critique, Correct and Clarify) with Critique a Partial or Flawed Response. For the second question, provide students with a flawed response such as "Mixing 18 ml of blue water with 13 ml of red water would result in the same shade of purple because I added 10 ml of each color."

Lesson Synthesis

The important take-aways from this lesson are:

- To create more batches of a color recipe that will come out to be the same shade of the color, multiply each ingredient by the same number.

- We can think of equivalent ratios as representing different numbers of batches of the same recipe.

Remind students of the work done and observations made in this lesson. Some questions to guide the discussion might include:

- How did you decide that 10 ml blue and 30 ml yellow would make 2 batches of 5 ml blue and 15 ml yellow? (Multiply each part by 2.)
- How did you decide that 15 ml blue and 45 ml yellow would make 3 batches? (Multiply each part by 3.)
- How did we know that $5 : 15$, $10 : 30$, and $15 : 45$ were equivalent? (They created the same shade of green. Also, $10 : 30$ has both parts of the original recipe multiplied by 2, and $15 : 45$ has both parts of the original recipe multiplied by 3.)

4.4 Orange Water

Cool Down: 5 minutes

Addressing

- 6.RP.A.1

Student Task Statement

A recipe for orange water says, “Mix 3 teaspoons yellow water with 1 teaspoon red water.” For this recipe, we might say: “The ratio of teaspoons of yellow water to teaspoons of red water is $3 : 1$.”

1. Write a ratio for 2 batches of this recipe.
2. Write a ratio for 4 batches of this recipe.
3. Explain why we can say that any two of these three ratios are equivalent.

Student Response

1. The ratio of teaspoons of yellow to teaspoons of red is $6 : 2$ (or any sentence that accurately states this ratio). Note: a statement like “The ratio of yellow to red is $6 : 2$” describes the association between the colors but does not indicate the amount of stuff in the mixture.
2. The ratio of teaspoons of yellow to teaspoons of red is $12 : 4$ (or any sentence that accurately states this ratio).
3. These are equivalent ratios because they describe different numbers of batches of the same recipe. To make 2 batches, multiply the amount of each color by 2. To make 4 batches, multiply the amount of each color by 4. As long as you multiply the amounts for both colors by the same number, you will get a ratio that is equivalent to the ratio in the recipe.

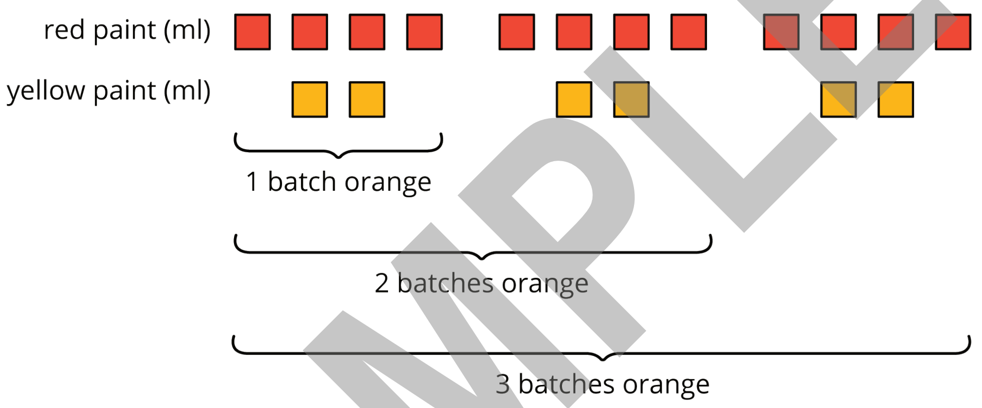
Student Lesson Summary

When mixing colors, doubling or tripling the amount of each color will create the same shade of the mixed color. In fact, you can always multiply the amount of *each* color by *the same number* to create a different amount of the same mixed color.

For example, a batch of dark orange paint uses 4 ml of red paint and 2 ml of yellow paint.

- To make two batches of dark orange paint, we can mix 8 ml of red paint with 4 ml of yellow paint.
- To make three batches of dark orange paint, we can mix 12 ml of red paint with 6 ml of yellow paint.

Here is a diagram that represents 1, 2, and 3 batches of this recipe.



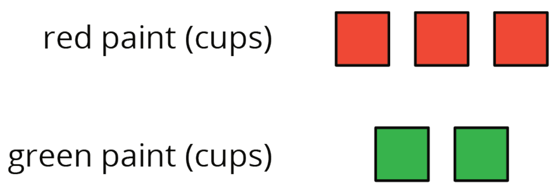
We say that the ratios $4 : 2$, $8 : 4$, and $12 : 6$ are *equivalent* because they describe the same color mixture in different numbers of batches, and they make the same shade of orange.

Lesson 4 Practice Problems

Problem 1

Statement

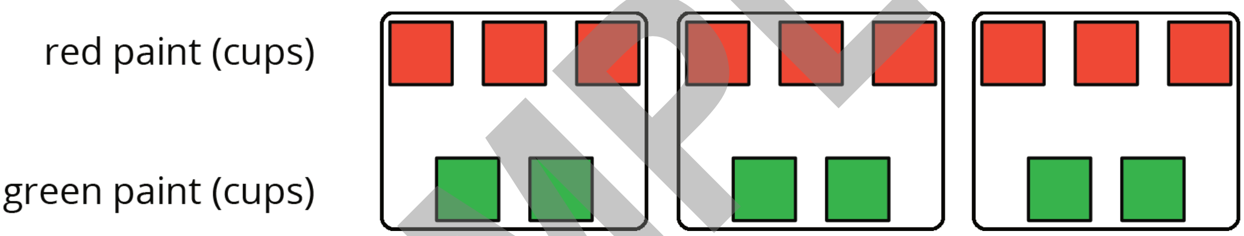
Here is a diagram showing a mixture of red paint and green paint needed for 1 batch of a particular shade of brown.



Add to the diagram so that it shows 3 batches of the same shade of brown paint.

Solution

Answers vary. Sample response:



Problem 2

Statement

Diego makes green paint by mixing 10 tablespoons of yellow paint and 2 tablespoons of blue paint. Which of these mixtures produce the same shade of green paint as Diego's mixture? Select **all** that apply.

A. For every 5 tablespoons of blue paint, mix in 1 tablespoon of yellow paint.

B. Mix tablespoons of blue paint and yellow paint in the ratio $1 : 5$.

C. Mix tablespoons of yellow paint and blue paint in the ratio 15 to 3.

D. Mix 11 tablespoons of yellow paint and 3 tablespoons of blue paint.

E. For every tablespoon of blue paint, mix in 5 tablespoons of yellow paint.

Solution

["B", "C", "E"]

Problem 3

Statement

To make 1 batch of sky blue paint, Clare mixes 2 cups of blue paint with 1 gallon of white paint.

a. Explain how Clare can make 2 batches of sky blue paint.

b. Explain how to make a mixture that is a darker shade of blue than the sky blue.

c. Explain how to make a mixture that is a lighter shade of blue than the sky blue.

Solution

a. Mix 4 cups of blue paint and 2 gallons of white paint.

b. Answers vary. Sample response: 3 cups of blue paint and 1 gallon of white paint. Mixing the same amount of white paint with *more* blue paint will make a darker shade of blue.

c. Answers vary. Sample response: 2 cups of blue paint and 2 gallons of white paint. Mixing the same amount of blue paint with *more* white paint will make a lighter shade of blue.

Problem 4

Statement

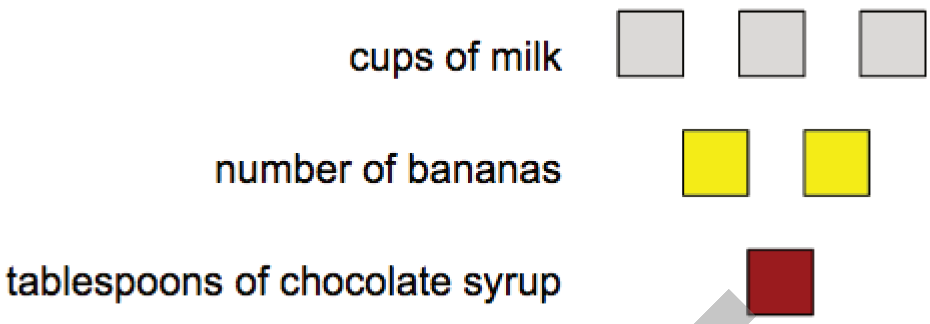
A smoothie recipe calls for 3 cups of milk, 2 frozen bananas and 1 tablespoon of chocolate syrup.

a. Create a diagram to represent the quantities of each ingredient in the recipe.

b. Write 3 different sentences that use a ratio to describe the recipe.

Solution

a. Answers vary. Sample response:



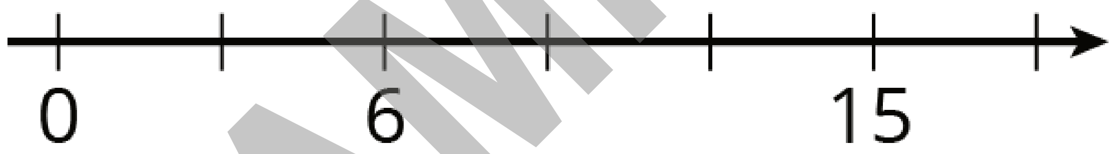
2. Answers vary. Sample response: The ratio of cups of milk to number of bananas is $3 : 2$, the ratio of bananas to tablespoons of chocolate syrup is 2 to 1, for every tablespoon of chocolate syrup, there are 3 cups of milk.

(From Grade6, Unit 2, Lesson 2.)

Problem 5

Statement

Write the missing number under each tick mark on the number line.



Solution

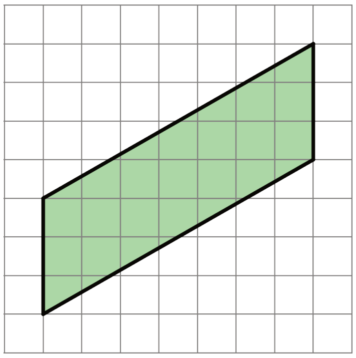
0, 3, 6, 9, 12, 15, 18 (intervals of 3)

(From Grade6, Unit 2, Lesson 1.)

Problem 6

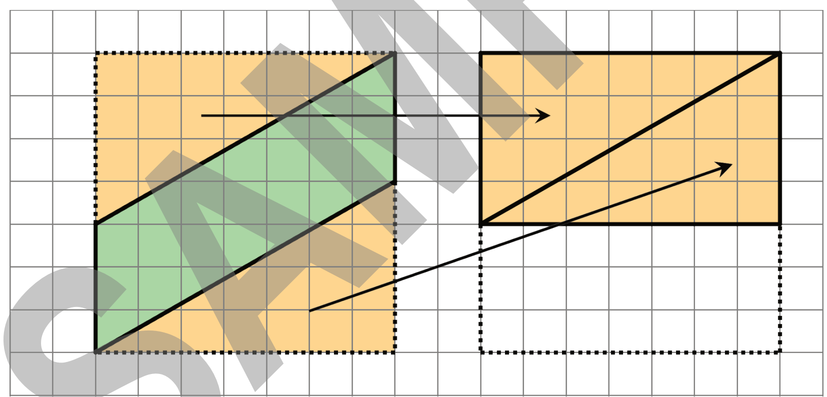
Statement

Find the area of the parallelogram. Show your reasoning.



Solution

21 square units. Reasoning varies. Sample reasoning: Draw a square around the parallelogram; its area is 49 square units, because $7 \cdot 7 = 49$. Rearrange the triangles above and below the parallelogram to form a rectangle; the area of this rectangle is 28 square units, because $4 \cdot 7 = 28$. Subtracting the area of the triangles from the area of the square, we have 21 square units. $49 - 28 = 21$.



(From Grade6, Unit 1, Lesson 4.)

Problem 7

Statement

Complete each equation with a number that makes it true.

a. $11 \cdot \frac{1}{4} =$ ________

b. $7 \cdot \frac{1}{4} =$ ________

c. $13 \cdot \frac{1}{27} =$ ________

d. $13 \cdot \frac{1}{99} =$ ________

e. $x \cdot \frac{1}{y} =$ ________
(As long as y does not equal 0.)

Solution

a. $\frac{11}{4}$ (or equivalent)

b. $\frac{7}{4}$ (or equivalent)

c. $\frac{13}{27}$ (or equivalent)

d. $\frac{13}{99}$ (or equivalent)

e. $\frac{x}{y}$ (or equivalent)

(From Grade6, Unit 2, Lesson 1.)

Grade 6, Unit 2, Lesson 17: A Fermi Problem

Goals

- Apply reasoning developed throughout this unit to an unfamiliar problem.
- Decide what information is needed to solve a real-world problem.
- Make simplifying assumptions about a real-world situation.

Learning Targets

- I can apply what I have learned about ratios and rates to solve a more complicated problem.
- I can decide what information I need to know to be able to solve a real-world problem about ratios and rates.

Lesson Narrative

This unit concludes with an opportunity for students to apply the reasoning developed so far to solve an unfamiliar, Fermi-type problem. Students must take a problem that is not well-posed and make assumptions and approximations to simplify the problem (MP4) so that it can be solved, which requires sense making and perseverance (MP1). To understand what the problem entails, students break down larger questions into more-manageable sub-questions. They need to make assumptions, plan an approach, and reason with the mathematics they know.

Engineers, computer scientists, physicists, and economists often make simplifying assumptions as they tackle complex problems involving mathematical modeling. Later in the year, unit 9 provides more exploration with solving Fermi problems, which are examples of mathematical modeling (MP4).

Alignments

Building On

- 4.MD: Grade 4 - Measurement and Data
- 4.MD.A: Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
- 5.MD: Grade 5 - Measurement and Data

Addressing

- 6.RP.A: Understand ratio concepts and use ratio reasoning to solve problems.
- 6.RP.A.3: Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

Required Materials

tools for creating a visual display

Student Learning Goals

Let's solve a Fermi problem.

17.1 Fix It!

Warm Up: 10 minutes

This activity encourages students to apply ratio reasoning to solve a problem they might encounter naturally outside a mathematics classroom. The warm up invites open-ended thinking that is validated by mathematical reasoning, which is the type of complex thinking needed to solve Fermi problems in the following activities.

Addressing

- 6.RP.A.3

Launch

Arrange students in groups of 2. Display the image for all to see.

Optionally, instead of the abstract image, you could bring in a clear glass, milk, and cocoa powder. Pour 1 cup of milk into the glass, add 5 tablespoons of cocoa powder, and introduce the task that way.

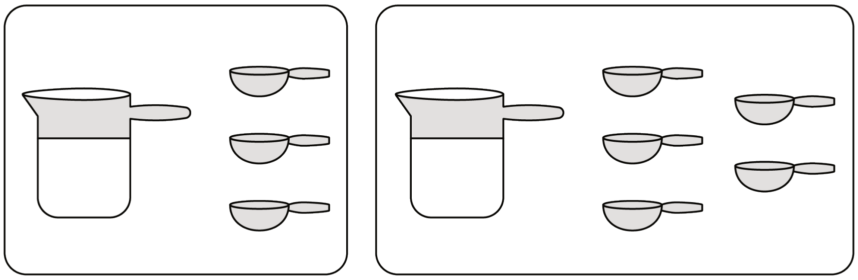
Tell students to give a signal when they have an answer and a strategy. Give students 2 minutes of quiet think time.

Support for Students with Disabilities

Conceptual Processing: Eliminate Barriers. Begin with a physical demonstration of the actions that occur in a situation.

Student Task Statement

Andre likes a hot cocoa recipe with 1 cup of milk and 3 tablespoons of cocoa. He poured 1 cup of milk but accidentally added 5 tablespoons of cocoa.



1. How can you fix Andre's mistake and make his hot cocoa taste like the recipe?
2. Explain how you know your adjustment will make Andre's hot cocoa taste the same as the one in the recipe.

Student Response

1. Answers vary. Possible strategies: Add 1 more tablespoon of cocoa and 1 cup of milk or add $\frac{2}{3}$ cup of milk.
2. The ratios for the recipe and for the fixed mixture are equivalent.

Activity Synthesis

Invite students to share their strategies with the class and record them for all to see. After each explanation, ask the class if they agree or disagree and how they know two hot cocoas will taste the same.

17.2 Who Was Fermi?

15 minutes
In this activity, students are introduced to the type of thinking useful for Fermi problems. The purpose of this activity is not to come up with an answer, but rather to see different ways to break a Fermi problem down into smaller questions that can be measured, estimated, or calculated.

Much of the appeal of Fermi problems is in making estimates for things that in modern times we *could* easily look up. To make this lesson more fun and interesting, challenge students to work without performing any internet searches.

As students work, notice the range of their estimates and the sub-questions they formulate to help them answer the large questions. Some examples of productive sub-questions might be:

- What information do we *know*?
- What information can be *measured*?
- What information cannot be measured but can be *calculated*?
- What *assumptions* should we make?

Building On

- 4.MD.A
- 5.MD

Launch

Open the activity with one or two questions that your students may find thought-provoking. Some ideas:

- “How many times does your heart beat in a year?”
- “How many hours of television do you watch in a year?”
- “Some research has shown that it takes 10,000 hours of practice for a person to achieve the highest level of performance in any field—sports, music, art, chess, programming, etc. If you aspire to be a top performer in a field you love—as Michael Jordan in basketball, Tiger Woods in golf, Maya Angelou in literature, etc., how many years would it take you to meet that 10,000-hour benchmark if you start now? How old would you be?”

Give students a moment to ponder a question and make a rough estimate. Then, share that the questions above are called “Fermi problems,” named after Enrico Fermi, an Italian physicist who loved to think up and discuss problems that are impossible to measure directly, but can be roughly estimated using known facts and calculations. Here are some other examples of Fermi problems:

- “How long would it take to paddle across the Pacific Ocean?”
- “How much would it cost to replace all the windows on all the buildings in the United States?”

Share the questions above or select a few other Fermi-type questions that are likely to intrigue your students. Have some resources on hand to support the investigation on your chosen questions (e.g., have a globe handy if the question about paddling across the Pacific is on your short list). As a class, decide on one question to pursue. For this activity, consider giving students the option to either work independently or in groups of two.

Support for Students with Disabilities

Executive Functioning: Eliminate Barriers. Provide a project checklist that chunks the various steps of the project into a set of manageable tasks.

Student Task Statement

1. Record the Fermi question that your class will explore together.
2. Make an estimate of the answer. If making an estimate is too hard, consider writing down a number that would definitely be too low and another number that would definitely be too high.

3. What are some smaller sub-questions we would need to figure out to reasonably answer our bigger question?

4. Think about how the smaller questions above should be organized to answer the big question. Label each smaller question with a number to show the order in which they should be answered. If you notice a gap in the set of sub-questions (i.e., there is an unlisted question that would need to be answered before the next one could be tackled), write another question to fill the gap.

Student Response

Answers vary depending on the question explored. For "How long would it take to paddle across the Pacific Ocean?" some sub-questions might be:

- What is the distance across the Pacific Ocean?
- At what speed can you paddle a boat?
- Do we assume that someone paddles continuously, or that they take breaks to sleep?

Activity Synthesis

First, ask students to share their estimates. Note the lowest and highest estimates, and point out that it is perfectly acceptable for an estimate to be expressed as a range of values rather than a single value.

Ask students to share some of their smaller questions. Then, discuss how you might come up with answers to these smaller questions, which likely revolve around what information is known, can be measured, or can be computed. Also, discuss how our assumptions about the situation affect how we solve the problem.

17.3 Researching Your Own Fermi Problem

30 minutes

This activity asks students to choose or pose a Fermi problem and solve it, with the aim of promoting the reasoning and tools developed in this unit. Students brainstorm potential problems, choose one, and—after your review—use a graphic organizer to help them formulate the sub-questions that could support their problem solving. They go on to solve their chosen Fermi problem.

To encourage ratio reasoning and the use of tools such as double number lines and tables, look for problems that involve *two* quantities. Questions that involve one quantity can be solved with multi-step multiplication and without ratio reasoning (e.g., "How many pens are there at the school?" involves only one quantity—the number of pens). But a problem such as "How much would it cost to replace all the windows on all buildings in the U.S.?" or "How long would it take to paddle across the Pacific Ocean?" involves accounting for two quantities at the same time (cost and number of windows, or time and distance across the Pacific) and is more likely to elicit ratio reasoning. Keep this in mind as you help students sift through their ideas.

Building On

- 4.MD
- 5.MD

Addressing

- 6.RP.A

Instructional Routines

- **Group Presentations (6–8 ONLY)**
  Some activities instruct students to work in small groups to solve a problem with mathematical modeling, invent a new problem, design something, or organize and display data, and then create a visual display of their work. Teachers need to help groups organize their work so that others can follow it, and then facilitate different groups' presentation of work to the class. Teachers can develop specific questioning skills to help more students make connections and walk away from these experiences with desired mathematical learning. For example, instead of asking if anyone has any questions for the group, it is often more productive to ask a member of the class to restate their understanding of the group's findings in their own words.

Launch

Explain to students that they will now brainstorm some Fermi problems they are interested in answering and select one to solve. Consider sharing a few more examples of Fermi problems to jumpstart their thinking:

- How much would it cost to charge all the students' cell phones in the school for a month?
- How much does it cost to operate a car for a year?
- How long would it take to make a sandwich for everyone living in our town?
- How long would it take to read the dictionary out loud?
- How long would it take to give every dog in America a bath?

Tell students that once they have a few good ideas, they should pause and get your attention so that you could help to decide on the one problem to pursue.

Arrange students in groups of 2, if desired. Provide tools for creating a visual display.

Support for Students with Disabilities

Executive Functioning: Eliminate Barriers. Provide a project checklist that chunks the various steps of the project into a set of manageable tasks.

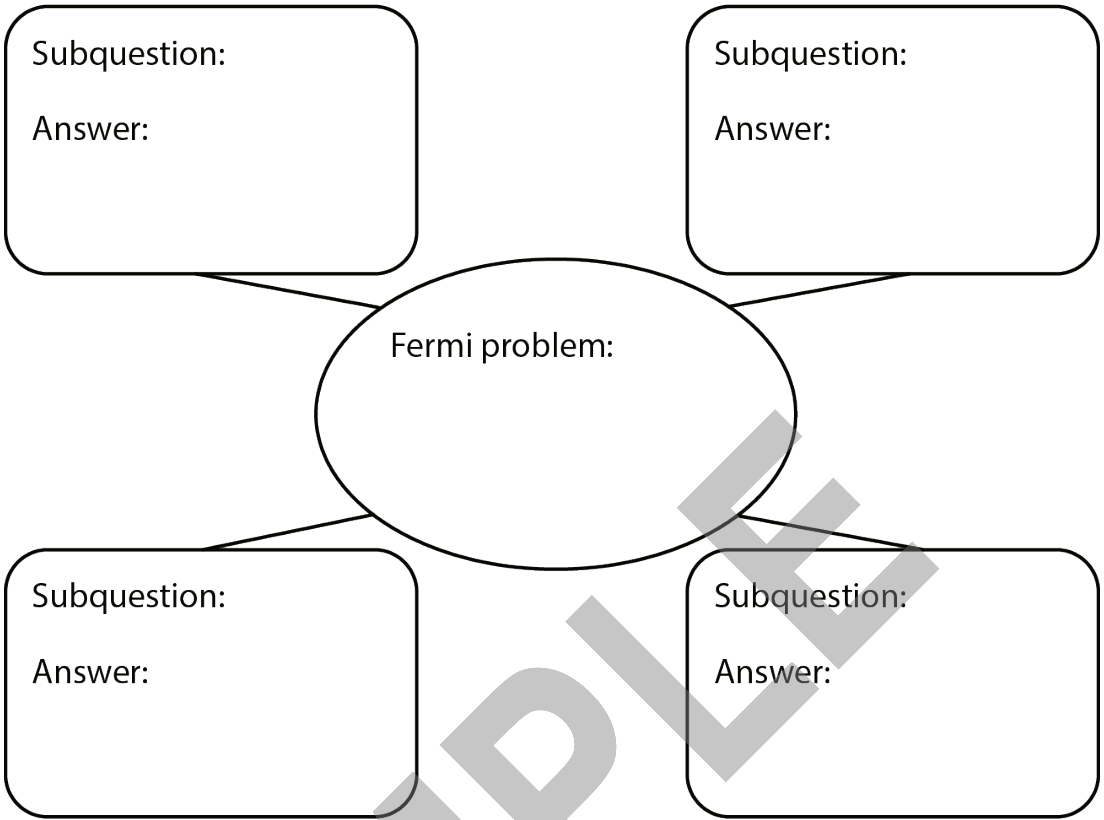
Receptive/Expressive Language: Peer Tutors. Pair students with their previously identified peer tutors to aid in comprehension and expression of understanding.

Anticipated Misconceptions

Students may think of problems that do not lend themselves to ratio reasoning because they only involve one quantity. If they have trouble coming up with any good options, offer them some examples. It may also be helpful to have a list of sample problems that students could refer to in creating their own problem.

Student Task Statement

1. Brainstorm at least five Fermi problems that you want to research and solve. If you get stuck, consider starting with "How much would it cost to . . .?" or "How long would it take to . . .?"
2. Pause here so your teacher can review your questions and approve one of them.
3. Use the graphic organizer to break your problem down into sub-questions.



4. Find the information you need to get closer to answering your question. Measure, make estimates, and perform any necessary calculations. If you get stuck, consider using tables or double number line diagrams.

5. Create a visual display that includes your Fermi problem and your solution. Organize your thinking so it can be followed by others.

Student Response

Answers vary.

Activity Synthesis

Display students' posters or visual presentations throughout the classroom. Consider asking some students (or all, if time permits) to present their problems and solutions to the class. Notice and highlight instances of ratio and rate reasoning, particularly productive use of double number lines or tables.

Lesson Synthesis

The debrief and presentation of student work provides opportunities to summarize takeaways from this lesson. Aside from opportunities to point out how ratio reasoning and the use of representations can help us tackle difficult problems, this lesson makes explicit some aspects of mathematical modeling. Highlight instances where students had to make an estimate in order to proceed, figured out what additional information they would need to make progress, or made simplifying assumptions.

Grade 6, Unit 8, Lesson 6: Histograms

Goals

- Recognize that a histogram allows us to see characteristics of the distribution that are hard to see in the raw data.
- Recognize that histograms provide an effective way to summarize large data sets, where is it more difficult to construct a dot plot.
- Understand how a histogram graphically summarizes a data set.

Learning Targets

- I can recognize when a histogram is an appropriate graphical display of a data set.
- I can use a histogram to get information about the distribution of data and explain what it means in a real-world situation.

Lesson Narrative

In this lesson students are introduced to **histograms**. They learn that, like a dot plot, a histogram can be used to show the distribution of a numerical data set, but unlike a dot plot, a histogram shows the frequencies of groups of values, rather than individual values. Students analyze the structures of dot plots and histograms displaying the same data sets and determine what information is easier to to understand from each type of display (MP7). Students read and interpret histograms in context (MP2) to prepare them to create a histogram.

Alignments

Addressing

- 6.SP.A.1: Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. For example, “How old am I?” is not a statistical question, but “How old are the students in my school?” is a statistical question because one anticipates variability in students' ages.
- 6.SP.A.3: Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.
- 6.SP.B.4: Display numerical data in plots on a number line, including dot plots, histograms, and box plots.
- 6.SP.B.5.b: Describing the nature of the attribute under investigation, including how it was measured and its units of measurement.

Required Materials

straightedges

Student Learning Goals

Let's explore how histograms represent data sets.

6.1 Dog Show (Part 1)

Warm Up: 5 minutes

The purpose of this warm-up is to connect the analytical work students have done with dot plots in previous lessons with statistical questions. This activity reminds students that we gather, display, and analyze data in order to answer statistical questions. This work will be helpful as students contrast dot plots and histograms in subsequent activities.

Addressing

- 6.SP.A.1
- 6.SP.A.3

Launch

Arrange students in groups of 2. Give students 1 minute of quiet work time, followed by 2 minutes to share their responses with a partner. Ask students to decide, during partner discussion, if each question proposed by their partner is a statistical question that can be answered using the dot plot. Follow with a whole-class discussion.

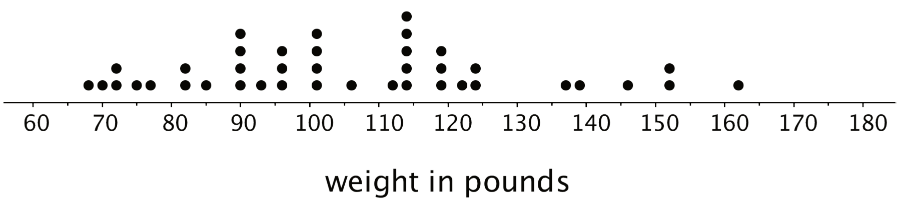
If students have trouble getting started, consider giving a sample question that can be answered using the data on the dot plot (e.g., "How many dogs weigh more than 100 pounds?")

Support for Students with Disabilities

Conceptual Processing: Processing Time. Provide the image to students who benefit from extra processing time to review prior to implementation of this activity.

Student Task Statement

Here is a dot plot showing the weights, in pounds, of 40 dogs at a dog show.



1. Write two statistical questions that can be answered using the dot plot.
2. What would you consider a typical weight for a dog at this dog show? Explain your reasoning.

Student Response

1. Answers vary. Sample questions:
    - How many dogs weigh exactly 70 pounds?
    - How many dogs weigh more 80 pounds but less than 150 pounds?
    - How much does the heaviest dog at the dog show weigh?
    - How many times as heavy as the lightest dog is the heaviest dog?
    - How alike or different are the weights of the dog at the show?
2. Answers vary. Sample responses:
    - About 114 pounds, because the largest percentage of the dots are at 114, and it seems to be about where the center of the data is.
    - About 100 pounds, because about half of the dogs are 100 pounds or lighter, and half are heavier than 100 pounds.

Activity Synthesis

Ask students to share questions they agreed were statistical questions that could be answered using the dot plot. Record and display their responses for all to see. If there is time, consider asking students how they would find the answer to some of the statistical questions.

Ask students to share a typical weight for a dog at this dog show and why they think it is typical. Record and display their responses for all to see. After each student shares, ask the class if they agree or disagree.

6.2 Dog Show (Part 2)

10 minutes

This activity introduces students to **histograms**. By now, students have developed a good sense of dot plots as a tool for representing distributions. They use this understanding to make sense of a different form of data representation. The data set shown on the first histogram is the same one from the preceding warm-up, so students are familiar with its distribution. This allows them to focus on making sense of the features of the new representation and comparing them to the corresponding dot plot.

Note that in all histograms in this unit, the left-end boundary of each bin or bar is included and the right-end boundary is excluded. For example, the number 5 would not be included in the 0–5 bin, but would be included in the 5–10 bin.

Addressing

- 6.SP.B.4
- 6.SP.B.5.b

Instructional Routines

- **MLR7: Compare and Connect**
  To foster students' meta-awareness as they identify, compare, and contrast different mathematical approaches, representations, and language. Teachers should demonstrate thinking out loud (e.g., exploring why we one might do or say it this way, questioning an idea, wondering how an idea compares or connects to other ideas or language), and students should be prompted to reflect and respond. This routine supports meta-cognitive and meta-linguistic awareness, and also supports mathematical conversation.

Launch

Explain to students that they will now explore histograms, another way to represent numerical data. Give students 3–4 minutes of quiet work time, and then 2–3 minutes to share their responses with a partner. Follow with a whole-class discussion.

Support for English Language Learners

Heavier Support: MLR 7 (Compare and Connect). Ask students to determine similarities between the representations and the differences. Students should be prompted to use the questions they answered about each graph to talk about similarities and differences.

Support for Students with Disabilities

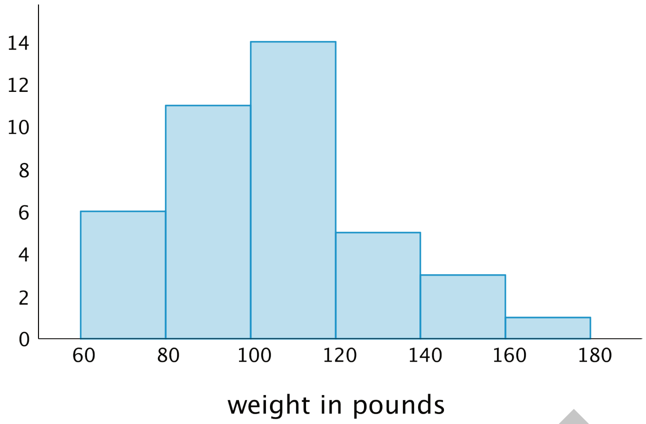
Receptive/Expressive Language: Processing Time. Students who benefit from extra processing time would also be aided by MLR 7 (Compare and Connect).

Executive Functioning: Graphic Organizers. Provide a Venn diagram with which to compare the similarities and differences during MLR 7 (Compare and Connect).

Executive Functioning: Visual Aids. Add **histogram** to the classroom vocabulary anchor chart.

Student Task Statement

Here is a **histogram** that shows some dog weights in pounds.



Each bar includes the left-end value but not the right-end value. For example, the first bar includes dogs that weigh 60 pounds and 68 pounds but not 80 pounds.

1. Use the histogram to answer the following questions.

    a. How many dogs weigh at least 100 pounds?

    b. How many dogs weigh exactly 70 pounds?

    c. How many dogs weigh at least 120 and less than 160 pounds?

    d. How much does the heaviest dog at the show weigh?

    e. What would you consider a typical weight for a dog at this dog show? Explain your reasoning.

2. Discuss with a partner:

    - If you used the dot plot to answer the same five questions you just answered, how would your answers be different?
    - How are the histogram and the dot plot alike? How are they different?

Student Response

1. a. 23 dogs.

    b. Unknown. It could be anywhere between 0 and 6.

    c. 8 dogs.

    d. The exact weight cannot be determined, but it weighs at least 160 pounds but less than 180 pounds.

    e. Answers vary. Sample response: Around 100 pounds. The largest percentage (35%) of the weights fall in the third bar (at least 100 pounds and less than 120 pounds), and it is approximately the middle of the data.

2. Answers vary. Sample responses:
    - They are alike in that they are both built on number lines, show the same total number of data values, and show how the values are spread out. They are different in that the dot plot shows individual data points and the histogram groups the data points together.
    - With the dot plot we can see the values of individual points and tell how many there are. With the histogram, we can't tell how many data points have a specific values; we only know how many points fall into a specific range.

Activity Synthesis

Ask a few students to briefly share their responses to the first set of questions to make sure students are able to read and interpret the graph correctly.

Focus the whole-class discussion on the last question. Select a few students or groups to share their observations about whether or how their answers to the statistical questions would change if they were to use a dot plot to answer them, and about how histograms and dot plots compare. If not already mentioned by students, highlight that, in a histogram:

- Data values are grouped into "bins" and represented as vertical bars.
- The height of a bar reflects the combined frequency of the values in that bin.
- A histogram uses a number line.

At this point students do not yet need to see the merits or limits of each graphical display; this work will be done in upcoming lessons. Students should recognize, however, how the structures of the two displays are different (MP7) and start to see that the structural differences affect the insights we are able to glean from the displays.

6.3 Population of States

20 minutes

In this activity, students continue to develop their understanding of histograms. They begin to notice that a dot plot may not be best for representing a data set with a lot of variability (or where few values are repeated) or when a data set has a large number of values. Histograms may help us visualize a distribution more clearly in these situations. Students organize a data set into "bins" and draw a histogram to display the distribution.

As students work and discuss, listen for explanations for why certain questions might be easy, hard, or impossible to answer using each graphical display.

Addressing

- 6.SP.B.4

Launch

Give students a brief overview of census and population data, as some students may not be familiar with them. Refer to the dot plot of the population data and discuss questions such as:

- “How many total dots are there?” (51)
- “What's the population of the state with the largest population? Do you know what state that is?” (Between 37 and 38 million. It's California.)
- “Look at the leftmost dot. What state might it represent? Approximately what is its population?” (The leftmost dot represents Wyoming, with a population of around half a million.)
- “Do you know the approximate population of our state? Where do you think we are in the dot plot?”

Explain to students that they will now draw a histogram to represent the population data. Remind them that histograms organize data values into “bins” or groups. In this case, the bins sizes are already decided for them. Then, arrange students in groups of 3–4. Provide access to straightedges. Give students 10–12 minutes to complete the activity. Encourage them to discuss their work within their group as needed.

Support for Students with Disabilities

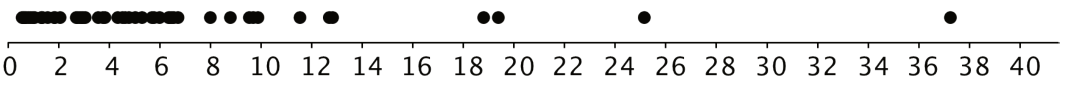
Conceptual Processing: Processing Time. Review an image or video about the Census in order to activate prior knowledge of the context of the problem.

Fine Motor Skills: Peer Tutors. Pair students with their previously identified peer tutors and allow students who struggle with fine motor skills to dictate creating histograms as needed.

Executive Functioning: Eliminate Barriers. Chunk this task into more manageable parts (e.g., presenting one question at a time), which will aid students who benefit from support with organizational skills in problem solving.

Student Task Statement

Every ten years, the United States conducts a census, which is an effort to count the entire population. The dot plot shows the population data from the 2010 census for each of the fifty states and the District of Columbia (DC).



population of states in millions

1. Here are some statistical questions about the population of the fifty states and DC. How difficult would it be to answer the questions using the *dot plot*?

   In the middle column, rate each question with an E (easy to answer), H (hard to answer), or I (impossible to answer). Be prepared to explain your reasoning.

statistical question	using the dot plot	using the histogram
a. How many states have populations greater than 15 million?		
b. Which states have populations greater than 15 million?		
c. How many states have populations less than 5 million?		
d. What is a typical state population?		
e. Are there more states with fewer than 5 million people, or more states with between 5 and 10 million people?		
f. How would you describe the distribution of state populations?		

2. Here are the population data for all states and the District of Columbia from the 2010 census. Use the information to complete the table.

Alabama	4.78
Alaska	0.71
Arizona	6.39
Arkansas	2.92
California	37.25
Colorado	5.03
Connecticut	3.57
Delaware	0.90
District of Columbia	0.60
Florida	18.80
Georgia	9.69
Hawaii	1.36
Idaho	1.57

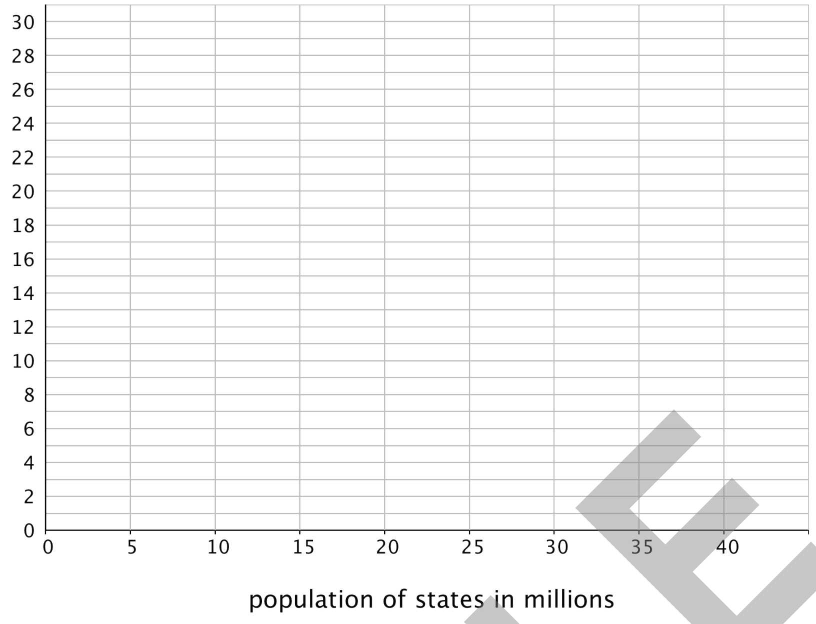
Illinois	12.83
Indiana	6.48
Iowa	3.05
Kansas	2.85
Kentucky	4.34
Louisiana	4.53
Maine	1.33
Maryland	5.77
Massachusetts	6.55
Michigan	9.88
Minnesota	5.30
Mississippi	2.97
Missouri	5.99

Montana	0.99
Nebraska	1.83
Nevada	2.70
New Hampshire	1.32
New Jersey	8.79
New Mexico	2.06
New York	19.38
North Carolina	9.54
North Dakota	0.67
Ohio	11.54
Oklahoma	3.75
Oregon	3.83
Pennsylvania	12.70

Rhode Island	1.05
South Carolina	4.63
South Dakota	0.81
Tennessee	6.35
Texas	25.15
Utah	2.76
Vermont	0.63
Virginia	8.00
Washington	6.72
West Virginia	1.85
Wisconsin	5.69
Wyoming	0.56

population (millions)	frequency
0–5	
5–10	
10–15	
15–20	
20–25	
25–30	
30–35	
35–40	

3. Use the grid and the information in your table to create a histogram.



4. Return to the statistical questions at the beginning of the activity. Which ones are now easier to answer?

   In the last column of the table, rate each question with an E (easy), H (hard), and I (impossible) based on how difficult it is to answer them. Be prepared to explain your reasoning.

Student Response

1. a. Easy (E). Unless some dots are lying directly on top of one another, there are four states with a population greater than 15 million.

   b. Impossible (I). Since the dots are not labeled, it is impossible to tell which states have a population greater than 15 million.

   c. Impossible (I). Since the dots are so close together below 5 million, it's impossible to count how many there are.

   d. Hard (H). Since so many dots are indistinguishable, it's hard to determine a typical state population.

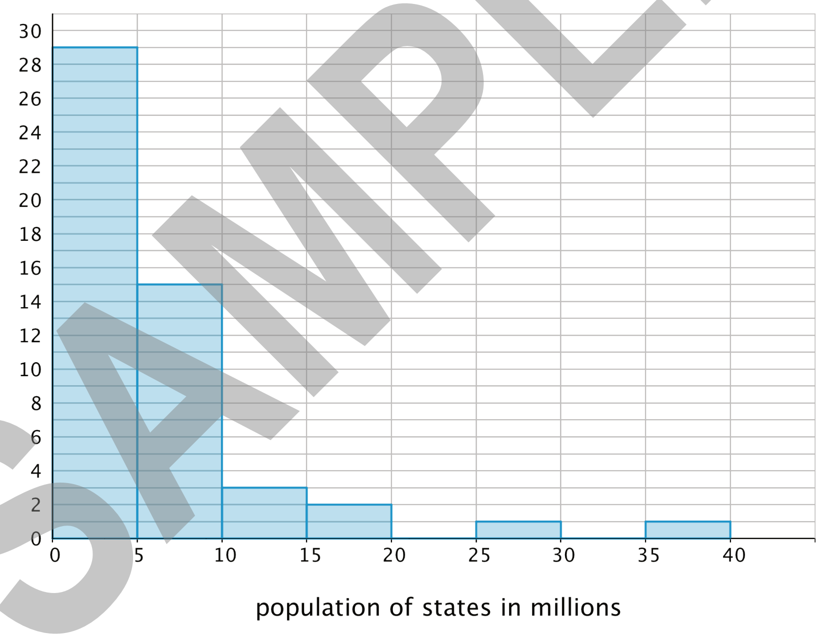
   e. Hard (H). It appears that there are more dots for populations that are less than 5 million than for those between 5 and 10 million, but we can't be sure because dots might be right on top of each other.

   f. Hard (H). Since the dots overlap a lot, it is difficult to give a good estimate for the center and spread.

2.

population in millions	frequency
0–5	29
5–10	15
10–15	3
15–20	2
20–25	0
25–30	1
30–35	0
35–40	1

3.



4. Revisiting the questions from the first problem:
   a. Still easy (E).

   b. Still impossible (I), based on the histogram alone.

   c. Now it is easy (E) to tell how many states had a population below 5 million. It was previously impossible (I).

   d. From the histogram, we can estimate that a typical state population has fewer than 10 million, but it is hard (H) to be more precise than that at this point. It was previously hard (H).

e. Using the histogram it is easy (E) to tell how many states have fewer than 5 million people and how many have between 5 and 10 million (there are more states in the smaller population category). It was previously hard (H).

f. It is easier (E) to describe the data distribution more precisely because the histogram shows the population sizes in intervals of 5 million people.

Are You Ready for More?

Think of two more statistical questions that can be answered using the data about populations of states. Then, decide whether each question can be answered using the dot plot, the histogram, or both.

Student Response

Answers vary.

Activity Synthesis

Much of the discussion about how to construct histograms should have happened in small groups. Address unresolved questions about drawing histograms if they are relatively simple. Otherwise, consider waiting until students have more opportunities to draw histograms in upcoming lessons.

Focus the discussion on comparing the effectiveness of dot plots and histograms to help us answer statistical questions.

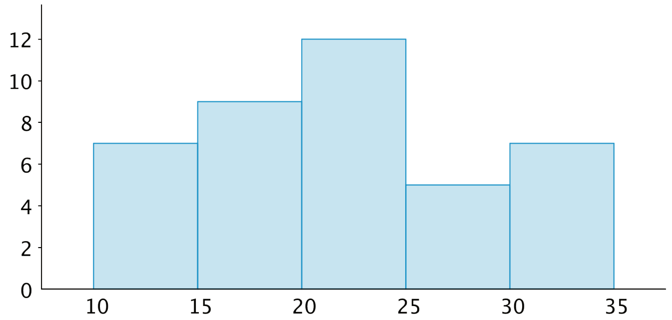
Select a few students or groups to share how their ratings of “easy,” “hard,” and “impossible,” changed when they transitioned from using dot plots to using histograms to answer statistical questions about populations of states. Then, discuss and compare the two displays more generally. Solicit as many ideas and observations as possible to these questions:

- “What are some benefits of histograms?”
- “When might histograms be preferable to dot plots?”
- “When might dot plots be preferable to histograms?”

Students should walk away from the activity recognizing that in cases when a data set has many numerical values, especially if the values do not repeat, a histogram can give us a better visualization of the distribution. In such a case, creating a dot plot can be very difficult to do including finding a scale that can meaningfully display the data while a histogram will be easier to create and display the information in a way that is easier to understand at a glance.

Lesson Synthesis

In this lesson, we learn about a different way to represent the distribution of numerical data—using a **histogram**. This histogram, for instance, represents the distribution for the weights of some dogs.



- “What could the smallest dog weigh? The largest?” (10 kilograms up to almost 50 kilograms)
- “What does the bar between 25 and 30 tell you?” (5 dogs weigh between 25 and just under 30 kilograms)
- “What can you say about the dogs who weigh between 10 and 20 kg?” (There are 18 total dogs in this range including 7 dogs between 10 and 15 kg and 11 between 15 and 20 kg)
- “In general, what information does a histogram allow us to see? How is it different from a dot plot?” (A bigger picture of the distribution is shown in the histogram, but some of the detail is lost when compared to a dot plot. For example, this histogram does not show the weight of any individual dogs.)
- “When might it be more useful to use a histogram than a dot plot?” (When the data is very spread out, when there are not very many data points with the same value, or when an overall idea of the distribution is more important than a detailed view.)

6.4 Rain in Miami

Cool Down: 5 minutes
Addressing

- 6.SP.B.4

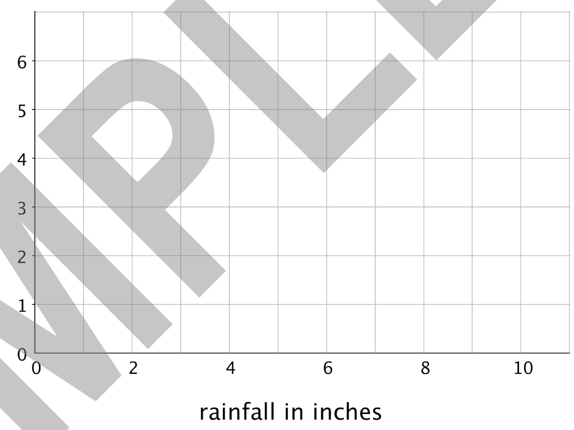
Student Task Statement

The table shows the average amount of rainfall, in inches, for each month in Miami, Florida.

month	rainfall (inches)	month	rainfall (inches)
January	1.61	July	6.5
February	2.24	August	8.9
March	2.99	September	9.84
April	3.14	October	6.34
May	5.35	November	3.27
June	9.69	December	2.05

1. Complete the table and use it to make a histogram.

rainfall (inches)	frequency
0–2	
2–4	
4–6	
6–8	
8–10	

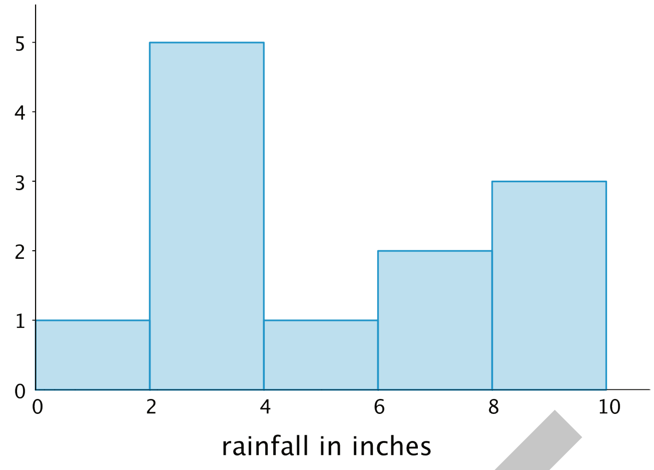


2. What is a typical amount of rainfall in one month in Miami?

Student Response

1.

rainfall (inches)	frequency
0–2	1
2–4	5
4–6	1
6–8	2
8–10	3

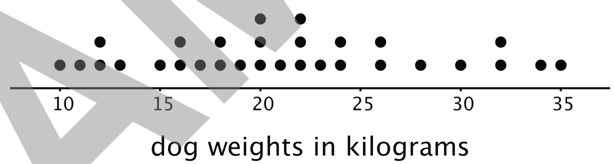


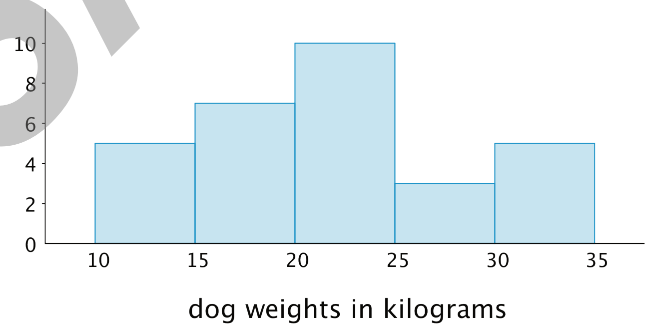
2. Answers vary. Sample response: It is difficult to say what typical amount of rainfall is. A typical month could be between 2 and 4 inches, this happens in 5 months. But for another 5 months, the rainfall is more than 6 inches, between 6 and 10 inches.

Student Lesson Summary

In addition to using dot plots, we can also represent distributions of numerical data using **histograms.**

Here is a dot plot that shows the weights, in kilograms, of 30 dogs, followed by a histogram that shows the same distribution.



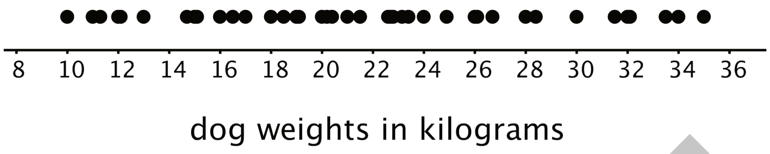


In a histogram, data values are placed in groups or “bins” of a certain size, and each group is represented with a bar. The height of the bar tells us the frequency for that group.

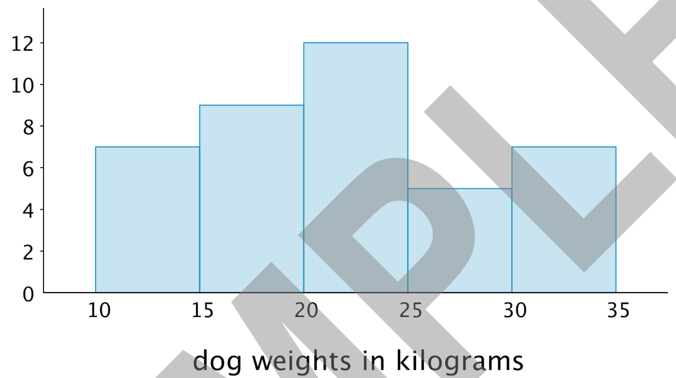
For example, the height of the tallest bar is 10, and the bar represents weights from 20 to less than 25 kilograms, so there are 10 dogs whose weights fall in that group. Similarly, there are 3 dogs that weigh anywhere from 25 to less than 30 kilograms.

Notice that the histogram and the dot plot have a similar shape. The dot plot has the advantage of showing all of the data values, but the histogram is easier to draw and to interpret when there are a lot of values or when the values are all different.

Here is a dot plot showing the weight distribution of 40 dogs. The weights were measured to the nearest 0.1 kilogram instead of the nearest kilogram.



Here is a histogram showing the same distribution.



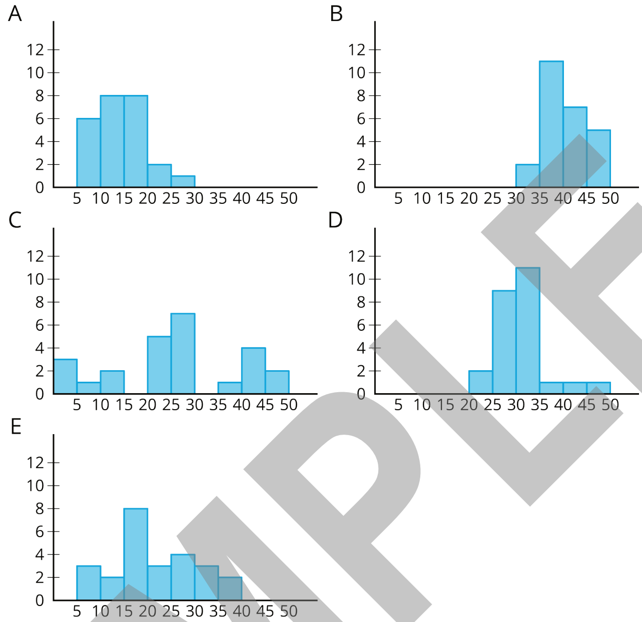
In this case, it is difficult to make sense of the distribution from the dot plot because the dots are so close together and all in one line. The histogram of the same data set does a much better job showing the distribution of weights, even though we can't see the individual data values.

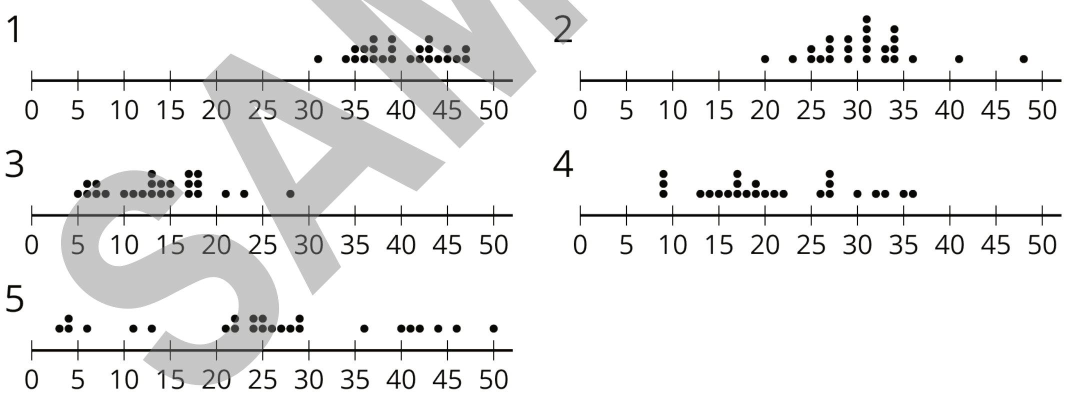
Lesson 6 Practice Problems

Problem 1

Statement

Match histograms A through E to dot plots 1 through 5 so that each match represents the same data set.





Solution

1. B

2. D

3. A

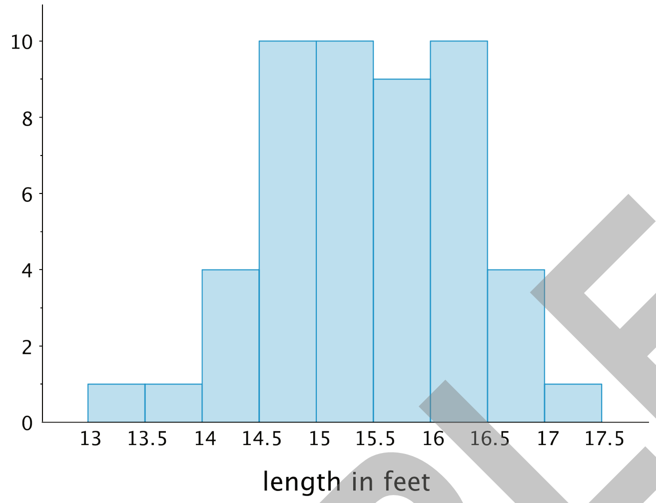
4. E

5. C

Problem 2

Statement

Here is a histogram that summarizes the lengths, in feet, of a group of adult female sharks. Select **all** the statements that are true, according to the histogram.



A. A total of 9 sharks were measured.

B. A total of 50 sharks were measured.

C. The longest shark that was measured was 10 feet long.

D. Most of the sharks that were measured were over 16 feet long.

E. Two of the sharks that were measured were less than 14 feet long.

Solution

["B", "E"]

Problem 3

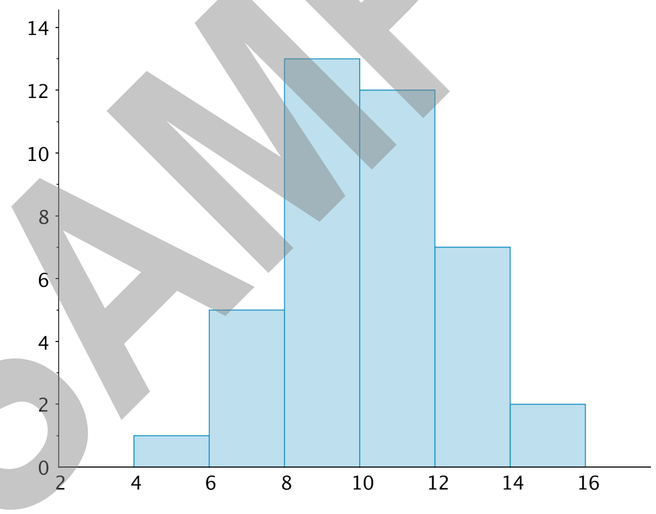
Statement

This table shows the times, in minutes, it took 40 sixth-grade students to run 1 mile.

time (minutes)	frequency
4 to less than 6	1
6 to less than 8	5
8 to less than 10	13
10 to less than 12	12
12 to less than 14	7
14 to less than 16	2

Draw a histogram for the information in the table.

Solution



Problem 4

Statement

$(-2, 3)$ is one vertex of a square on a coordinate plane. Name three points that could be the other vertices.

Solution

Answers vary. Sample response: $(2, 3)$, $(2, -3)$, $(-2, -3)$

(From Grade6, Unit 7, Lesson 12.)

Grade 7, Unit 1, Lesson 1: What are Scaled Copies?

Goals

- Use informal language to describe characteristics of scaled copies.
- Visually distinguish scaled copies of a figure.

Learning Targets

- I can describe some characteristics of a scaled copy.
- I can tell whether or not a figure is a scaled copy of another figure.

Lesson Narrative

This lesson introduces students to the idea of a **scaled copy** of a picture or a figure. Students learn to distinguish scaled copies from those that are not—first informally, and later, with increasing precision. They may start by saying that scaled copies have the same shape as the original figure, or that they do not appear to be distorted in any way, though they may have a different size. Next, they notice that the lengths of segments in a scaled copy vary from the lengths in the original figure in a uniform way. For instance, if a segment in a scaled copy is half the length of its counterpart in the original, then all other segments in the copy are also half the length of their original counterparts. Students work toward articulating the characteristics of scaled copies quantitatively (e.g., “all the segments are twice as long,” “all the lengths have shrunk by one third,” or “all the segments are one-fourth the size of the segments in the original”), articulating the relationships carefully (MP6) along the way.

The lesson is designed to be accessible to all students regardless of prior knowledge, and to encourage students to make sense of problems and persevere in solving them (MP1) from the very beginning of the course.

Alignments

Addressing

- 7.G.A.1: Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

Required Materials

pre-printed slips, cut from copies of the blackline master

Required Preparation

You will need the Pairs of Scaled Polygons blackline master for this lesson. Print and cut slips A–J for the Pairs of Scaled Polygons activity. Prepare 1 copy for every 2 students. If possible, copy each complete set on a different color of paper, so that a stray slip can quickly be put back.

Student Learning Goals

Let's explore scaled copies.

1.1 Printing Portraits

Warm Up: 10 minutes

This opening task introduces the term **scaled copy**. It prompts students to observe several copies of a picture, visually distinguish scaled and unscaled copies, and articulate the differences in their own words. Besides allowing students to have a mathematical conversation about properties of figures, it provides an accessible entry into the concept and gives an opportunity to hear the language and ideas students associate with scaled figures.

Students are likely to have some intuition about the term "to scale," either from previous work in grade 6 (e.g., scaling a recipe, or scaling a quantity up or down on a double number line) or from outside the classroom. This intuition can help them identify scaled copies.

Expect them to use adjectives such as "stretched," "squished," "skewed," "reduced," etc., in imprecise ways. This is fine, as students' intuitive definition of scaled copies will be refined over the course of the lesson. As students discuss, note the range of descriptions used. Monitor for students whose descriptions are particularly supportive of the idea that lengths in a scaled copy are found by multiplying the original lengths by the same value. Invite them to share their responses later.

Addressing

- 7.G.A.1

Instructional Routines

- **MLR1: Stronger and Clearer Each Time**
  To provide a structured and interactive opportunity for students to revise and refine both their ideas and their verbal and written output. This routine provides a purpose for student conversation as well as fortifies output. The main idea is to have students think or write individually about a response, use a structured pairing strategy to have multiple opportunities to refine and clarify the response through conversation, and then finally revise their original written response. Throughout this process, students should be pressed for details, and encouraged to press each other for details.

- **Think pair share**
  What: Students have quiet time to think about a problem and work on it individually, and then time to share their response or their progress with a partner. Once these partner

conversations have taken place, some students are selected to share their thoughts with the class.

Why: This is a teaching routine useful in many contexts whose purpose is to give all students enough time to think about a prompt and form a response before they are expected to try to verbalize their thinking. First they have an opportunity to share their thinking in a low-stakes way with one partner, so that when they share with the class they can feel calm and confident, as well as say something meaningful that might advance everyone's understanding. Additionally, the teacher has an opportunity to eavesdrop on the partner conversations so that she can purposefully select students to share with the class.

Launch

Arrange students in groups of 2. Give students 2–3 minutes of quiet think time and a minute to share their response with their partner.

If using the digital activity, have students work in groups of 2–3 to complete the activity. They should have quiet time in addition to share time, while solving the problem and developing language to describe scaling.

Support for English Language Learners

Lighter Support. As students share with a partner, notice any gesturing students use to describe the pictures. Refer to these student gestures during the synthesis to model how to use language to describe the gestures.

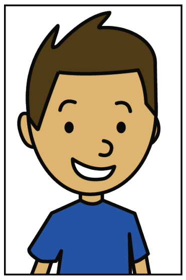
Heavier Support: MLR 1 (Stronger and Clearer Each Time). Use this routine for question 3. Have students work in pairs on questions 1, 2, and 3. Then have students meet with two different partners to make their answer to question 3 stronger and clearer.

Support for Students with Disabilities

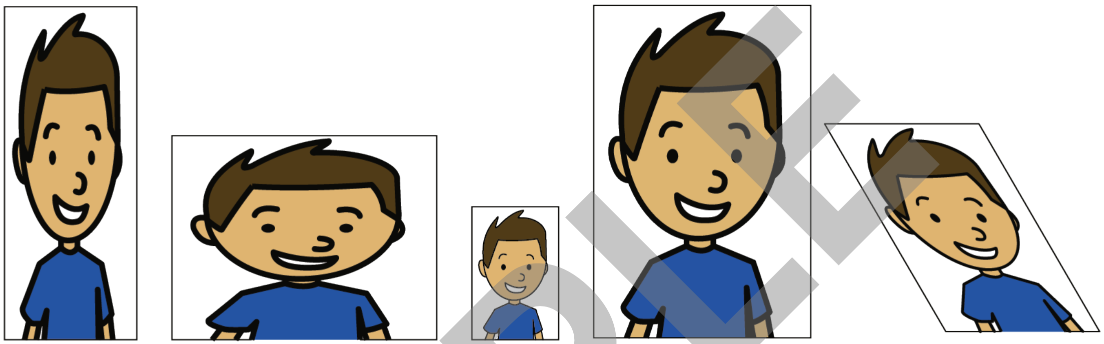
Fine Motor Skills: Assistive Technology. Provide access to the digital version of this activity.

Student Task Statement

Here is a portrait of a student.



1. Look at Portraits A–E. How is each one the same as or different from the original portrait of the student?



A B C D E

2. Some of the Portraits A–E are **scaled copies** of the original portrait. Which ones do you think are scaled copies? Explain your reasoning.

3. What do you think "scaled copy" means?

Student Response

1. Answers vary. Sample response:

    - Similarities: Pictures A–E are all based on the same original portrait. They all show the same boy wearing a blue shirt and brown hair. They all have the same white background.
    - Differences: They all have different sizes; some have different shapes. Pictures A, B, and E have been stretched or somehow distorted. C and D are not stretched or distorted but are each of a different size than the original.

2. C and D are scaled copies. Sample explanation:

    - A, B, and E are not scaled copies because they have changed in shape compared to the original portrait. Portrait A is stretched vertically, so the vertical side is now much longer than the horizontal side. B is stretched out sideways, so the horizontal sides are now longer than the vertical. E seems to have its upper left and lower right corners stretched out in opposite directions. The portrait is no longer a rectangle.

- C is a smaller copy and D is a larger copy of the original, but their shapes remain the same.

3. Answers vary. Sample definitions:

    - A scaled copy is a copy of a picture that changes in size but does not change in shape.
    - A scaled copy is a duplicate of a picture with no parts of it distorted, though it could be larger, smaller, or the same size.
    - A scaled copy is a copy of a picture that has been enlarged or reduced in size but nothing else changes.

Activity Synthesis

Select a few students to share their observations. Record and display students' explanations for the second question. Consider organizing the observations in terms of how certain pictures are or are not distorted. For example, students may say that C and D are scaled copies because each is a larger or smaller version of the picture, but the face (or the sleeve, or the outline of the picture) has not changed in shape. They may say that A, B, and E are not scaled copies because something other than size has changed. If not already mentioned in the discussion, guide students in seeing features of C and D that distinguish them from A, B, and E.

Invite a couple of students to share their working definition of scaled copies. Some of the students' descriptions may not be completely accurate. That is appropriate for this lesson, as the goal is to build on and refine this language over the course of the next few lessons until students have a more precise notion of what it means for a picture or figure to be a scaled copy.

Support for English Language Learners

Heavier Support. Use gestures to support student understanding of vocabulary such as "slanted," "skewed," "horizontal," "distorted," or "stretching."

1.2 Scaling F

10 minutes

This task enables students to describe more precisely the characteristics of scaled copies and to refine the meaning of the term. Students observe copies of a line drawing on a grid and notice how the lengths of line segments and the angles formed by them compare to those in the original drawing.

Students engage in MP7 in multiple ways in this task. Identifying distinguishing features of the scaled copies means finding similarities and differences in the shapes. In addition, the fact that corresponding parts increase by the *same* scale factor is a vital structural property of scaled copies.

For the first question, expect students to explain their choices of scaled copies in intuitive, qualitative terms. For the second question, students should begin to distinguish scaled and unscaled copies in more specific and quantifiable ways. If it does not occur to students to look at lengths of segments, suggest they do so.

As students work, monitor for students who notice the following aspects of the figures. Students are not expected to use these mathematical terms at this point, however.

- The original drawing of the letter F and its scaled copies have equivalent width-to-height ratios.
- We can use a scale factor (or a multiplier) to compare the lengths of different figures and see if they are scaled copies of the original.
- The original figure and scaled copies have corresponding angles that have the same measure.

Addressing

- 7.G.A.1

Instructional Routines

- **MLR2: Collect and Display**
  To capture students' oral words and phrases into a stable, collective reference. The intent of this routine is to stabilize the fleeting language that students use during partner, small-group, or whole-class activities in order for student's own output to be used as a reference in developing their mathematical language. The teacher listens for, and scribes, the student output using written words, diagrams and pictures; this collected output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. This routine provides feedback for students in a way that increases accessibility while simultaneously supporting meta-awareness of language.

- **Think pair share**
  What: Students have quiet time to think about a problem and work on it individually, and then time to share their response or their progress with a partner. Once these partner conversations have taken place, some students are selected to share their thoughts with the class.

  Why: This is a teaching routine useful in many contexts whose purpose is to give all students enough time to think about a prompt and form a response before they are expected to try to verbalize their thinking. First they have an opportunity to share their thinking in a low-stakes way with one partner, so that when they share with the class they can feel calm and confident, as well as say something meaningful that might advance everyone's understanding. Additionally, the teacher has an opportunity to eavesdrop on the partner conversations so that she can purposefully select students to share with the class.

Launch

Keep students in the same groups. Give them 3–4 minutes of quiet work time, and then 1–2 minutes to share their responses with their partner. Tell students that how they decide whether each of the seven drawings is a scaled copy may be very different than how their partner decides. Encourage students to listen carefully to each other's approach and to be prepared to share their strategies. Use gestures to elicit from students the words "horizontal" and "vertical" and ask groups to agree internally on common terms to refer to the parts of the F (e.g., "horizontal stems").

Support for English Language Learners

Lighter Support. Remind students to agree on shared names for parts of the F so they can understand one another when describing the drawings.

Support for Students with Disabilities

Fine Motor Skills:

- *Peer Tutors.* Pair students with their previously identified peer tutors and allow students who struggle with fine motor skills to dictate drawing scaled copies as needed.
- *Assistive Technology.* Provide access to the digital version of this activity.

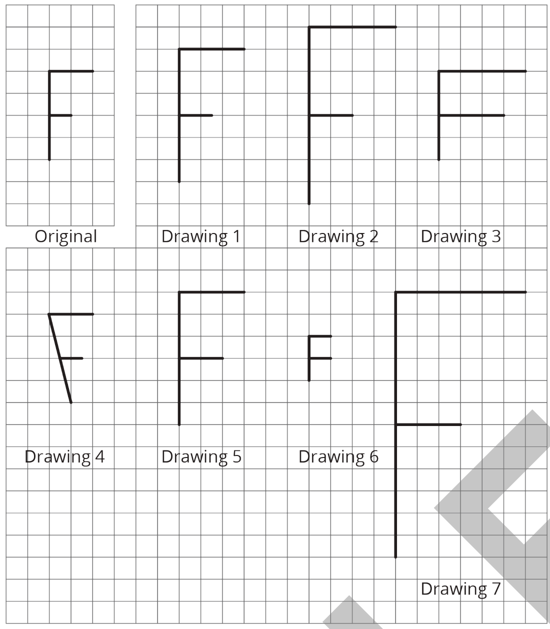
Anticipated Misconceptions

Students may make decisions by "eyeballing" rather than observing side lengths and angles. Encourage them to look for quantifiable evidence and notice lengths and angles.

Some may think vertices must land at intersections of grid lines (e.g., they may say Drawing 4 is not a scaled copy because the endpoints of the shorter horizontal segment are not on grid crossings). Address this during the whole-class discussion, after students have a chance to share their observations about segment lengths.

Student Task Statement

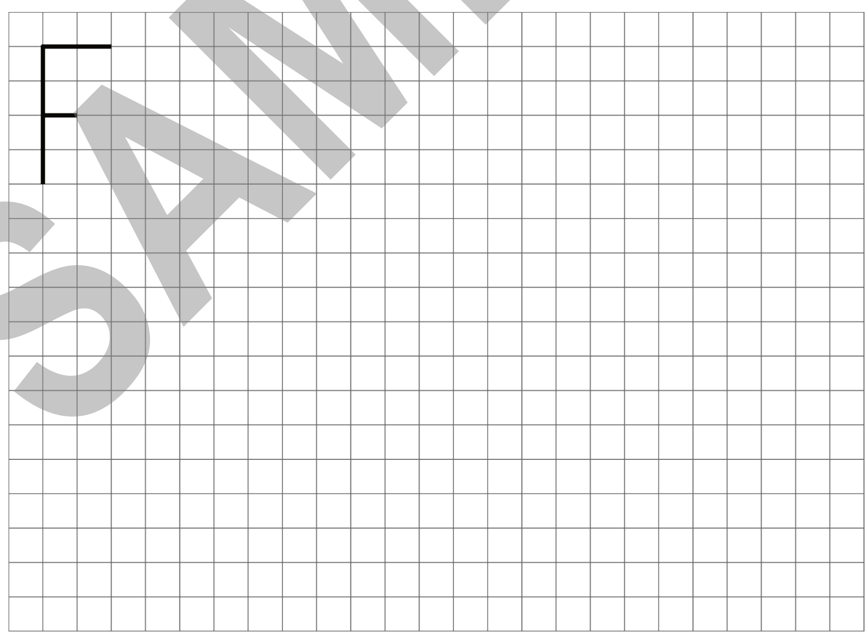
Here is an original drawing of the letter F and some other drawings.



1. Identify **all** the drawings that are scaled copies of the original letter F. Explain how you know.

2. Examine all the scaled copies more closely, specifically the lengths of each part of the letter F. How do they compare to the original? What do you notice?

3. On the grid, draw a different scaled copy of the original letter F.



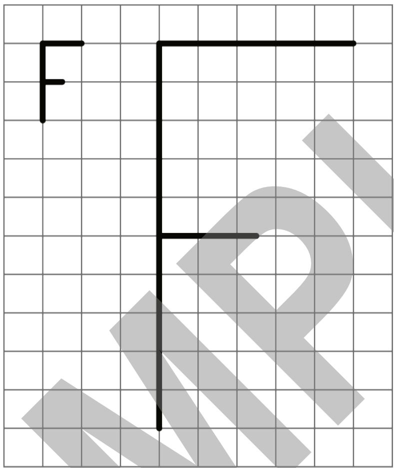
Student Response

1. Drawings 1, 2, and 7 are scaled copies of the original drawing. Explanations vary. Sample explanation: I know because they are not stretched differently in one direction. They are enlarged evenly in both vertical and horizontal directions.

2. Answers vary. Sample responses:

    - In the scaled copies, every segment is the same number of times as long as the matching segment in the original drawing.
    - In the scaled copies, all segments keep the same relationships as in the original. The original drawing of F is 4 units tall. Its top horizontal segment is 2 units wide and the shorter horizontal segment is 1 unit. In Drawing 1, the F is 6 units tall and 3 units wide; in Drawing 2, it is 8 units tall and 4 units wide, and in Drawing 7, it is 8 units tall and 4 units wide. In each scaled copy, the width is half of the height, just as in the original drawing of F, and the shorter horizontal segment is half of the longer one.

3. Drawings vary. Sample response:



Activity Synthesis

Display the seven copies of the letter F for all to see. For each copy, ask students to indicate whether they think each one is a scaled copy of the original F. Record and display the results for all to see. For contested drawings, ask 1–2 students to briefly say why they ruled these out.

Discuss the identified scaled and unscaled copies.

- What features do the scaled copies have in common? (Be sure to invite students who were thinking along the lines of scale factors and angle measures to share.)
- How do the other copies fail to show these features? (Sometimes lengths of sides in the copy use different multipliers for different sides. Sometimes the angles in the copy do not match the angles in the original.)

If there is a misconception that scaled copies must have vertices on intersections of grid lines, use Drawing 1 (or a relevant drawing by a student) to discuss how that is not the case.

Some students may not be familiar with words such as “twice,” “double,” or “triple.” Clarify the meanings by saying “two times as long” or “three times as long.”

Support for English Language Learners

Use MLR 2 (Collect and Display) to keep a running list of observations and sketches made by students and the explanation for why a drawing is scaled or not. Label diagrams that tell the scale factor as appropriate.

1.3 Pairs of Scaled Polygons

15 minutes

In this activity, students hone their understanding of scaled copies by working with more complex figures. Students work with a partner to match pairs of polygons that are scaled copies. The polygons appear comparable to one another, so students need to look very closely at all side lengths of the polygons to tell if they are scaled copies.

As students confer with one another, notice how they go about looking for a match. Monitor for students who use precise language (MP6) to articulate their reasoning (e.g., “The top side of A is half the length of the top side of G, but the vertical sides of A are a third of the lengths of those in G.”).

You will need the Pairs of Scaled Polygons blackline master for this activity.

Addressing

- 7.G.A.1

Instructional Routines

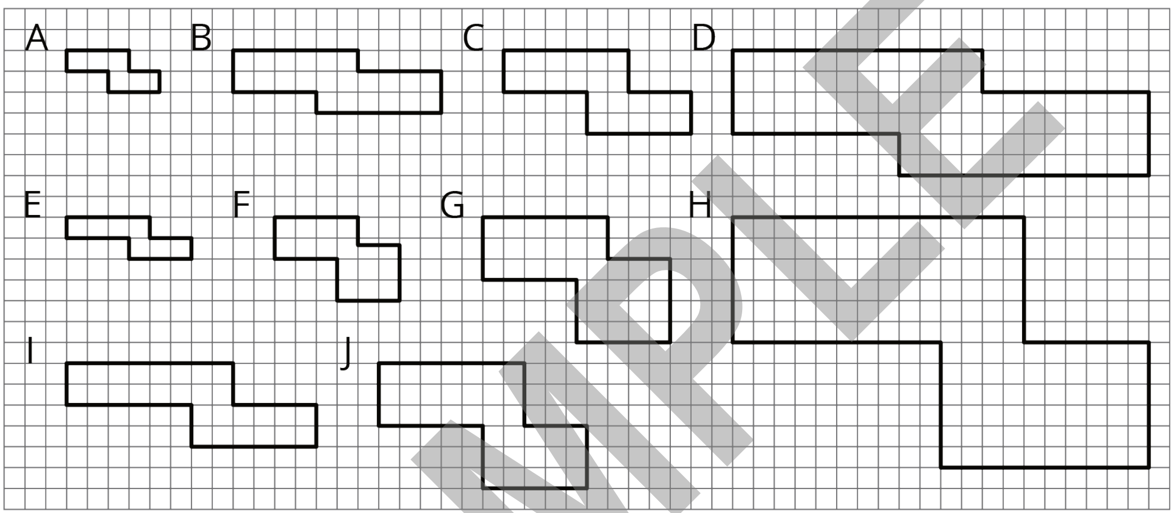
- **Take turns**
  What: Students work with a partner or small group. They take turns in the work of the activity, whether it be spotting matches, explaining, justifying, agreeing or disagreeing, or asking clarifying questions. If they disagree, they are expected to support their case and listen to their partner’s arguments. The first few times students engage in these activities, the teacher should demonstrate, with a partner, how the discussion is expected to go. Once students are familiar with these structures, less set-up will be necessary. While students are working, the teacher can ask students to restate their question more clearly or paraphrase what their partner said.

  Why: Building in an expectation, through the routine, that students explain the rationale for their choices and listen to another's rationale deepens the understanding that can be achieved through these activities. Specifying that students take turns deciding, explaining, and listening limits the phenomenon where one student takes over and the other does not participate. Taking turns can also give students more opportunities to construct logical arguments and critique others’ reasoning (MP3).

Launch

Demonstrate how to set up and do the matching activity. Choose a student to be your partner. Mix up the cards and place them face-up. Tell them that each polygon has one and only one match (i.e., for each polygon, there is one and only one scaled copy of the polygon). Select two cards and then explain to your partner why you think the cards do or do not match. Demonstrate productive ways to agree or disagree (e.g., by explaining your mathematical thinking, asking clarifying questions, etc.).

Arrange students in groups of 2. Give each group a set of 10 slips cut from the blackline master. Encourage students to refer to a running list of statements and diagrams to refine their language and explanations of how they know one figure is a scaled copy of the other.



Support for Students with Disabilities

Conceptual Processing: Processing Time. Begin with a demonstration of one match, which will provide access for students who benefit from clear and explicit instructions.

Fine Motor Skills:

- *Peer Tutors.* Pair students with their previously identified peer tutors and allow students who struggle with fine motor skills to dictate drawing scaled copies as needed.
- *Assistive Technology.* Provide access to the digital version of this activity.

Anticipated Misconceptions

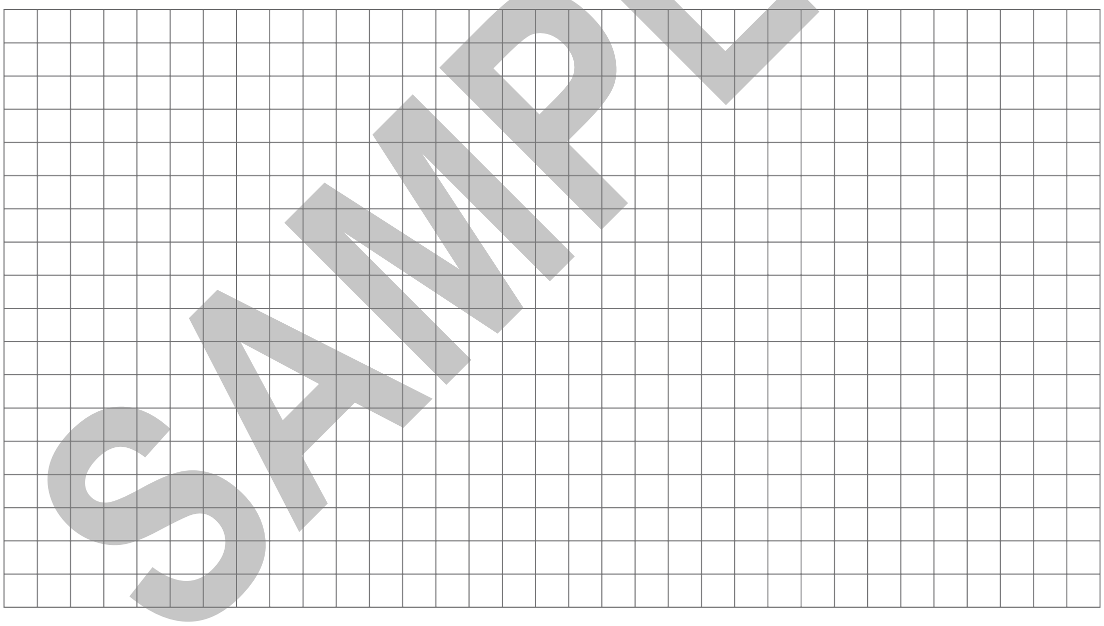
Some students may think a figure has more than one match. Remind them that there is only one scaled copy for each polygon and ask them to recheck all the side lengths.

Some students may think that vertices must land at intersections of grid lines and conclude that, e.g., G cannot be a copy of F because not all vertices on F are on such intersections. Ask them to consider how a 1-unit-long segment would change if scaled to be half its original size. Where must one or both of its vertices land?

Student Task Statement

Your teacher will give you a set of cards that have polygons drawn on a grid. Mix up the cards and place them all face up.

1. Take turns with your partner to match a pair of polygons that are scaled copies of one another.
    a. For each match you find, explain to your partner how you know it's a match.
    b. For each match your partner finds, listen carefully to their explanation, and if you disagree, explain your thinking.
2. When you agree on all of the matches, check your answers with the answer key. If there are any errors, discuss why and revise your matches.
3. Select one pair of polygons to examine further. Draw both polygons on the grid. Explain or show how you know that one polygon is a scaled copy of the other.



Student Response

1. The following polygons are scaled versions of one another:
    - A and C
    - B and D
    - E and I
    - F and G
    - H and J

2. No answer needed.
3. Answers vary. Sample explanation for A and C: All the side lengths in C are twice as long as the lengths of the matching sides in A.

Are You Ready for More?

Is it possible to draw a polygon that is a scaled copy of both Polygon A and Polygon B? Either draw such a polygon, or explain how you know this is impossible.

Student Response

It's impossible to draw a polygon that is a scaled copy of both Polygon A and Polygon B. Sample explanations:

- If I draw a polygon that is a scaled copy of A, all the side lengths would be the same number of times larger or smaller than A, but they won't be the same number of times larger or smaller than B.
- A and B are not scaled copies of each other, so if I draw a scaled copy of one, it will not be a scaled copy of the other.

Activity Synthesis

The purpose of this discussion is to draw out concrete methods for deciding whether or not two polygons are scaled copies of one another, and in particular, to understand that just eyeballing to see whether they look roughly the same is not enough to determine that they are scaled copies.

Display the image of all the polygons. Ask students to share their pairings and guide a discussion about how students went about finding the scaled copies. Ask questions such as:

- When you look at another polygon, what exactly did you check or look for? (General shape, side lengths)
- How many sides did you compare before you decided that the polygon was or was not a scaled copy? (Two sides can be enough to tell that polygons are not scaled copies; all sides are needed to make sure a polygon is a scaled copy.)
- Did anyone check the angles of the polygons? Why or why not? (No; the sides of the polygons all follow grid lines.)

If students do not agree about some pairings after the discussion, ask the groups to explain their case and discuss which of the pairings is correct. Highlight the use of quantitative descriptors such as "half as long" or "three times as long" in the discussion. Ensure that students see that when a figure is a scaled copy of another, all of its segments are the same number of times as long as the corresponding segments in the other.

Support for English Language Learners

Heavier Support. As students answer questions in the discussion, refer to the display of students' statements and sketches. Ask if any of the words or drawings should be refined.

Lesson Synthesis

In this lesson, we encountered copies of a figure that are both scaled and not scaled. We saw different versions of a portrait of a student and of a letter F, as well as a variety of polygons that had some things in common.
In each case, we decided that some were scaled copies of one another and some were not.
Consider asking students:

- What is a **scaled copy**?
- What are some characteristics of scaled copies? How are they different from figures that are not scaled copies?
- What specific information did you look for when determining if something was a scaled copy of an original?

While initial answers need not be particularly precise at this stage of the unit (for example, “scaled copies look the same but are a different size”), guide the discussion toward making careful statements that one could test. The lengths of segments in a scaled copy are related to the lengths in the original figure in a consistent way. For instance, if a segment in a scaled copy is half the length of its counterpart in the original, then all other segments in the copy are also half the length of their original counterparts. We might say, “All the segments are twice as long,” or “All the segments are one-third the size of the segments in the original.”

1.4 Scaling L

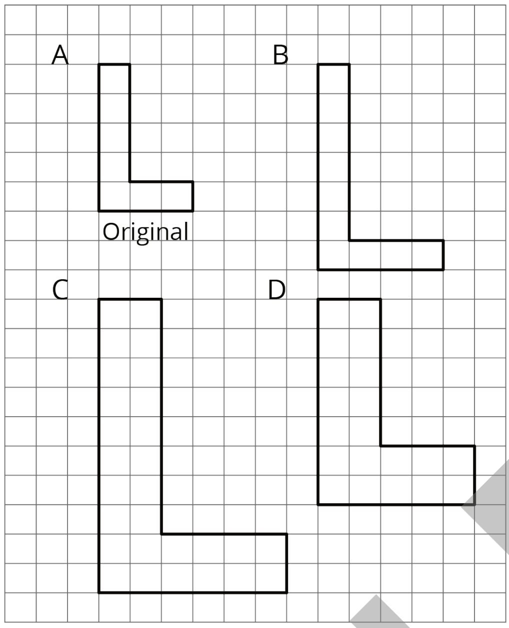
Cool Down: 5 minutes

Addressing

- 7.G.A.1

Student Task Statement

Are any of the figures B, C, or D scaled copies of figure A? Explain how you know.



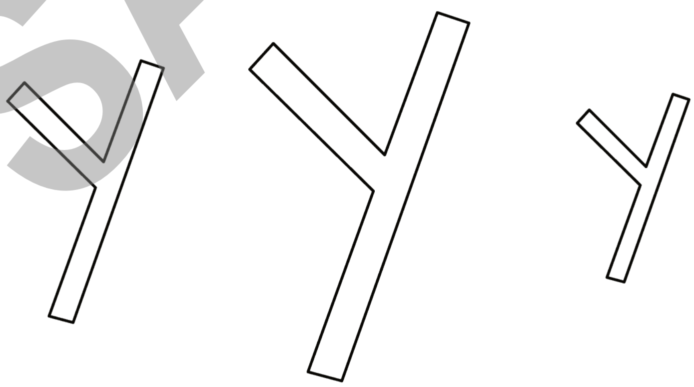
Student Response

Only figure C is a scaled copy of figure A. Sample explanation: In figure C, the length of each segment of the letter L is twice the length of the matching segment in A. In B, none of the segments are double the length. In figure D, some segments are double in length and some are not. So the block letters in B and D are not enlarged evenly.

Student Lesson Summary

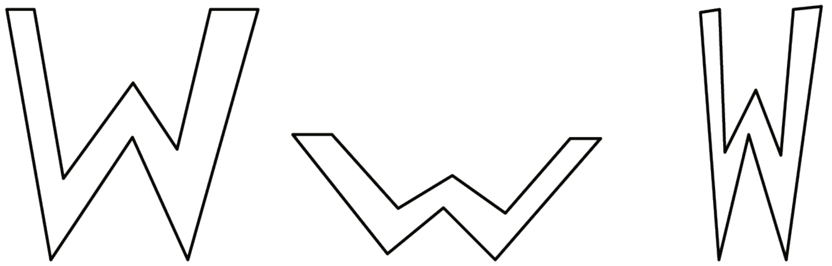
What is a **scaled copy**of a figure? Let's look at some examples.

The second and third drawings are both scaled copies of the original Y.



Original

However, here, the second and third drawings are *not* scaled copies of the original W.



Original

The second drawing is spread out (wider and shorter). The third drawing is squished in (narrower, but the same height).

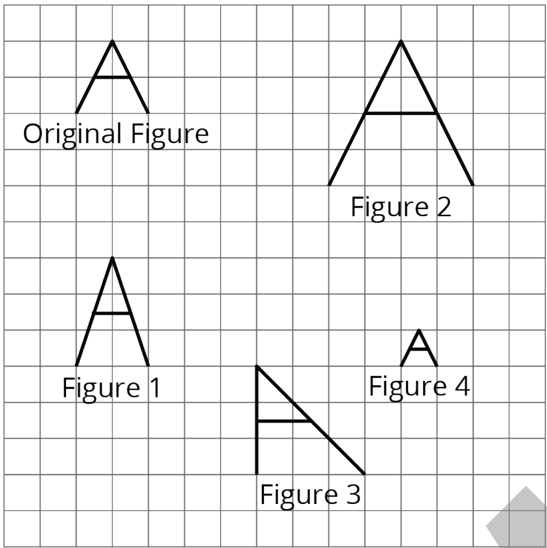
We will learn more about what it means for one figure to be a scaled copy of another in upcoming lessons.

Lesson 1 Practice Problems

Problem 1

Statement

Here is a figure that looks like the letter A, along with several other figures. Which figures are scaled copies of the original A? Explain how you know.



Solution

Figures 2 and 4 are scaled copies. Sample explanations:

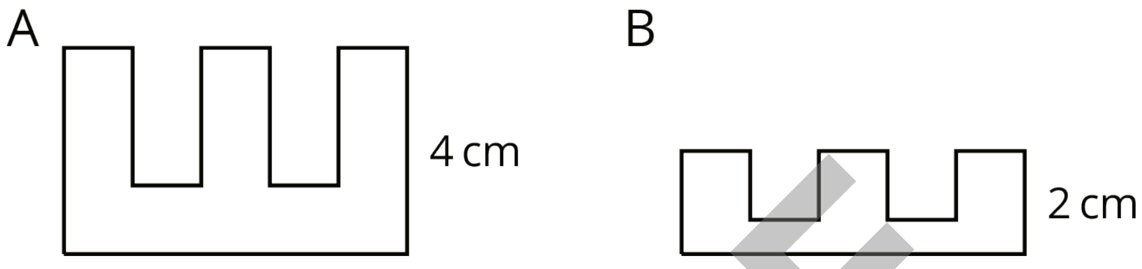
- The original A fits inside a square. The horizontal segment is halfway up the height of the square. The tip of the A is at the midpoint of the horizontal side of the square.
- Figure 1 inside a rectangle, not a square, so it is not a scaled copy. Figure 3 fits inside a square but the shape is different than the original letter A, since one of the legs of the A in Figure 3 is now vertical, so it also is not a scaled copy.
- Figure 2 is twice as high and twice as wide as the original A, and Figure 4 is half as tall and as wide, but in both figures the locations of the horizontal segment and the tip of the letter A still match the original.

Problem 2

Statement

Tyler says that Figure B is a scaled copy of Figure A because all of the peaks are half as tall.

Do you agree with Tyler? Explain your reasoning.



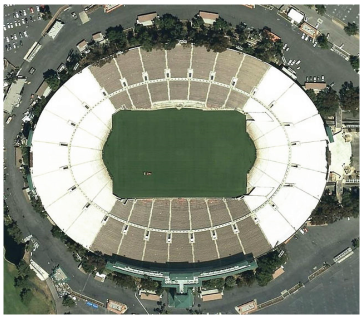
Solution

No. For the smaller figure to be a scaled copy, the figure would have to be half as wide as well.

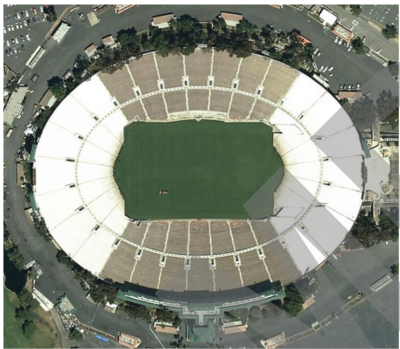
Problem 3

Statement

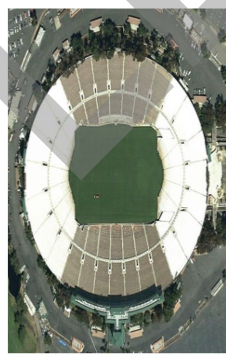
Here is a picture of the Rose Bowl Stadium in Pasadena, CA.



Here are some copies of the picture. Select **all** the pictures that are scaled copies of the original picture.



A



B

C

D

Solution

["A", "D"]

Problem 4

Statement

Complete each equation with a number that makes it true.

a. $5 \cdot \_\_\_\_\_\_ = 15$

b. $4 \cdot \_\_\_\_\_\_ = 32$

c. $6 \cdot \_\_\_\_\_\_ = 9$

d. $12 \cdot \_\_\_\_\_\_ = 3$

Solution

a. 3

b. 8

c. 1.5, $\frac{3}{2}$, or equivalent

d. 0.25, $\frac{1}{4}$, or equivalent

Grade 7, Unit 1, Lesson 13: Draw It to Scale

Goals

- Choose an appropriate scale to represent an actual object or distance given limited drawing size.
- Create a scale drawing of a real-world two-dimensional object.

Learning Targets

- I can create a scale drawing of my classroom.
- When given requirements on drawing size, I can choose an appropriate scale to represent an actual object.

Lesson Narrative

This culminating lesson is optional. Students use what they have learned in this unit to create a scale floor plan of their classroom.

The lesson is organized into three main parts:

- Part 1: Plan and measure. Each student sketches a rough floor plan of the classroom. In groups, they decide on necessary measurements to take, plan the steps and the tools for measuring, and carry out their plan (MP1).
- Part 2: Calculate and draw. Students select the paper to use for drawing, decide on a scale, and work individually to create their drawings. They choose their scale and method strategically, given their measurements and the constraints of their paper.
- Part 3: Reflect and discuss. In small groups, students explain their work, discuss and compare their floor plans (MP3), and evaluate the decisions they made in creating the scale drawing (MP4). As a class, they reflect on how the choice of scale, units, and paper affected the drawing process and the floor plans created.

Depending on the instructional choices made, this lesson could take one or more class meetings. The amount of time needed for each part might vary depending on factors such as:

- The size and complexity of the classroom, and whether measuring requires additional preparation or steps (e.g., moving furniture, taking turns, etc.).
- What the class or individual students decide to include in the floor plans.
- How much organizational support is given to students.
- How student work is ultimately shared with the class (not at all, informally, or with formal presentations).

Consider further defining the scope of work for students and setting a time limit for each part of the activity to focus students' work and optimize class time.

This activity can be modified so that students draw floor plans for different parts of the school—the cafeteria, the gym, the school grounds, and so on—and their drawings could later be assembled as a scale floor plan of the school. If this version is chosen, coordinate the scale used by all students before they begin to draw.

Alignments

Addressing

- 7.G.A.1: Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

Required Materials

Blank paper

graph paper

measuring tools

Required Preparation

Make any available linear measuring tools available, which might include rulers, yardsticks, meter sticks, and tape measures, in centimeters and inches.

Prepare at least three different types of paper for each group, which could include:

- $8\frac{1}{2} \times 11$ printer paper
- 11×17 printer paper
- centimeter graph paper
- $\frac{1}{4}$-inch graph paper
- $\frac{1}{5}$-inch graph paper

Student Learning Goals

Let's draw a floor plan.

13.1 Which Measurements Matter?

Warm Up: 5 minutes

This warm-up prepares students to create a scale floor plan of the classroom. Students brainstorm and make a list of the aspects of the classroom to include in a floor plan and the measurements to take.

Students are likely to note built-in fixtures, like walls, windows, and doors, as important components to measure. They may also include movable objects like furniture. As students work, identify those who list positions of objects (e.g., where a blackboard is on a wall, how far away the teacher's desk is from the door, etc.). Invite them to share later.

Addressing

- 7.G.A.1

Instructional Routines

- **MLR2: Collect and Display**
  To capture students' oral words and phrases into a stable, collective reference. The intent of this routine is to stabilize the fleeting language that students use during partner, small-group, or whole-class activities in order for student's own output to be used as a reference in developing their mathematical language. The teacher listens for, and scribes, the student output using written words, diagrams and pictures; this collected output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. This routine provides feedback for students in a way that increases accessibility while simultaneously supporting meta-awareness of language.

Launch

Tell students they will be creating a scale drawing of the classroom. Their first job is to think about what parts of the classroom to measure for the drawing. Give students 2 minutes of quiet think time to make a list, followed by 3 minutes of whole-class discussion. Ask students to be specific about the measurements they would include on the list.

Support for Students with Disabilities

Strengths-based Approach:

- This activity leverages many natural strengths of students with ADHD, LD, and other concrete learners in terms of its integration of real-world context and personal student interest.
- This may be an opportunity for the teacher to highlight this strength in class and allow an individual with disability to lead peer interactions/discussions, increasing buy-in.

Executive Functioning: Visual Aids. Create an anchor chart (i.e., classroom measurements) publicly displaying important definitions, rules, formulas, or concepts for future reference.

Student Task Statement

Which measurements would you need in order to draw a scale floor plan of your classroom? List which parts of the classroom you would measure and include in the drawing. Be as specific as possible.

Student Response

Answers vary. Sample responses:

- The lengths of walls
- The size and location of windows and doors
- The size and location of fixed and movable furniture
- The measurements of different floor materials in the classroom

Activity Synthesis

Invite students to share their responses with the class, especially those who included measurements between objects in their lists. Record and display students' responses for all to see and to serve as a reference during the main activity. Consider organizing students' responses by type rather than by items (e.g., listing "furniture" instead of "chairs," "desks," etc.). Some guiding questions:

- Which parts of the classroom must be included in a scale floor plan? Which parts are less important?
- What measurements do we need?
- Besides lengths of walls and objects, what else would be helpful? (If no one mentioned the positions of objects, ask how we know where to place certain objects on the drawing.)
- Should we include vertical measurements? Why or why not?

Support for English Language Learners

Consider using MLR 2: Collect and Display.

13.2 Creating a Floor Plan (Part 1)

Optional: 15 minutes
The purpose of this activity is for students to make preparations to create their scale drawings. They sketch a rough floor plan of the classroom.
In groups, they plan the steps for making measurements and then carry out their plan.

Some things to notice as students work:

- As they draw their sketch, encourage them to focus on big-picture elements and not on details. It is not important that the sketch is neat or elaborate. What matters more is that it does not omit important features like the door.
- As they make plans for measuring and recording, encourage them to work systematically to minimize omissions and errors.
- Urge students to measure twice and record once. It is better to take a little more time to double check the measurements than to find out during drawing that they are off.

Addressing

- 7.G.A.1

Instructional Routines

- **MLR8: Discussion Supports**
  To support rich discussions about mathematical ideas, representations, contexts, and strategies. The examples provided can be combined and used together with any of the other routines. They include multi-modal strategies for helping students comprehend complex language and ideas, and can be used to make classroom communication accessible, to foster meta-awareness of language, and to demonstrate strategies students can use to enhance their own communication and construction of ideas.
- **Think pair share**
  What: Students have quiet time to think about a problem and work on it individually, and then time to share their response or their progress with a partner. Once these partner conversations have taken place, some students are selected to share their thoughts with the class.

  Why: This is a teaching routine useful in many contexts whose purpose is to give all students enough time to think about a prompt and form a response before they are expected to try to verbalize their thinking. First they have an opportunity to share their thinking in a low-stakes way with one partner, so that when they share with the class they can feel calm and confident, as well as say something meaningful that might advance everyone's understanding. Additionally, the teacher has an opportunity to eavesdrop on the partner conversations so that she can purposefully select students to share with the class.

Launch

Give students 1–2 minutes to read the task statement individually and to ask any clarifying questions. Consider displaying a floor plan sketch of another room in the school. Emphasize that the sketch serves a similar purpose as an outline in writing. It does not need to be to scale, accurate, or elaborate, but it should show all the important pieces in the right places so it can be a reference in creating the scale drawing.

Arrange students in groups of 2–4. Smaller groups means that each individual student can be more involved in the measuring process, which is a benefit, but consider that it might also make the measuring process more time consuming (as it would mean more groups moving about in a confined space).

Distribute blank paper and give students 4–5 minutes to draw a sketch and to share it with a partner. Provide access to measuring tools. Give students another 4–5 minutes to plan in groups, and then time to measure (which may vary depending on size of classroom and other factors).

Support for Students with Disabilities

Executive Functioning: Eliminate Barriers. Provide a task checklist that makes all the required components of the activity explicit.

Fine Motor Skills: Peer Tutors. Pair students with their previously identified peer tutors and allowing students who struggle with fine motor skills to dictate drawing or sketching as needed.

Student Task Statement

1. On a blank sheet of paper, make a *rough sketch* of a floor plan of the classroom. Include parts of the room that the class has decided to include or that you would like to include. Accuracy is not important for this rough sketch, but be careful not to omit important features like a door.
2. Trade sketches with a partner and check each other's work. Specifically, check if any parts are missing or incorrectly placed. Return their work and revise your sketch as needed.
3. Discuss with your group a plan for measuring. Work to reach an agreement on:
    - Which classroom features must be measured and which are optional.
    - The units to be used.
    - How to record and organize the measurements (on the sketch, in a list, in a table, etc.).
    - How to share the measuring and recording work (or the role each group member will play).
4. Gather your tools, take your measurements, and record them as planned. Be sure to double-check your measurements.
5. Make your own copy of all the measurements that your group has gathered, if you haven't already done so. You will need them for the next activity.

Student Response

Answers vary.

Activity Synthesis

After groups finish measuring, ask them to make sure that every group member has a copy of the measurements before moving on to the next part.

Consider briefly discussing what was challenging about doing the measuring. A few important issues which may come up include:

- Making sure that the measuring device stays in a straight line.
- It is hard to be accurate when the measuring device needs to be used *multiple* times in order to find the length of something long, such as a wall.
- Taking turns with other groups that are trying to measure the same thing.
- The measurements are not exact and need to be rounded.

Support for English Language Learners

Lighter Support: MLR 8 (Discussion Supports). As students describe their process for measuring features of the room, revoice student ideas to demonstrate mathematical language use by restating a statement as a question in order to clarify, apply appropriate language, and involve more students. Press for details in students' explanations by requesting that students challenge an idea, elaborate on an idea, or give an example of their measuring process. Show central concepts multi-modally by using different types of sensory inputs: acting out scenarios or inviting students to do so, using gestures, and talking about the context of what they were measuring.

13.3 Creating a Floor Plan (Part 2)

Optional: 15 minutes

In this activity, students use the measurements they just gathered to create their scale floor plans. Each student selects one of the paper options, decides on a scale to use, and works individually to create their drawing.

Support students as they reason about scale, scaled lengths, and how to go about creating the drawing. Encourage all to pay attention to units as they calculate scaled lengths. Ask students to think about the different ways that we can write a scale. If they struggle, remind students that a scale can be written in different units or written without units.

Support for Students with Disabilities

Executive Functioning: Eliminate Barriers. Chunk this task into more manageable parts (e.g., one section of the room at a time), which will aid students who benefit from support with organizational skills in problem solving.

Fine Motor Skills: Peer Tutors. Pair students with their previously identified peer tutors and allow students who struggle with fine motor skills to dictate drawing or sketching as needed.

Addressing

- 7.G.A.1

Instructional Routines

- **MLR2: Collect and Display**
  To capture students' oral words and phrases into a stable, collective reference. The intent of this routine is to stabilize the fleeting language that students use during partner, small-group, or whole-class activities in order for student's own output to be used as a reference in developing their mathematical language. The teacher listens for, and scribes, the student output using written words, diagrams and pictures; this collected output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. This routine provides feedback for students in a way that increases accessibility while simultaneously supporting meta-awareness of language.

Launch

Distribute at least three different types of paper for each group, which could include:

- $8\frac{1}{2} \times 11$ printer paper
- 11×17 printer paper
- Centimeter graph paper
- $\frac{1}{4}$-inch graph paper
- $\frac{1}{5}$-inch graph paper

Ask each group member to select a paper for their drawing. Encourage variation in paper selections. Explain that they should choose an appropriate scale based on the size of their paper, the size of the classroom, and their chosen units of measurement. This means that the floor plan must fit on the paper and not end up too small (e.g., if the paper is 11×17 inches, the floor plan should not be the size of a postcard).

Give students quiet time to create their floor plan. If the classroom layout is fairly complex, consider asking students to pause after they have completed a certain portion of the drawing (e.g., the main walls of the classroom) so their work may be checked. Alternatively, give them a minute to share their drawing-in-progress with a partner and discuss any issues.

Support for English Language Learners

Heavier Support: MLR 2 (Collect and Display) with Gather and Show Student Discourse. During group work of creating floor plans, circulate and listen to students talk, and jot notes about common or important phrases (e.g., scale, size, units, etc.), together with helpful sketches or diagrams. Pay particular attention to how students are determining scales for their floor plans. Scribe students' words and sketches on a visual display to refer back to during whole-class discussions throughout this lesson and the rest of the unit.

Anticipated Misconceptions

Some students may pick a scale and start drawing without considering how large their completed floor plan will be. Encourage students to consider the size of their paper in order to determine an appropriate scale before they start drawing.

Student Task Statement

Your teacher will give you several paper options for your scale floor plan.

1. Determine an appropriate scale for your drawing based on your measurements and your paper choice. Your floor plan should fit on the paper and not end up too small.
2. Use the scale and the measurements your group has taken to draw a scale floor plan of the classroom. Make sure to:
    - Show the scale of your drawing.
    - Label the key parts of your drawing (the walls, main openings, etc.) with their actual measurements.
    - Show your thinking and organize it so it can be followed by others.

Student Response

Answers vary.

Are You Ready for More?

1. If the flooring material in your classroom is to be replaced with 10-inch by 10-inch tiles, how many tiles would it take to cover the entire room? Use your scale drawing to approximate the number of tiles needed.
2. How would using 20-inch by 20-inch tiles (instead of 10-inch by 10-inch tiles) change the number of tiles needed? Explain your reasoning.

Student Response

1. Answers vary.
2. It would reduce the number of tiles. Each 20-by-20 tile covers 4 times the area of each 10-by-10 tile, so it would take about $\frac{1}{4}$ as many tiles.

Activity Synthesis

Small-group and whole-class reflections will occur in the next activity.

13.4 Creating a Floor Plan (Part 3)

Optional: 15 minutes

In the final phase of the drawing project, students reflect on and revise their work. Students who chose the same paper option confer in small groups to analyze and compare their floor plans. They discuss their decisions, evaluate the accuracy of their drawings, and then revise them as needed.

After revision, students debrief as a class and discuss how the choice of scale, units, and paper affected the drawing process and the floor plans they created.

Addressing

- 7.G.A.1

Instructional Routines

- **Group Presentations (6–8 ONLY)**
  Some activities instruct students to work in small groups to solve a problem with mathematical modeling, invent a new problem, design something, or organize and display data, and then create a visual display of their work. Teachers need to help groups organize their work so that others can follow it, and then facilitate different groups' presentation of work to the class. Teachers can develop specific questioning skills to help more students make connections and walk away from these experiences with desired mathematical learning. For example, instead of asking if anyone has any questions for the group, it is often more productive to ask a member of the class to restate their understanding of the group's findings in their own words.

- **MLR1: Stronger and Clearer Each Time**
  To provide a structured and interactive opportunity for students to revise and refine both their ideas and their verbal and written output. This routine provides a purpose for student conversation as well as fortifies output. The main idea is to have students think or write individually about a response, use a structured pairing strategy to have multiple opportunities to refine and clarify the response through conversation, and then finally revise their original written response. Throughout this process, students should be pressed for details, and encouraged to press each other for details.

Launch

Arrange students who use the same type and size of paper into small groups. Give them 8–10 minutes to share and explain their drawings. Display and read aloud questions such as the following. Ask students to use them to guide their discussion.

- What scale did you use? How did you decide on the scale?
- Do the scaled measurements in each drawing seem accurate? Do they represent actual measurements correctly?
- Did the scale seem appropriate for the chosen paper? Why or why not?
- What was the first thing you drew in your drawing? Why?
- How did you decide on the objects to show in your drawing?
- What aspects of your drawings are different?
- How could each floor plan be revised to better represent the classroom?

Support for English Language Learners

Heavier Support: MLR 1 (Stronger and Clearer Each Time). After writing down ideas for how their floor plans could be improved, have students switch partners to share and listen to more ideas. After meeting with 1–2 more partners, students can revise their written answers.

Support for Students with Disabilities

Receptive/Expressive Language: Processing Time. Students who benefit from extra processing time would also be aided by MLR 1 (Stronger and Clearer Each Time).

Student Task Statement

1. Trade floor plans with another student who used the same paper size as you. Discuss your observations and thinking.
2. Trade floor plans with another student who used a different paper size than you. Discuss your observations and thinking.
3. Based on your discussions, record ideas for how your floor plan could be improved.

Student Response

Answers vary.

Activity Synthesis

Before debriefing as a class, give students 4–5 minutes of quiet time to reflect. Ask them to write down ideas for revising their floor plan and strategies for creating accurate scale drawings based on their conversation.

Though much of the discussion will take place within the groups, debrief as a class so students can see floor plans created at a variety of scales and on different paper types or sizes. Display a range of scale drawings for all to see and discuss the following questions. (Alternatively, consider posting all students' work for a gallery walk and ask students to reflect on these questions.)

- What are the differences in these drawings?
- How did different scales impact the final drawing?
- How did the size of paper impact the choice of scale?
- What choices were really important when creating the scale drawing?
- Would these choices be the same if you were doing a different room in the school? Or some other building?

Grade 8, Unit 3, Lesson 1: Understanding Proportional Relationships

Goals

- Graph a proportional relationship from a verbal description.
- Remember that a graph representing a proportional relationship is a line through (0,0) and $(1, k)$.
- Remember that $\frac{y}{x}$ is called the constant of proportionality, k.

Learning Targets

- I can graph a proportional relationship from a story.
- I can use the constant of proportionality to compare the pace of different animals.

Lesson Narrative

This lesson is the first of four where students work with proportional relationships from a grade 8 perspective. Embedded alongside their work with proportional relationships, students learn about graphing from a blank set of axes. Attending to precision in labeling axes, choosing an appropriate scale, and drawing lines are skills students work with in this lesson and refine over the course of this unit and in units that follow (MP6).

The purpose of this lesson is to get students thinking about what makes a “good” graph by first considering what are the components of a graph (e.g., labels, scale) and then adding scale to graphs of the pace of two bugs. Students also graph a line based on a verbal description of a relationship and compare the newly graphed line to already graphed proportional relationships.

This lesson includes graphs with elapsed time in seconds on the vertical axis and distance traveled in centimeters on the horizontal axis. It is common for people to believe that time is always the independent variable, and should therefore always be on the horizontal axis. This is a really powerful heuristic. The problem is, it isn’t true.

In general, a context that involves a relationship between two quantities does not dictate which quantity is the independent variable and which is the dependent variable: that is a choice made by the modeler. Consider this situation: A runner is traveling one mile every 10 minutes. There is more than one way to represent this situation.

- We can say the number of miles traveled, d, depends on the number of minutes that have passed, t, and write $d = 0.1t$. This way of expressing the relationship might be more useful for questions like, "How far does the runner travel in 35 minutes?"

- We can also say that the number of minutes that have passed, t, depends on the number of miles traveled, d, and write $t = 10d$. This way of expressing the relationship might be more useful for questions like, "How long does it take the runner to travel 2 miles?"

These are both linear relationships. The rate of change in the first corresponds to speed (0.1 miles per minute), and the rate of change in the second corresponds to pace (10 minutes per mile). Both have meaning, and both could be of interest. It is up to the modeler to decide what kinds of questions she wants to answer about the context and which way of expressing the relationship will be most useful in answering those questions.

Alignments

Building On

- 7.RP.A.2: Recognize and represent proportional relationships between quantities.

Addressing

- 8.EE.B: Understand the connections between proportional relationships, lines, and linear equations.

Building Towards

- 8.EE.B.5: Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.

Student Learning Goals

Let’s study some graphs.

1.1 Notice and Wonder: Two Graphs

Warm Up: 5 minutes
The purpose of this warm-up is to get a conversation started about what features a graph needs. In the following activities, students will put these ideas to use by adding scale to some axes with two proportional relationships graphed on it.

Building Towards

- 8.EE.B.5

Instructional Routines

- **Notice and wonder**
What: This routine can appear as a warm-up or in the launch or synthesis of a classroom activity. Students are shown some media or a mathematical representation. The prompt to students is “What do you notice? What do you wonder?” Students are given a few minutes to write down things they notice and things they wonder. After students have had a chance to write down their responses, the teacher asks several students to share things they noticed and

things they wondered; these are recorded by the teacher for all to see. Usually, the teacher steers the conversation to wondering about something mathematical that the class is about to focus on.

Where: Appears frequently in warm-ups but also appears in launches to classroom activities.

Why: The purpose is to make a mathematical task accessible to all students with these two approachable questions. By thinking about them and responding, students gain entry into the context and might get their curiosity piqued. Taking steps to become familiar with a context and the mathematics that might be involved is making sense of problems (MP1). Note: *Notice and Wonder* and *I Notice/I Wonder* are trademarks of NCTM and the Math Forum and used in these materials with permission.

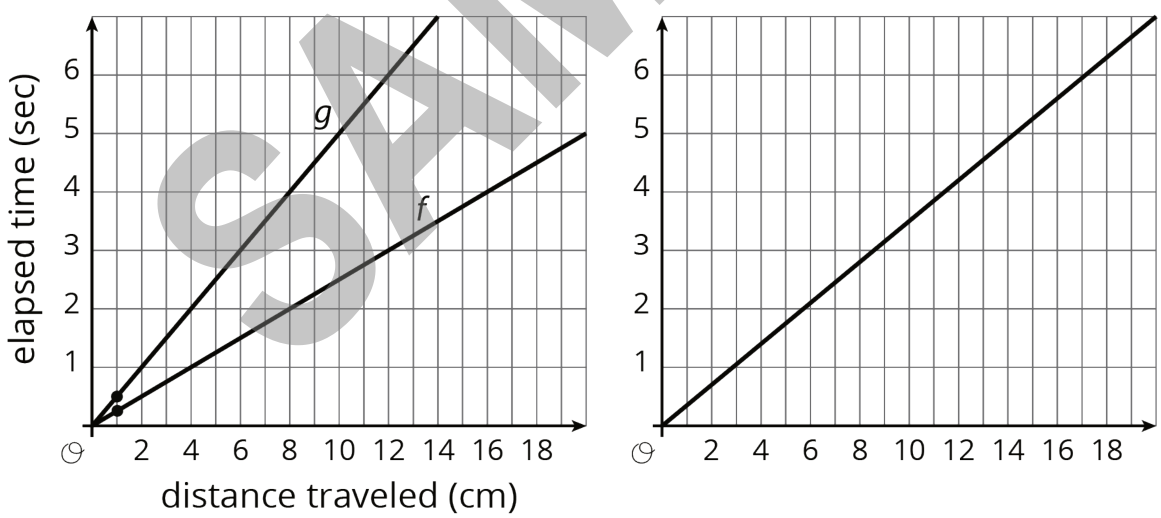
Launch

Tell students they will look at a picture, and their job is to think of at least one thing they notice and at least one thing they wonder about the picture. Display the problem for all to see and ask students to give a signal when they have noticed or wondered about something.

Support for Students with Disabilities

Executive Functioning: Graphic Organizers. Provide a t-chart for students to record what they notice and wonder prior to being expected to share these ideas with others.

Student Task Statement



What do you notice? What do you wonder?

Student Response

Things students may notice:

- The second set of axes are not labeled

- If the first graph is about speed, then f is twice as fast as g.
- Graph g is something going a speed of 2 centimeters every second
- Graph f is something going a pace of about 0.25 seconds per 1 centimeter.

Things students may wonder:

- What do the two points mean?
- Why does one graph show two lines while the other only has one?
- What do g and f represent?

Activity Synthesis

Invite students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. After each response, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to the images each time. If the missing labels are not mentioned, make sure to bring them up.

1.2 Moving Through Representations

15 minutes
In this activity, students investigate the paces of two different bugs. Using the chart at the start of the activity, students answer questions about pace, decide on a scale for the axes, and mark and label the time needed to travel 1 cm for each bug (unit rate).

Identify students who use different scales on the axes to share during the Activity Synthesis. For example, some students may count by 1s on the distance axis while others may count by 0.5s.

Addressing

- 8.EE.B

Building Towards

- 8.EE.B.5

Instructional Routines

- **MLR5: Co-Craft Questions and Problems**
  To allow students to get inside of a context before feeling pressure to produce answers, and to create space for students to produce the language of mathematical questions themselves. Through this routine, students are able to use conversation skills as well as develop meta-awareness of the language used in mathematical questions and problems. Teachers should push for clarity and revoice oral responses as necessary.

Launch

Arrange students in groups of 2. Before students start working, ensure that they understand that each bug's position is measured at the front of their head. So for example, in the second diagram, the ladybug has moved 4 centimeters and the ant has moved 6 centimeters.

Ask students to review the images and the first problem in the activity and give a signal when they have finished. Invite students to share their ideas about which bug is represented by line u and which bug is represented by line v. (The ladybug is u, the ant is v.) If not brought up in students' explanations, draw attention to how the graph shows the *pace* of the two bugs—that is, the graph shows how much time it takes to go a certain distance, which is different than a graph of speed, which shows how much distance you go for a certain amount of time.

Give students work time to complete the remaining problems with their partner followed by a whole-class discussion.

Support for English Language Learners

Heavier Support: MLR 5 (Co-Craft Questions and Problems). Before answering the questions in this activity, display the prompt and ladybug and ant diagrams (without the line graphs). Have students write possible mathematical questions about the situation. Have pairs of students compare their questions and then invite pairs to share their questions with the class. Then reveal the line graph and ask students to work on questions 1–4.

Support for Students with Disabilities

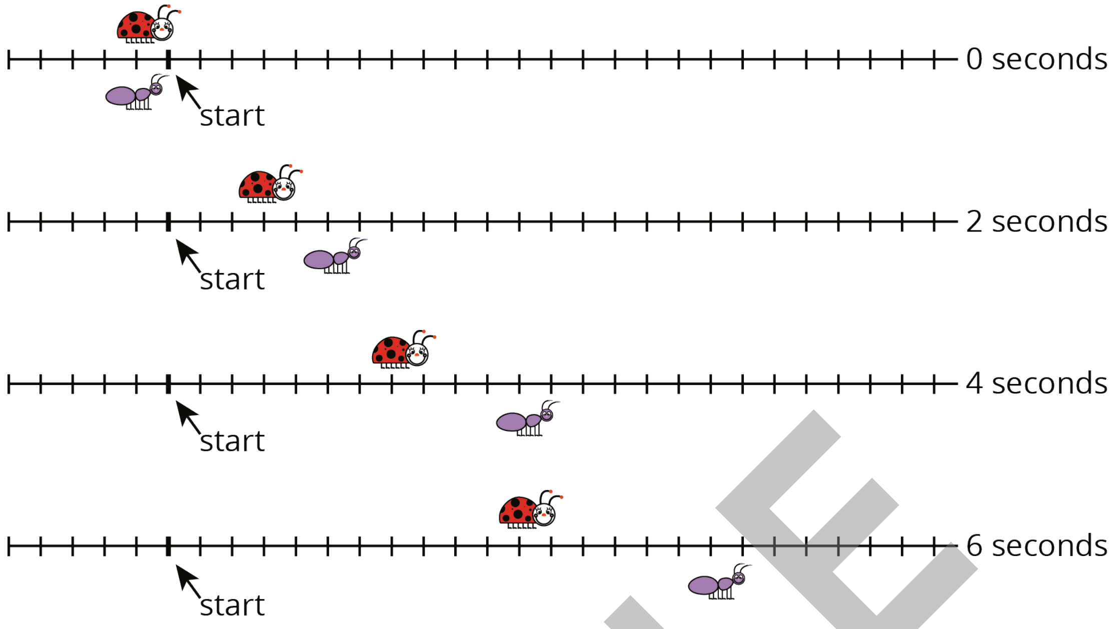
Conceptual Processing: Processing Time. Check in with individual students, as needed, to assess for comprehension during each step of the activity.

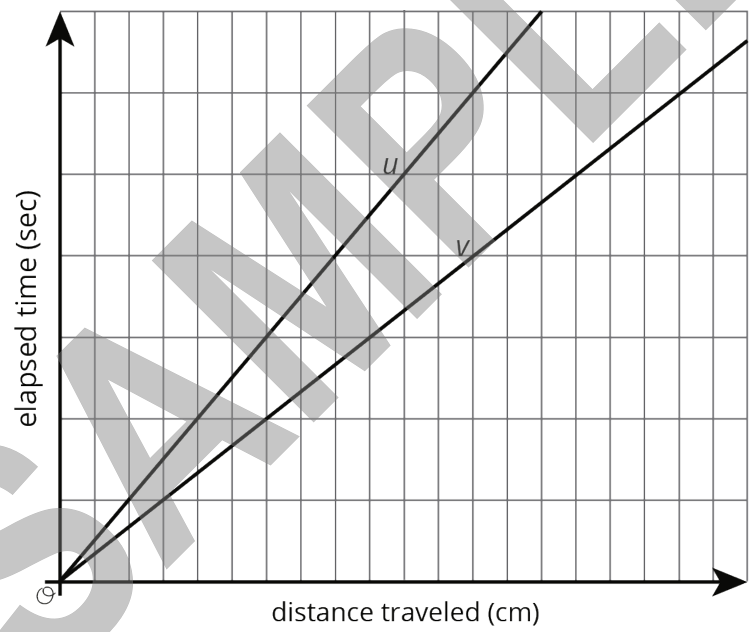
Anticipated Misconceptions

Students might confuse pace with speed and interpret a steeper line as meaning the ladybug is moving faster. Monitor students to ensure that they attend to the time and distance on the tick mark diagrams and plot points as $(distance, time)$ with time on the y-axis and distance on the x-axis. Reinforce language of how many seconds per a given interval of distance. Make explicit that twice as fast means half the pace.

Student Task Statement

A ladybug and ant move at constant speeds. The diagrams with tick marks show their positions at different times. Each tick mark represents 1 centimeter.





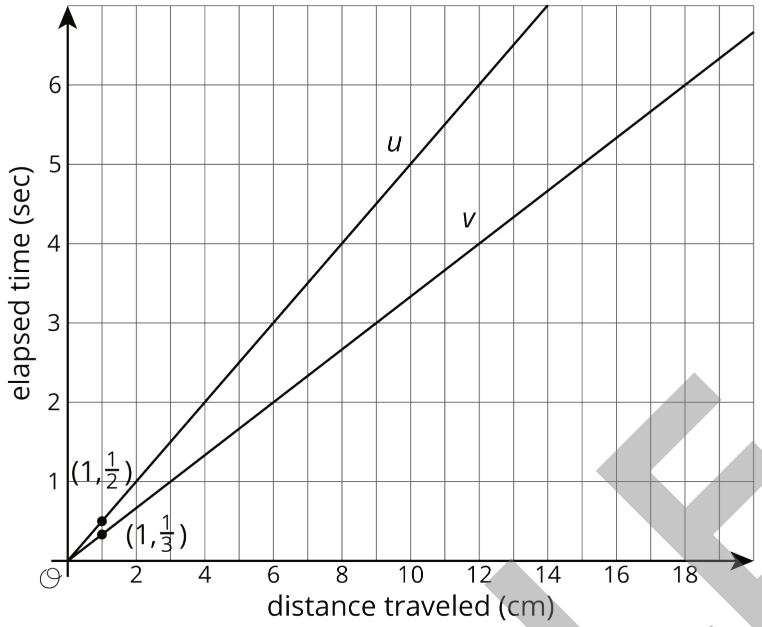
1. Lines u and v also show the positions of the two bugs. Which line shows the ladybug's movement? Which line shows the ant's movement? Explain your reasoning.

2. How long does it take the ladybug to travel 12 cm? The ant?

3. Scale the vertical and horizontal axes by labeling each grid line with a number. You will need to use the time and distance information shown in the tick-mark diagrams.

4. Mark and label the point on line u and the point on line v that represent the time and position of each bug after travelling 1 cm.

Student Response



1. Ladybug: line u, ant: line v
2. Ladybug: 6 seconds, ant: 4 seconds
3. See graph.
4. See graph.

Are You Ready for More?

1. How fast is each bug traveling?
2. Will there ever be a time when the purple bug (ant) is twice as far away from the start as the red bug (ladybug)? Explain or show your reasoning.

Student Response

1. The red bug (ladybug) is traveling at 2 cm/sec and the purple bug (ant) is traveling at 3 cm/sec.
2. No, the purple bug (ant) is always half as much again as far from the start as the red bug (lady bug).

Activity Synthesis

Display the images from the problem for all to see. Begin the discussion by inviting students to share their solutions for how long it takes each bug to travel 12 cm. Encourage students to reference one or both of the images as they explain their thinking.

Ask previously selected students to share their graphs with added scale and how they decided on what scale to use. If possible, display these graphs for all to see. There are many correct ways to choose a scale for this situation, though some may have made it difficult for students to plot the answer to the final problem. If this happened, highlight these graphs and encourage students to read all problems when they are making decisions about how to construct a graph. Since this activity had a problem asking for information about 1 cm, it makes sense to count by 1s (or even something smaller!) on the distance axis.

1.3 Moving Twice as Fast

15 minutes

In this activity, students use the representations from the previous activity and add a third bug that is moving twice as fast as the ladybug. Students are also asked to write equations for all three bugs. An important aspect of this activity is students making connections between the different representations.

Monitor for students using different strategies to write their equations. For example, some students may reason from the unit rates they can see on their graphs and write equations in the form of $y = kx$, where k is the unit rate (constant of proportionality). Others may write equations of the form $\frac{y}{x} = \frac{b}{a}$, where (a, b) is a point on the line. Select several of these students to share during the discussion.

Addressing

- 8.EE.B

Building Towards

- 8.EE.B.5

Instructional Routines

- **Anticipate, monitor, select, sequence, connect**
  What: These are the *5 Practices for Orchestrating Productive Mathematical Discussions* (Smith and Stein, 2011). In this curriculum, much of the work of anticipating, sequencing, and connecting is handled by the materials in the activity narrative, launch, and synthesis sections. Teachers need to prepare for and conduct whole-class discussions.

  Where: Many classroom activities lend themselves to this structure.

  Why: In a problem-based curriculum, many activities can be described as “do math and talk about it,” but the 5 Practices lend more structure to these activities so that they more reliably result in students making connections and learning new mathematics.

- **MLR3: Critique, Correct, and Clarify**
  To give students a piece of mathematical writing that is not their own to analyze, reflect on, and develop. The intent is to prompt student reflection with an incorrect, incomplete, or ambiguous written argument or explanation, and for students to improve upon the written work by correcting errors and clarifying meaning. This routine fortifies output and engages students in meta-awareness. Teachers can demonstrate with meta-think-alouds and press for details when necessary.

Launch

Keep students in the same groups. Give 5–7 minutes work time followed by a whole-class discussion.

Support for Students with Disabilities

Fine Motor Skills: Peer Tutors. Pair students with their previously identified peer tutors and allow students who struggle with fine motor skills to dictate graphing as needed.

Support for English Language Learners

Heavier Support: MLR 3 (Critique, Correct, and Clarify). Use this to support students with the principles of proportionality. For question 4, provide an incomplete statement like, "I looked at how far the ladybug went and made my bug go farther" or a flawed statement like "I put my bug 2 tick marks ahead of the ant." Have students discuss with a partner how they could correct or clarify the statement to be more accurate. The goal is to have students express orally the concept and relationships of proportionality.

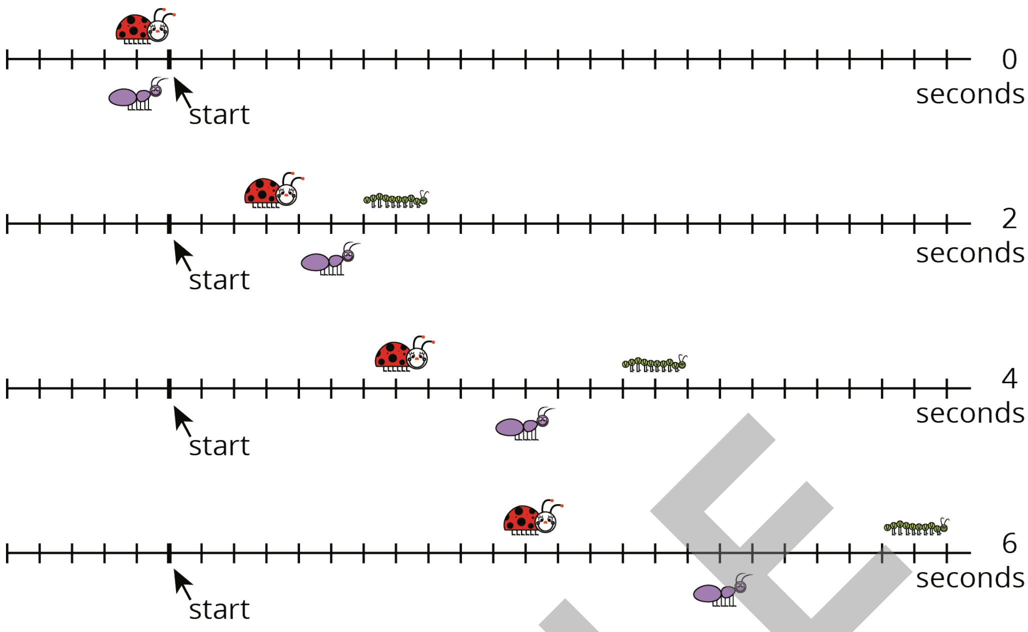
Student Task Statement

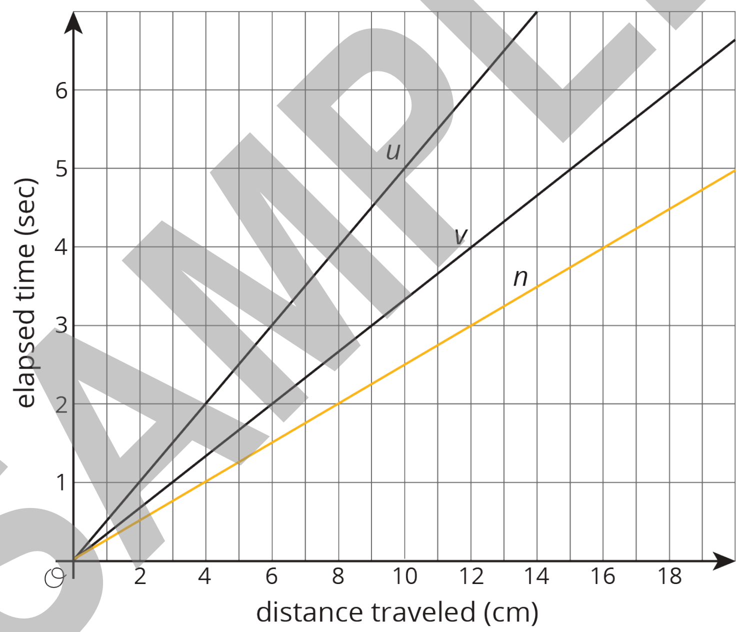
Refer to the tick-mark diagrams and graph in the earlier activity when needed.

1. Imagine a bug that is moving twice as fast as the ladybug. On each tick-mark diagram, mark the position of this bug.
2. Plot this bug's positions on the coordinate axes with lines u and v, and connect them with a line.
3. Write an equation for each of the three lines.

Student Response

1.





2. See graph. Line n represents a bug moving double the ladybug's distance in the same amount of time.

3. Answers vary. Possible response: Equations are ladybug: $y = \frac{1}{2}x$, ant: $y = \frac{1}{3}x$, new bug: $y = \frac{1}{4}x$ (twice as fast as ladybug), where x represents the distance traveled and y represents elapsed time.

Activity Synthesis

Display both images from the previous task for all to see. Invite previously selected students to share their equations for each bug. Sequence students so that the most common strategies are first. Record the different equations created for each bug and display these for all to see.

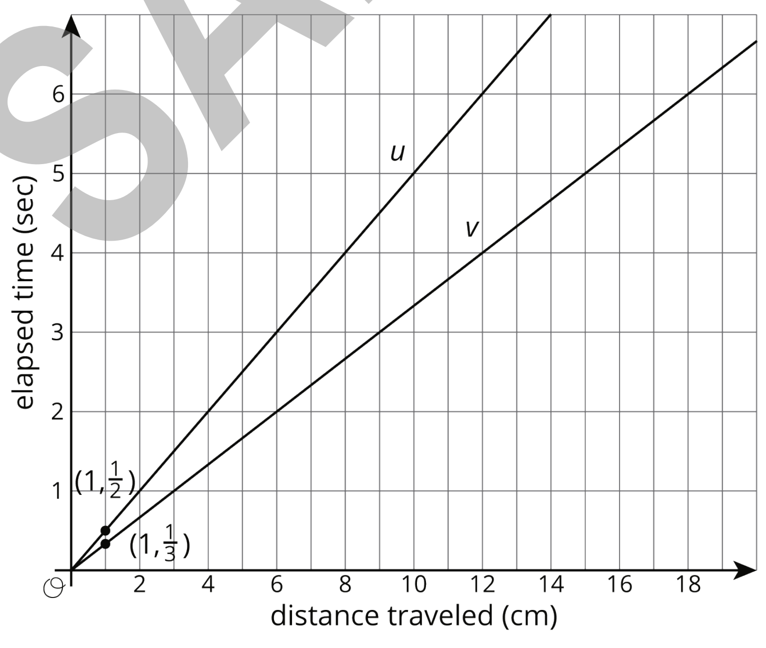
As students share their reasoning about the equation for the third bug, highlight strategies that support using the equation (original is k and new one is $\frac{1}{2}k$) and graph (less steep, still constant proportionality, half point values). If no students write an equation of the form $y = kx$, do so and remind students of the usefulness of k, the constant of proportionality, when reasoning about proportional relationships.

Consider asking the following questions to help students make connections between the different representations:

- "What features of the tick-mark diagrams, lines, and equations can you identify that would allow someone to figure out which bug is moving faster?" (The tick-mark diagrams give the coordinates of points that will go on the graph because they show how far each bug has gone after each amount of time. We can see the positions of the bugs on the tick-mark diagrams so we know which is faster. The graph shows how far they went for any amount of time and the slope helps to show which is faster. Both help compare the movements of the two or three bugs.)
- "The tick-mark diagrams show some of the bugs' movements, but not all of them. How can we use the graphs of the lines to get more complete information?" (The tick-mark diagrams only show time every 2 seconds. On the graph we can see the bugs' positions at any point in time.)
- "Are you convinced that your graph (or equation) supports the fact that the new bug is going twice as fast as the ladybug?"

Lesson Synthesis

Display a scaled graph of the two bugs for all to see. Remind students that line u is the ladybug and that line v is the ant.



Ask students:

- "What would the graph of a bug going 3 times faster than the ant look like?" (It would go through the points $(0,0)$, $(1, \frac{1}{9})$, and $(9,1)$.)
- "What would an equation showing the relationship between the bugs' distance and time look like?" (Since it is going 4 times faster and goes through the point $(9,1)$, it has constant of proportionality of $\frac{1}{9}$, which means one equation is $y = \frac{1}{9}x$.)
- "If we wanted to scale the graph so we could see how long it takes the ladybug to travel 50 cm, what numbers could we use on the vertical axis?" (The ladybug travels 50 cm in 25 seconds, so the vertical axis would need to extend to at least that value.)

1.4 Turtle Race

Cool Down: 5 minutes

Building On

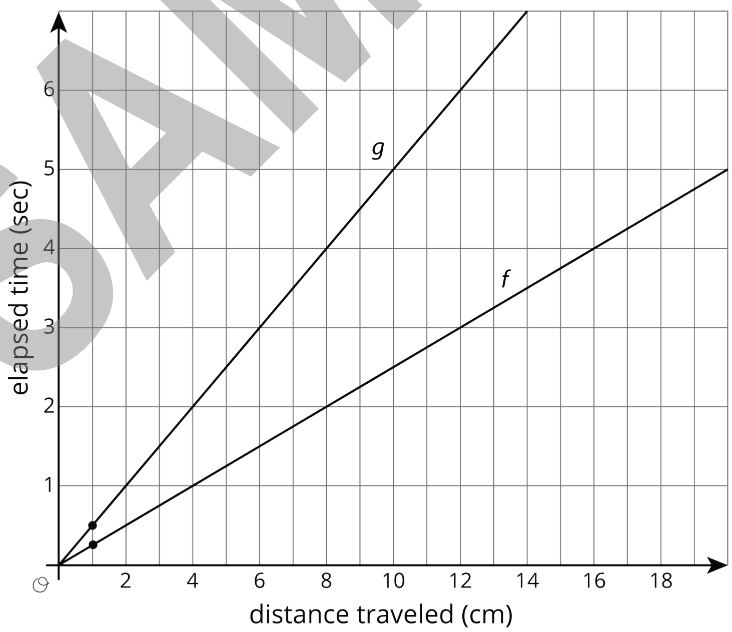
- 7.RP.A.2

Building Towards

- 8.EE.B.5

Student Task Statement

This graph represents the positions of two turtles in a race.



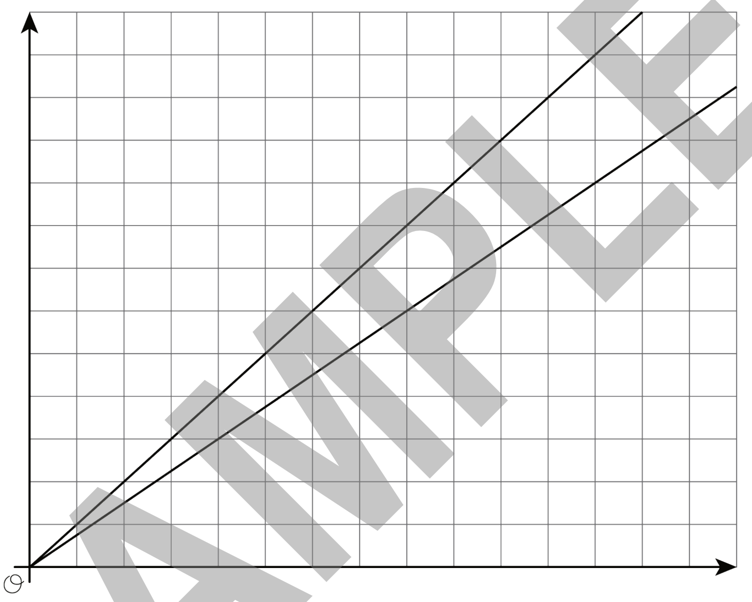
1. On the same axes, draw a line for a third turtle that is going half as fast as the turtle described by line g.
2. Explain how your line shows that the turtle is going half as fast.

Student Response

1. A line through $(0, 0)$, $(1, 1)$, $(2, 2)$, etc.
2. Looking at the values for 2 seconds, turtle g moves 4 cm and the third turtle moves only 2 cm. This third turtle covers half the distance in the same amount of time.

Student Lesson Summary

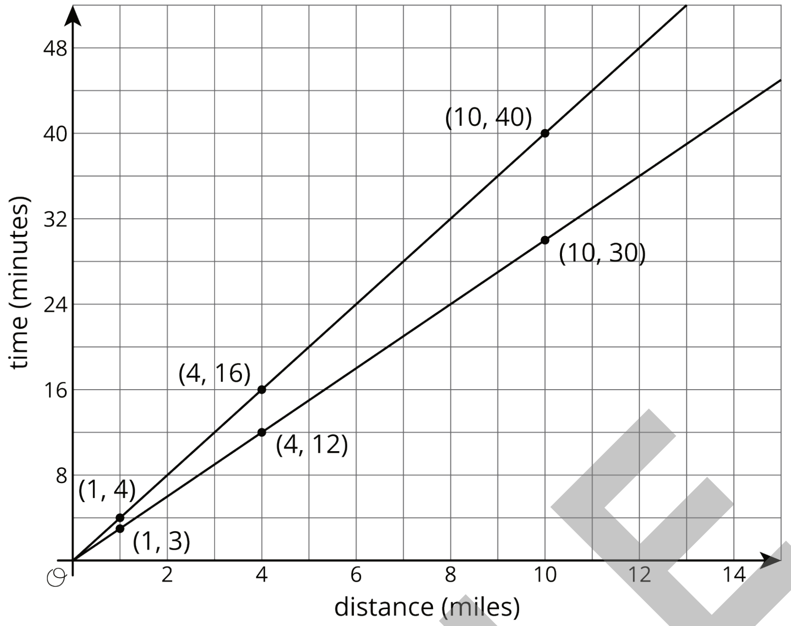
Graphing is a way to help us make sense of relationships. But the graph of a line on a coordinate axes without scale or labels isn't very helpful. For example, let's say we know that on longer bike rides Kiran can ride 4 miles every 16 minutes and Mai can ride 4 miles every 12 minutes. Here are the graphs of these relationships:



Without labels we can't even tell which line is Kiran and which is Mai! Without labels and a scale on the axes, we can't use these graphs to answer questions like:

1. Which graph goes with which rider?
2. Who rides faster?
3. If Kiran and Mai start a bike trip at the same time, how far are they after 24 minutes?
4. How long will it take each of them to reach the end of the 12 mile bike path?

Here are the same graphs, but now with labels and scale:



Revisiting the questions from earlier:

1. Which graph goes with each rider? If Kiran rides 4 miles in 16 minutes, then the point $(4, 16)$ is on his graph. If he rides for 1 mile, it will take 4 minutes. 10 miles will take 40 minutes. So the upper graph represents Kiran's ride. Mai's points for the same distances are $(1, 3)$, $(4, 12)$, and $(10, 30)$, so hers is the lower graph. (A letter next to each line would help us remember which is which!)

2. Who rides faster? Mai rides faster because she can ride the same distance as Kiran in a shorter time.

3. If Kiran and Mai start a bike trip at the same time, how far are they after 20 minutes? The points on the graphs at height 20 are 5 miles for Kiran and a little less than 7 miles for Mai.

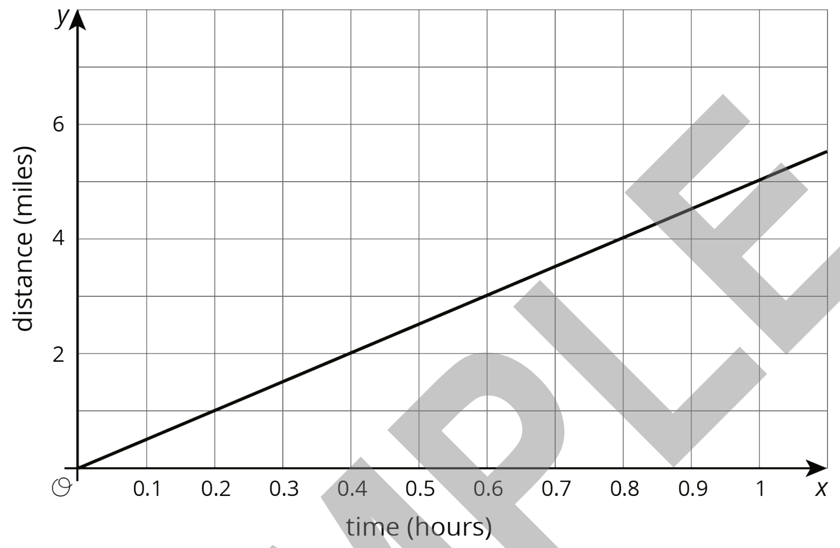
4. How long will it take each of them to reach the end of the 12 mile bike path? The points on the graphs at a horizontal distance of 12 are 36 minutes for Mai and 48 minutes for Kiran. (Kiran's time after 12 miles is almost off the grid!)

Lesson 1 Practice Problems

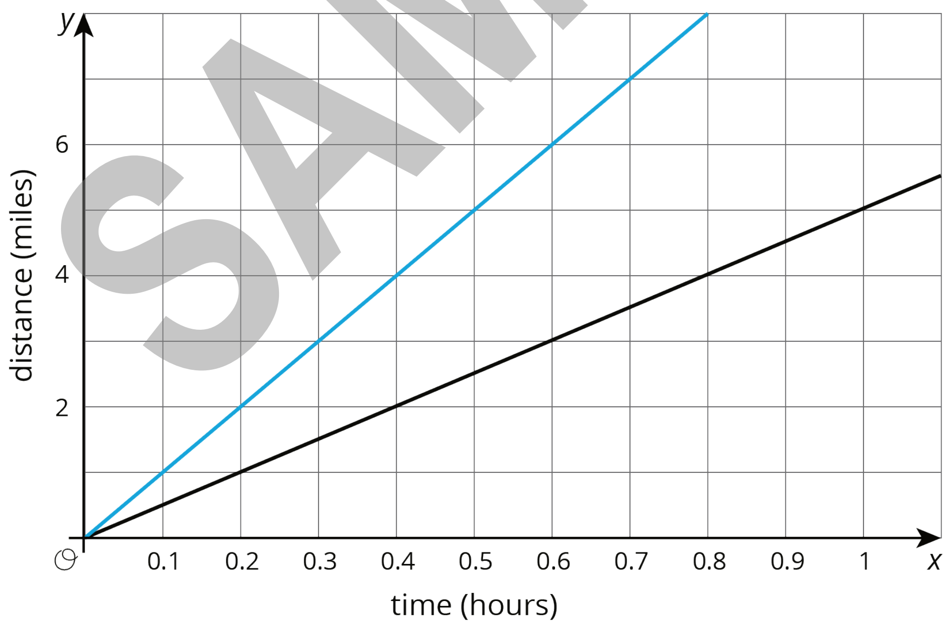
Problem 1

Statement

Priya jogs at a constant speed. The relationship between her distance and time is shown on the graph. Diego bikes at a constant speed twice as fast as Priya. Sketch a graph showing the relationship between Diego's distance and time.



Solution

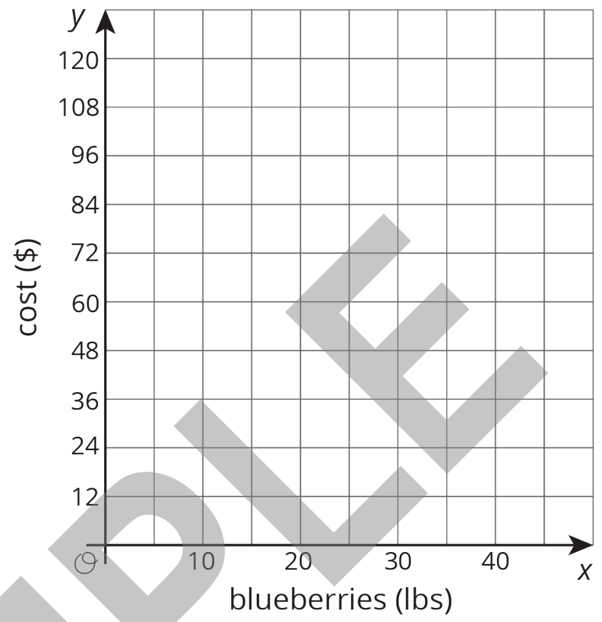


Problem 2

Statement

A you-pick blueberry farm offers 6 lbs of blueberries for \$16.50.

Sketch a graph of the relationship between cost and pounds of blueberries.



Solution

A ray that passes through $(0, 0)$ and $(6, 16.5)$.

Problem 3

Statement

A line contains the points $(-4, 1)$ and $(4, 6)$. Decide whether or not each of these points is also on the line:

a. $(0, 3.5)$

b. $(12, 11)$

c. $(80, 50)$

d. $(-1, 2.875)$

Solution

a. On the line

b. On the line

c. Not on the line

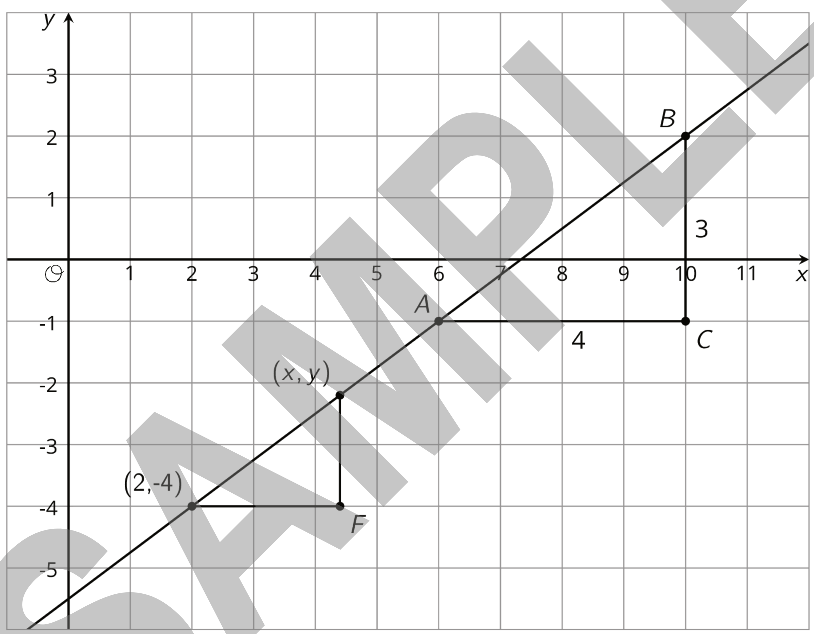
d. On the line

(From Grade8, Unit 2, Lesson 12.)

Problem 4

Statement

The points $(2, \text{-}4)$, (x, y), A, and B all lie on the line. Find an equation relating x and y.



Solution

$\frac{y+4}{x-2} = \frac{3}{4}$ (or equivalent)

(From Grade8, Unit 2, Lesson 11.)

Grade 8, Unit 3, Lesson 14: Using Linear Relations to Solve Problems

Goals

- Recognize that solutions to linear equations may be limited based on real-world constraints on the quantities.
- Solve and interpret solutions in contexts using multiple representations of non-proportional linear relationships.

Learning Targets

- I can write linear equations to reason about real-world situations.

Lesson Narrative

In this culminating lesson for the unit, students put what they have learned to work in solving real-world problems, using all the different forms of equations they have studied (MP4).

Alignments

Addressing

- 8.EE.B.6: Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.
- 8.EE.C.8.a: Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

Student Learning Goals

Let's write equations for real-world situations and think about their solutions.

14.1 Buying Fruit

Warm Up: 5 minutes
Students write expressions and equations representing total cost. The purpose of this activity is to support students in writing the equation for the "Ordering Fish" activity.

Addressing

- 8.EE.B.6

Instructional Routines

- **MLR6: Three Reads**
  To ensure that students know what they are being asked to do, and to create an opportunity for students to reflect on the ways mathematical questions are presented. This routine supports reading comprehension of problems and meta-awareness of mathematical language. It also supports negotiating information in a text with a partner in mathematical conversation.

- **Think pair share**
  What: Students have quiet time to think about a problem and work on it individually, and then time to share their response or their progress with a partner. Once these partner conversations have taken place, some students are selected to share their thoughts with the class.

  Why: This is a teaching routine useful in many contexts whose purpose is to give all students enough time to think about a prompt and form a response before they are expected to try to verbalize their thinking. First they have an opportunity to share their thinking in a low-stakes way with one partner, so that when they share with the class they can feel calm and confident, as well as say something meaningful that might advance everyone's understanding. Additionally, the teacher has an opportunity to eavesdrop on the partner conversations so that she can purposefully select students to share with the class.

Launch

Arrange students in groups of 2. Display questions for all to see. Give 2 minutes quiet think time, followed by 2 minutes partner discussion then whole-class discussion.

Support for Students with Disabilities

Social-Emotional Functioning: Peer Tutors. Pair students with their previously identified peer tutors.

Receptive/Expressive Language: Processing Time. Students who benefit from extra processing time would also be aided by MLR 6 (Three Reads).

Anticipated Misconceptions

Some students may not be sure how to approach writing the scenario as an equation. For these students, suggest that they make a table of possible prices based on the amount of fruit purchased.

Student Task Statement

For each relationship described, write an equation to represent the relationship.

1. Grapes cost \$2.39 per pound. Bananas cost \$0.59 per pound. You have \$15 to spend on g pounds of grapes and b pounds of bananas.

2. A savings account has \$50 in it at the start of the year and \$20 is deposited each week. After x weeks, there are y dollars in the account.

Student Response

1. $2.39g + 0.59b = 15$
2. $y = 20x + 50$

Activity Synthesis

The purpose of this discussion is to have students explain strategies for writing equations for real-world scenarios. Ask students to share what they discussed with their partners by asking:

- “What did each of the variables mean in the situations?” (Since we want a price based on the number of pounds of fruit, b and g represent the amount of bananas and grapes purchased. For the savings account, the x was the number of weeks, and the y was number of dollars.)
- “Was the slope for each of these equations positive or negative? Why does that make sense with the scenario?” (For the fruit, the slope was negative, which makes sense because if you buy more of one fruit, you have to buy less of the other. For the savings account, the slope is positive, which makes sense because the more weeks go by, the more money will be in the account.)

14.2 Five Savings Accounts

25 minutes

Given a graph with five lines representing changes in bank account balance over time, students write equations and interpret how points represent solutions. The activity also connects to and contextualizes students’ prior understanding of slope and intercepts, and lays the foundation for the coming unit on systems of equations by considering what points of intersection of lines and non-intersecting lines represent.

Addressing

- 8.EE.B.6

Launch

Display the image from the lesson for all to see and ask the students to consider line a. Invite 2–3 students to describe in words what line a shows. If no students bring it up, tell students that they saw this line before in the warm-up, and they wrote an equation for it. Instruct students that for #1, they should not choose line a.

Arrange students in groups of 3–4. Groups work for about 10 minutes, followed by a whole-class discussion.

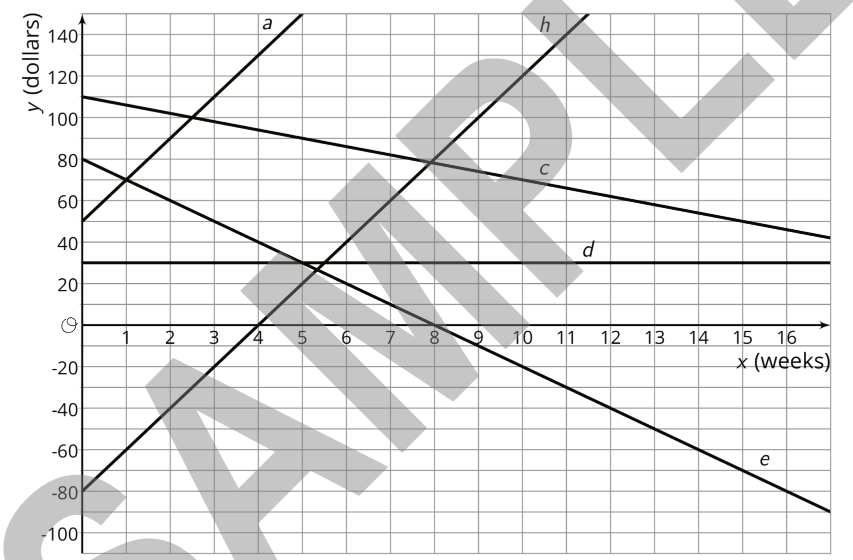
Support for English Language Learners

Heavier Support: MLR 4 (Info Gap). For question 1, have students take turns sharing their stories while their teammate guesses which line is represented. Encourage students to ask probing and clarifying questions if they are unsure which line matches a story.

Support for Students with Disabilities

Receptive/Expressive Language: Processing Time. Students who benefit from extra processing time would also be aided by MLR 4 (Info Gap).

Student Task Statement



Each line represents one person's weekly savings account balance from the start of the year.

1. Choose one line and write a description of what happens to that person's account over the first 17 weeks of the year. Do not tell your group which line you chose.
2. Share your story with your group and see if anyone can guess your line.
3. Write an equation for each line on the graph. What do the slope, m, and vertical intercept, b, in each equation mean in the situation?
4. For which equation is $(1, 70)$ a solution? Interpret this solution in terms of your story.
5. Predict the balance in each account after 20 weeks.

Student Response

1. Answers vary. Sample responses: Person a starts with \$50 and is saving money at the rate of \$20 per week. Person h owed \$80 and is paying it back at the rate of \$20 per week, then saving once the debt is paid off. Person c starts with \$110 and is spending money at the rate of \$20 every 5 weeks, or \$4 per week. Person d has \$30 and is neither saving or spending. Person e starts with \$80 and spends at the rate of \$10 per week.

2. Responses vary.

3. a: $y = 20x + 50$; h: $y = 20x – 80$; c: $y = -4x + 110$; d: $y = 30$; e: $y = -10x + 80$; For each equation, the slope tells the rate of change of saving (positive) or spending (negative). The value of b indicates the amount of money they started with, positive represents a saved balance, negative represents money they owe. Person d shows a slope of zero—neither saving or spending, so that they remain over time with the same amount that they start with.

4. We can see from the graph that lines a and e share a common point, or solution, at $x = 1$ week, where \$ for both. Sample explanation: at 1 week, each of these people had \$70 in their accounts.

5. Person a will have \$. Person h will have \$. Person c will have \$. Person d will have \$30. Person e will have \$.

Activity Synthesis

Students should understand that points on a line show solutions to the equation of the line. Discuss with students:

- “What can we say about the points where two lines cross?” (The accounts had the same amount of money at the same time.)
- “How do the slopes of the lines help to tell the story from the graph?” (The slope tells us whether a person is spending or saving each week.)
- “What does your answer to question 3 tell us about their rates of saving?” (By knowing the value of the slope, we can compare who is spending or saving more quickly or more slowly.)

14.3 Fabulous Fish

20 minutes

Students represent a scenario with an equation and use the equation to find solutions. They create a graph (either with a table of values or by using two intercepts), interpret points on the graph, and interpret points not on the graph (MP2).

Addressing

- 8.EE.C.8.a

Instructional Routines

- **MLR2: Collect and Display**
  To capture students' oral words and phrases into a stable, collective reference. The intent of this routine is to stabilize the fleeting language that students use during partner, small-group, or whole-class activities in order for student's own output to be used as a reference in developing their mathematical language. The teacher listens for, and scribes, the student output using written words, diagrams and pictures; this collected output can be organized, revoiced, or explicitly connected to other language in a display for all students to use. This routine provides feedback for students in a way that increases accessibility while simultaneously supporting meta-awareness of language.

Launch

Allow about 10 minutes quiet think time for questions 1 through 4, then have students work with a partner to discuss questions 4 and 5. Look for students who define the variables or label the axes differently. This can be an opportunity to discuss the importance of defining what quantities your variables represent and that different graphs can represent the same information.

Support for English Language Learners

Lighter Support: MLR 2 (Collect and Display). Listen for and record the language students use to discuss question 5. Organize and group similar strategies in the display for students to refer back to throughout the lesson.

Support for Students with Disabilities

Fine Motor Skills: Peer Tutors. Pair students with their previously identified peer tutors and allow students who struggle with fine motor skills to dictate graphing as needed.

Receptive/Expressive Language: Processing Time. Students who benefit from extra processing time would also be aided by MLR 2 (Collect and Display).

Anticipated Misconceptions

If students let x be pounds of salmon, then the equation would be $5x + 3y = 210$ and the coordinates would be reversed. The intercepts of the graph would be $(0, 70)$ and $(42, 0)$. This is a good place to mention the importance of defining what quantities your variables represent.

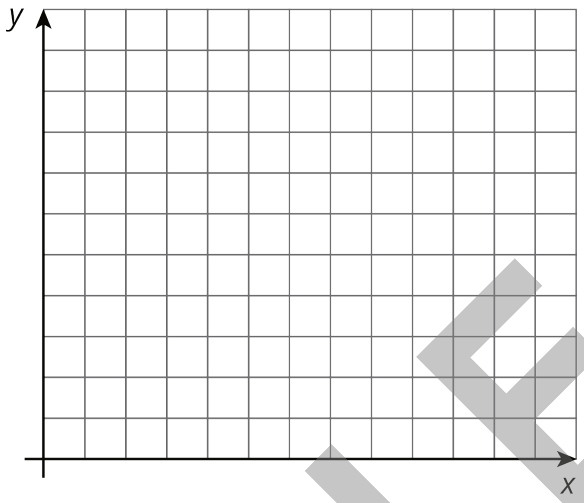
Student Task Statement

The Fabulous Fish Market orders tilapia, which costs \$3 per pound, and salmon, which costs \$5 per pound. The market budgets \$210 to spend on this order each day.

1. What are five different combinations of salmon and tilapia that the market can order?

2. Define variables and write an equation representing the relationship between the amount of each fish bought and how much the market spends.

3. Sketch a graph of the relationship. Label your axes.



4. On your graph, plot and label the combinations A—F.

A	B	C	D	E	F
5	19	27	25	65	55
36	30.6	25	27	6	4

Which of these combinations can the market order? Explain or show your reasoning.

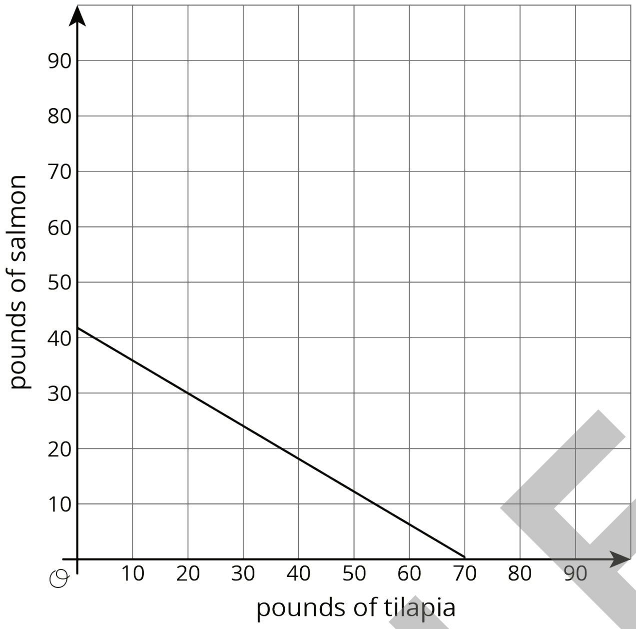
5. List two ways you can tell if a pair of numbers is a solution to an equation.

Student Response

1. Answers vary. Sample responses: $(0, 42)$, $(10, 36)$, $(20, 30)$, $(30, 24)$, $(50, 12)$, $(70, 0)$

2. Answers vary. Sample response: Let x be number of pounds of tilapia, let y be number of pounds of salmon: $3x + 5y = 210$.

3. Descriptions vary. Sample response: graph is a line that begins at $(0, 42)$ and slopes downward until reaching $(70, 0)$. It will not continue on indefinitely because negative pounds of fish does not make sense in the situation.



4. A does not work because $3(5) + 5(36)$ is 195, not 210.
   B works because $3(19) + 5(30.6)$ is 210.
   C does not work because $3(27) + 5(25)$ is 206, not 210.
   D works because $3(25) + 5(27)$ is 210.
   E does not work because $3(65) + 5(6)$ is 225, not 210.
   F does not work because $3(55) + 5(4)$ is 185, not 210.

5. Responses vary. Sample response: Solutions make the equation true and can be found on the graph of the equation.

Activity Synthesis

Ask students to share some strategies for graphing and features of their graphs. Invite 2–3 students to display their graphs. Consider asking:

- "Was the slope of the line positive or negative? Why does that makes sense in this situation?" (In this relationship, as one quantity increases the other must decrease in order to keep the sum $3x + 5y$ constant.)
- "What was your strategy for graphing the relationship?" (Plotting points from question 1, using the table, figuring out the intercepts and connecting the line)
- "Why does it make sense for the graph to be only in quadrant I?" (You cannot purchase a negative amount of fish, so the x and y values cannot be negative.)
- "How is this situation different from the apples and oranges problem in a previous lesson?" (Buying $\frac{1}{2}$ pound of fish is reasonable while buying $\frac{1}{2}$ of an apple probably is not.)

Briefly reiterate key concepts: If a point is not on the graph of the equation then it is not a solution. The ordered pairs that are solutions to the equation all make the equation true and are all found on the line that is the graph of the equation.

A discussion could also include the detail that orders that are less than $210 can also be considered to work, because there is money left over. That gives an opportunity to discuss the shape of the graph of $3x + 5y \leq 210$.

Lesson Synthesis

Ask students to consider the real-world situations described in this lesson. Discuss:

- "Give an example of a solution to an equation that doesn't make sense in the context it represents." (Some values might not make sense in the context, like negative values. A length cannot be negative, for example.)
- "If some values make sense in the equation but not in the context, how could this impact the graph?" (We might only draw part of the line or draw some of the points that lie on the line.)

Lesson 14 Practice Problems

Problem 1

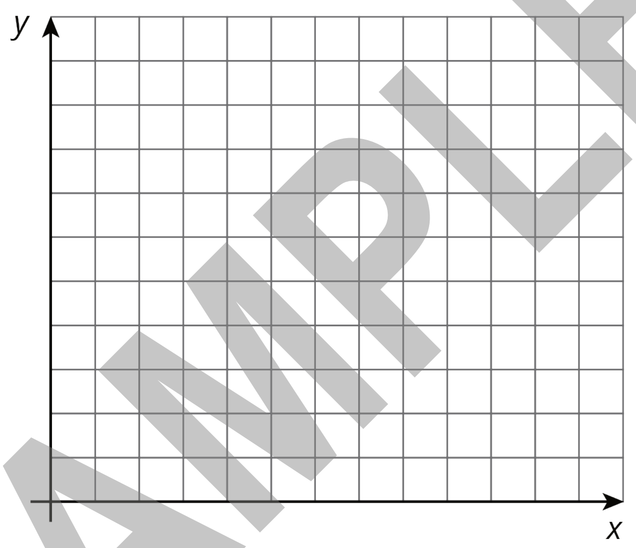
Statement

The owner of a new restaurant is ordering tables and chairs. He wants to have only tables for 2 and tables for 4. The total number of people that can be seated in the restaurant is 120.

a. Describe some possible combinations of 2-seat tables and 4-seat tables that will seat 120 customers. Explain how you found them.

b. Write an equation to represent the situation. What do the variables represent?

c. Create a graph to represent the situation.



d. What does the slope tell us about the situation?

e. Interpret the x and y intercepts in the situation.

Solution

a. No 2-seat and 30 4-seat, 10 2-seat and 25 4-seat, 40 2-seat and 10 4-seat. Explanations vary. Sample response: I decided on a number for the 2-seat tables, then figured out how many people that would be (multiply number of tables by 2) and subtracted that from 120. Then I divided by 4 to get the number of 4-seat tables needed for the remaining people.

b. Answers vary. Sample response: $2x + 4y = 120$. x represents the number of 2-seat tables and y represents the number of 4-seat tables.

c. Graph is the line connecting $(0, 30)$ and $(60, 0)$.

d. Answers vary. Sample response. The slope is $\frac{-1}{2}$. $\frac{-1}{2}$ tells us that for every one fewer 4-seat table we can use 2 2-seat tables.

e. The intercepts are $(0, 30)$ and $(60, 0)$. They tell us how many tables there will be if only 4-seat tables are used (30) or only 2-seat tables are used (60).

Problem 2

Statement

Triangle A is an isosceles triangle with two angles of measure x degrees and one angle of measure y degrees.

a. Find three combinations of x and y that make this sentence true.

b. Write an equation relating x and y.

c. If you were to sketch the graph of this linear equation, what would its slope be? How can you interpret the slope in the context of the triangle?

Solution

a. Answers vary; the key constraint is that the three angles must sum to 180 ($x + x + y = 180$). For example, $x = y = 60$, or $x = 30$ and $y = 120$ or $x = 45$ and $y = 90$.

b. $2x + y = 180$

c. -2. In the context of the triangle, for every 1 degree increase of x, y decreases by 2 degrees.

(From Grade8, Unit 3, Lesson 13.)

Problem 3

Statement

Select **all** the equations for which $(-6, -1)$ is a solution.

A. $y = 4x + 23$

B. $3x = \frac{1}{2}y$

C. $2x - 13y = 1$

D. $3y = \frac{1}{2}x$

E. $2x + 6y = -6$

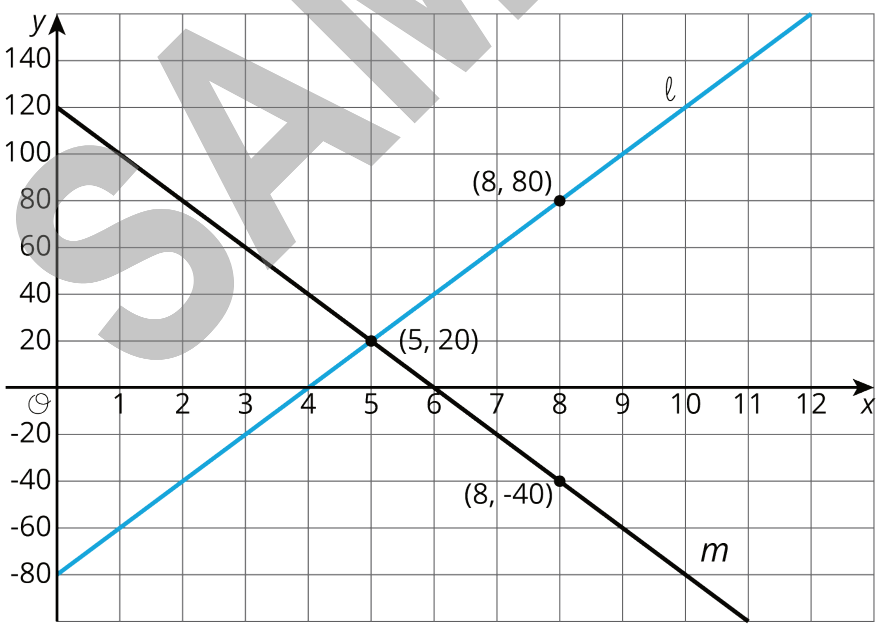
Solution

["A", "C", "D"]
(From Grade8, Unit 3, Lesson 12.)

Problem 4

Statement

Consider the following graphs of linear equations. Decide which line has a positive slope, and which has a negative slope. Then calculate each line's exact slope.



Solution

The line ℓ moves up on the y-axis as it moves to the right, so it has a positive slope. m has a negative slope since it moves downwards. The slope of ℓ is $\frac{80-20}{8-5} = \frac{60}{3} = 20$. The slope of m is $\frac{-40-20}{8-5} = \frac{-60}{3} = -20$.

(From Grade8, Unit 3, Lesson 10.)