



ACCELERATED

7

Unit

1

ACCELERATED
SAMPLE BOOK

Kendall Hunt

SAMPLE

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SAMPLE

Unit Narrative

Unit Narrative

Work with transformations of plane figures in this unit draws on earlier work with geometry and geometric measurement. Students began to learn about two- and three-dimensional shapes in kindergarten, and continued this work in grades 1 and 2, composing, decomposing, and identifying shapes. Students' work with geometric measurement began with length and continued with area. Students learned to "structure two-dimensional space," that is, to see a rectangle with whole-number side lengths as composed of an array of unit squares or composed of iterated rows or iterated columns of unit squares. In grade 3, students distinguished between perimeter and area. They connected rectangle area with multiplication, understanding why (for whole-number side lengths) multiplying the side lengths of a rectangle yields the number of unit squares that tile the rectangle. They used area diagrams to represent instances of the distributive property. In grade 4, students applied area and perimeter formulas for rectangles to solve real-world and mathematical problems, and learned to use protractors. In grade 5, students extended the formula for the area of rectangles to rectangles with fractional side lengths. In a previous course, students combined their knowledge of geometry and geometric measurement to produce formulas for the areas of parallelograms and triangles, using these formulas to find surface areas of polyhedra.

Through activities designed and sequenced to allow students to make sense of problems and persevere in solving them (MP1), students use and extend their knowledge of geometry and geometric measurement. They begin the unit by looking at pairs of cartoons, each of which illustrates a translation, rotation, or reflection. Students describe in their own words how to move one cartoon figure onto another. As the unit progresses, they solidify their understanding of these transformations, increase the precision of their descriptions (MP6), and begin to use associated terminology, recognizing what determines each type of transformation, for example, two points determine a translation.

In the first few lessons, students encounter examples of transformations in the plane, without the added structure of a grid or coordinates. The reason for this choice is to avoid limiting students' schema by showing the least restrictive examples of transformations. Specifically, students see examples of translations in any direction, rotations by any angle, and reflections over any arbitrary line. Through these examples, they begin to understand the features of these transformations without having their understanding limited to, for example, horizontal or vertical translations or rotations only by 90 or 180 degrees. Also, through the use of transparencies, students' initial understanding of transformations involves moving the entire plane, rather than just moving a given figure. Since all transformations are transformations of the plane, it is preferable for students to first encounter examples that involve moving the entire plane.

They identify and describe translations, rotations, and reflections, and sequences of these. In describing images of figures under rigid transformations on and off square grids and the coordinate plane, students use the terms "corresponding points," "corresponding sides," and

“image.” Students learn that angles and distances are preserved by any sequence of translations, rotations, and reflections, and that such a sequence is called a “rigid transformation.” They learn the definition of “congruent”: two figures are said to be congruent if there is a rigid transformation that takes one figure to the other. Students experimentally verify the properties of translations, rotations, and reflections, and use these properties to reason about plane figures, understanding informal arguments showing that the alternate interior angles cut by a transversal have the same measure and that the sum of the angles in a triangle is 180 degrees. The latter will be used in a subsequent unit on similarity and dilations. Students end the unit investigating whether sets of angle and side length measurements determine unique triangles (that is, triangles that are congruent), multiple triangles, or fail to determine triangles. Students also study and apply angle relationships, learning to understand and use the terms “complementary,” “supplementary,” “vertical angles,” and “unique” (MP6). Throughout the unit, students discuss their mathematical ideas and respond to the ideas of others (MP3, MP6).

Many of the lessons in this unit ask students to work on geometric figures that are not set in a real-world context. This design choice respects the significant intellectual work of reasoning about area. Tasks set in real-world contexts are sometimes contrived and hinder rather than help understanding. Moreover, mathematical contexts are legitimate contexts that are worthy of study. Students do have opportunities in the unit to tackle real-world applications. In the culminating activity of the unit, students examine and create different patterns formed by plane figures. This is an opportunity for them to apply what they have learned in the unit (MP4).

In this unit, several lesson plans suggest that each student have access to a geometry toolkit. These contain tracing paper, graph paper, colored pencils, scissors, ruler, protractor, and an index card to use as a straightedge or to mark right angles, giving students opportunities to develop their abilities to select appropriate tools and use them strategically to solve problems (MP5). Note that even students in a digitally enhanced classroom should have access to such tools. Apps and simulations should be considered additions to their toolkits, not replacements for physical tools.

Progression of Disciplinary Language

In this unit, teachers can anticipate students using language for mathematical purposes such as describing, generalizing, and justifying. Throughout the unit, students will benefit from routines designed to grow robust disciplinary language, both for their own sense-making and for building shared understanding with peers. Teachers can formatively assess how students are using language in these ways, particularly when students are using language to:

Describe

- movements of figures (Lessons 1 and 2)
- observations about transforming parallel lines (Lesson 8)
- transformations using corresponding points, line segments, and angles (Lesson 9)
- observations about angle measurements (Lesson 14)
- positioning and movement of side lengths and angles (Lesson 15)
- transformations found in tessellations and in designs with rotational symmetry (Lesson 18)

Generalize

- about categories for movement (Lesson 2)
- about rotating line segments 180° (Lesson 7)
- about the relationship between vertical angles (Lesson 8)
- about corresponding segments and length (Lesson 11)
- about alternate interior angles (Lesson 12)
- about the sum of angles in a triangle (Lesson 14)
- about categories for unique triangles (Lesson 16)

Justify

- whether or not rigid transformations could produce an image (Lesson 6)
- whether or not shapes are congruent (Lesson 10)
- whether or not polygons are congruent (Lesson 11)
- whether or not triangles can be created from given angle measurements (Lesson 13)
- whether or not shapes are identical copies (Lesson 15)
- whether or not measurements determine unique triangles (Lesson 17)

In addition, students are expected to explain and interpret directions for transforming figures and how to apply transformations to find specific images. Students are also asked to use language to compare rotations of a line segment, compare perimeters and areas of rectangles, and compare triangles in a set. Over the course of the unit, teachers can support students' mathematical understandings by amplifying (not simplifying) language used for all of these purposes as students demonstrate and develop ideas.

The table shows lessons where new terminology is first introduced, including when students are expected to understand the word or phrase receptively and when students are expected to produce the word or phrase in their own speaking or writing. Terms from the glossary appear bolded. Teachers should continue to support students' use of a new term in the lessons that follow the one in which it was first introduced.

lesson	new terminology	
	receptive	productive
7.1.1	vertex plane measure direction	slide turn
7.1.2	clockwise counterclockwise reflection rotation translation	opposite
7.1.3	image angle of rotation center (of rotation) line of reflection sequence of transformations distance	vertex clockwise counterclockwise reflect rotate translate
7.1.4	coordinate plane point segment coordinates x-axis y-axis	
7.1.5	polygon	angle of rotation center (of rotation) line of reflection
7.1.6	rigid transformation corresponding measurements preserve	reflection rotation translation measure point
7.1.7	midpoint	segment
7.1.8	vertical angles parallel intersect	distance



lesson	new terminology	
	receptive	productive
7.1.9		image rigid transformation midpoint parallel
7.1.10	congruent perimeter area	
7.1.11		right angle corresponding x-axis y-axis area
7.1.12	alternate interior angles transversal supplementary complementary	vertical angles congruent
7.1.13	straight angle	supplementary
7.1.14		alternate interior angles transversal straight angle
7.1.15	compass identical copy condition different triangle	
7.1.16		condition different triangle
7.1.17	unique triangle	protractor compass
7.1.18	tessellation symmetry	

Required Materials

Blank paper

Compasses

Copies of blackline master

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty

paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Graph paper

Isometric graph paper

Metal paper fasteners

brass brads

Pre-printed cards, cut from copies of the blackline master

Pre-printed slips, cut from copies of the blackline master

SAMPLE

Assessments

Assessment : Check Your Readiness

Teacher Instructions

Provide access to a ruler, protractor, and compass.

Student Instructions

You will need a ruler, protractor, and compass.

Problem 1

The content assessed in this problem is first encountered in Lesson 1: Moving in the Plane.

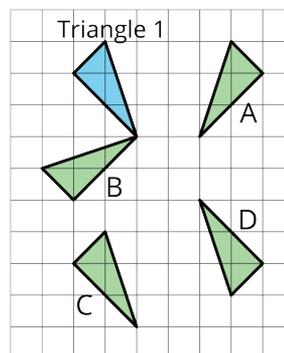
Some students already have an intuitive (if not rigorous) understanding of what rotations are. This item probes that understanding by having students identify rotated images of a given triangle. If students can answer this question correctly, then they already have a good intuition for rigid motions of the plane.

Triangle A is a reflection of Triangle 1. Triangle B is a rotation of Triangle 1. Triangle C is a translation of Triangle 1. Some students may note that it's possible to combine a slide with a 360 degree rotation to match up the triangles. This argument shows a decent understanding of both reflections and rotations, but is not technically correct since Triangle C cannot land exactly on Triangle 1 with a single rotation. Triangle D is a rotation of Triangle 1.

This language will formalize and the concept of rotations will be developed over the span of several lessons. If most students do well with this item, it may be possible to move more quickly through the first two lessons in the unit.

Statement

Select **all** the triangles that can be rotated to match up with Triangle 1.



- A. A
- B. B
- C. C
- D. D

Solution

["B", "D"]

Aligned Standards

8.G.A

Problem 2

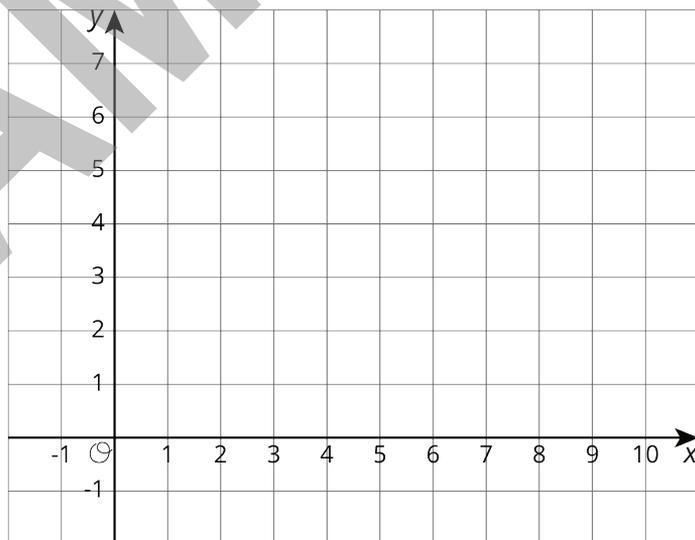
The content assessed in this problem is first encountered in Lesson 4: Coordinate Moves.

Students plot points on a coordinate grid and find distances between points sharing the same x -coordinate or the same y -coordinate. The last part of the problem assesses whether students can identify a parallelogram.

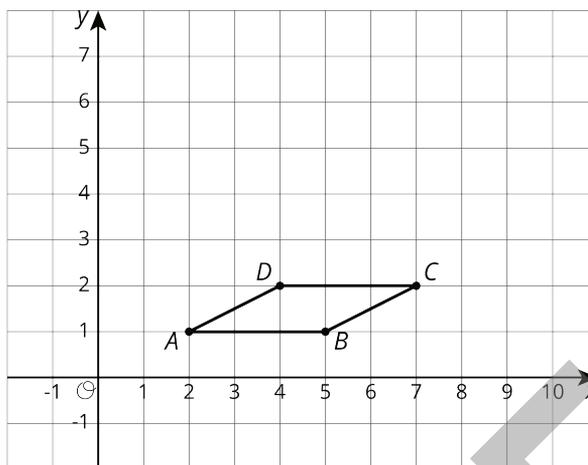
If most students struggle with this item, plan to use Lesson 4 Activity 1 to review the coordinate plane and to consider how to describe translations.

Statement

1. On the coordinate plane, plot and label these points: $A = (2, 1)$, $B = (5, 1)$, $C = (7, 2)$, $D = (4, 2)$
2. What is the length of CD ?
3. What kind of quadrilateral is $ABCD$?



Solution



1. See plotted points.
2. 3 units
3. Parallelogram

Aligned Standards

5.G.A.1, 6.G.A.3

Problem 3

The content assessed in this problem is first encountered in Lesson 12: Alternate Interior Angles.

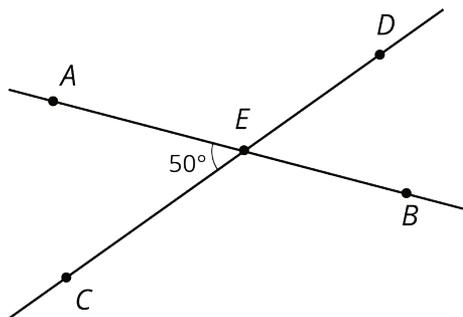
Students identify and use facts about adjacent and vertical angles to calculate angles. It is possible that students will use the fact that angle DEB is adjacent to angle AED to answer the second question.

If students demonstrate an understanding of the ideas of complementary and supplementary angles, it may be possible to move more quickly through the warm-up of the lesson.

Statement

Lines AB and CD intersect at E .

1. What is the measure of angle AED ? Explain how you know.
2. What is the measure of angle DEB ? Explain how you know.



Solution

1. 130° . Angles AEC and AED are supplementary, so their measures add up to 180° .
2. 50° . Angles AEC and DEB are vertical angles, so they have the same measure.

Aligned Standards

7.G.B.5

Problem 4

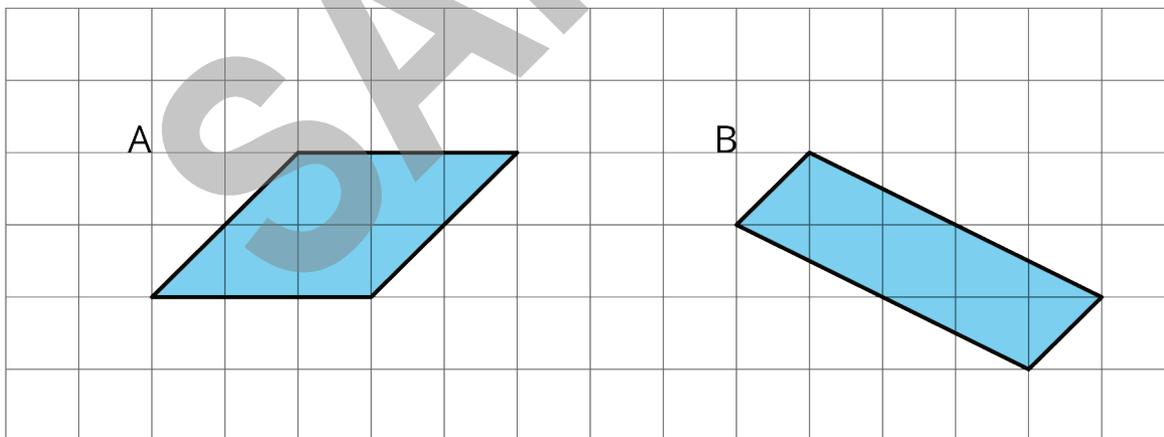
The content assessed in this problem is first encountered in Lesson 9: Composing Figures.

Students can use a variety of strategies to solve these problems. Some students may remember the formula for the area of a parallelogram, but it is not necessary that they do. Because of its positioning on the lattice, Parallelogram B will require a more creative strategy than using the area formula. In the unit, students will use area as one way to reason about whether two figures might be congruent.

If most students do well on this item, plan to encourage students to explore the Are You Ready for More in Lesson 9. If most students struggle with the parallelogram on the left, plan to use Lesson 9 Activity 2 to explore and review the relationship between strategies for finding the area of a triangle and the area of a parallelogram. If most students struggle with the parallelogram on the right, you may want to review strategies for finding area of figures that aren't oriented with the grid. However, this skill will be more important in future units.

Statement

Find the area of each parallelogram.



Solution

Parallelogram A: 6 square units. Sample reasoning:

- The parallelogram is 3 units wide and two units high, so its area is 6 square units.

- The shape can be decomposed into a 2-by-1 rectangle and 2 triangles that fit together to make a 2-by-2 square. So the total area is $2 + 4 = 6$ square units.

Parallelogram B: 6 square units. Sample reasoning: The parallelogram can be surrounded by a 3-by-5 rectangle, and the areas of the missing triangles, which are $\frac{1}{2} + \frac{1}{2} + 4 + 4$, can be subtracted.

Aligned Standards

7.G.A.2

Problem 5

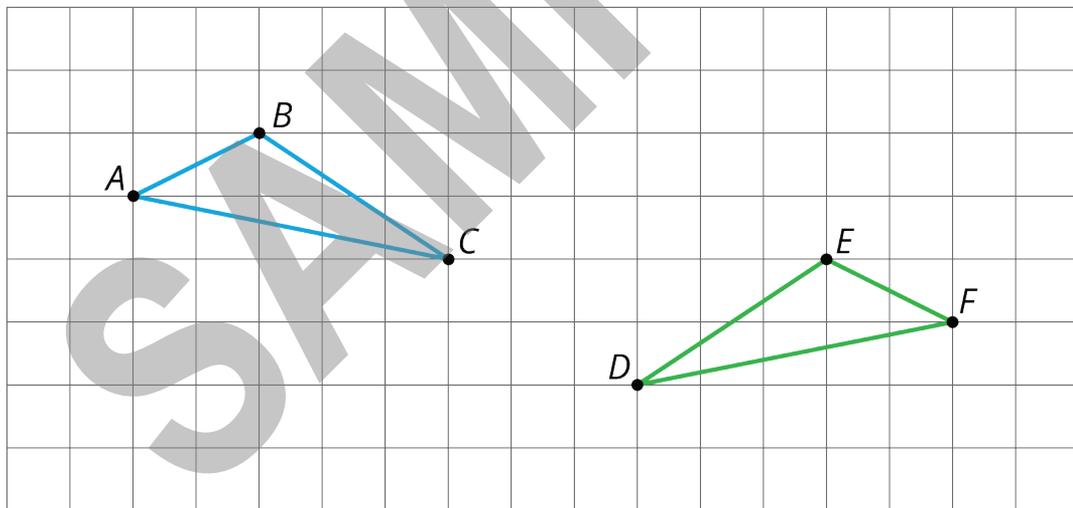
The content assessed in this problem is first encountered in Lesson 3: Making the Moves.

The formal idea of congruence is new in this unit, but the idea is intuitive. This item examines students' ability to visualize and verbalize the steps that map one figure to another.

If most students do well with this item, it may be possible to move fairly quickly through the first two lessons in the unit.

Statement

Here are two triangles:



Describe a way to move triangle ABC so that it matches up perfectly with triangle FED .

Solution

Answers vary. Sample response: Triangle ABC can be moved down two units and then flipped over a vertical line that lies halfway between the two triangles.

Aligned Standards

8.G.A.2

Problem 6

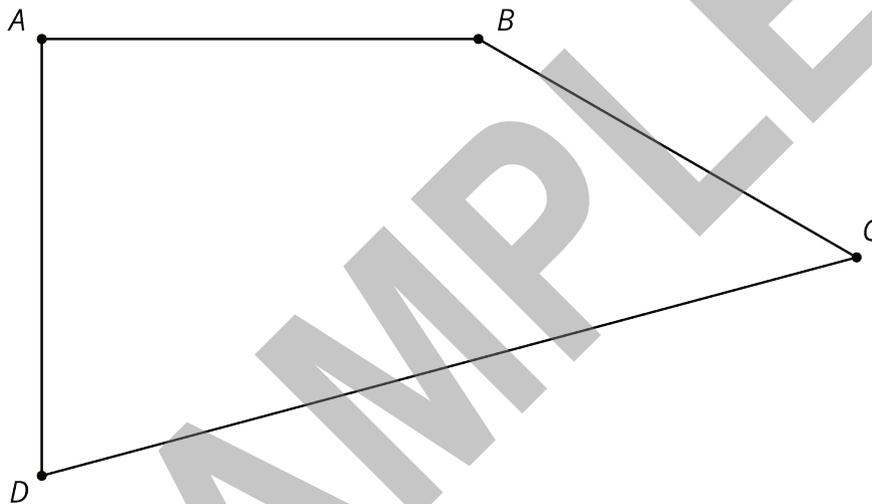
The content assessed in this problem is first encountered in Lesson 12: Alternate Interior Angles.

In several lessons in the unit, students are expected to measure angles with a protractor in order to make conjectures such as “vertical angles are equal.”

If most students struggle with this item, plan to incorporate practice with a protractor before Lesson 12. For example, during the warm-up of Lesson 2 when students study the rotation of a figure about a point.

Statement

Use a protractor to measure all four angles inside quadrilateral $ABCD$. Write the measure of each angle to the nearest whole degree.



Solution

$A = 90$, $B = 150$, $C = 45$, $D = 75$. Measurements within ± 5 degrees should be accepted.

Aligned Standards

4.MD.C.6

Problem 7

The content assessed in this problem is first encountered in Lesson 17: Drawing Triangles.

In this unit, students investigate the number of distinct polygons or triangles that can be drawn with given information: say, two sides and an angle. In some cases, students use protractors as a tool in constructing these shapes.

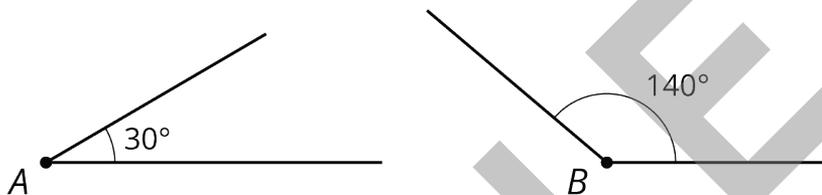
If most students struggle with this item, plan to incorporate drawing angles on tracing paper into earlier activities to verify measures of angles that are given. Students can place their tracing paper over the printed angles to check their drawings.

Statement

Use a protractor to draw an angle of each measure.

1. 30°
2. 140°

Solution



Aligned Standards

4.MD.C.6

Problem 8

The content assessed in this problem is first encountered in Lesson 4: Coordinate Moves.

The content assessed in this problem is first encountered in Lesson 7: Rotation Patterns.

The content assessed in this problem is first encountered in Lesson 1: Moving in the Plane.

These vocabulary words are all background for the geometric work of this unit. Watch for students drawing an obtuse angle instead of an obtuse triangle.

If most students struggle with the term parallel, plan to highlight the parallel lines that show up in the activities of Lesson 7 in preparation for the more direct work with parallel lines in Lesson 8.

If most students struggle with the term perpendicular, Lesson 4 Activity 3 provides an opportunity to review the term perpendicular.

If most students struggle with the terms acute and obtuse, plan to define acute and obtuse in Lesson 1 Activity 1. Consider making an anchor chart with names and types of angles that will be used throughout this unit.

Statement

1. Draw a pair of parallel lines.
2. Draw a pair of perpendicular lines.
3. Draw an acute angle.

- 4. Draw an obtuse triangle.

Solution

Answers vary.

Aligned Standards

4.G.A.1

SAMPLE

Assessment : Mid-Unit Assessment

Teacher Instructions

Give this assessment after lesson 9. Students will need access to a straightedge.

Student Instructions

You will need a straightedge for this assessment.

Problem 1

Students selecting A or B have basic misunderstandings about the way translations or reflections work. Students selecting C may have had trouble keeping track of the sequence, failing to notice that the shape has moved a total of 4 units to the right to compensate for the 4 units to the left. Students who do not select D may be thinking that the two rotations together add up to 360° .

Statement

All of these sequences of transformations would return a shape to its original position except?

- A. Translate 3 units up, then 3 units down.
- B. Reflect over line p , then reflect over line p again.
- C. Translate 1 unit to the right, then 4 units to the left, then 3 units to the right.
- D. Rotate 120° counterclockwise around center C , then rotate 220° counterclockwise around C again.

Solution

D

Aligned Standards

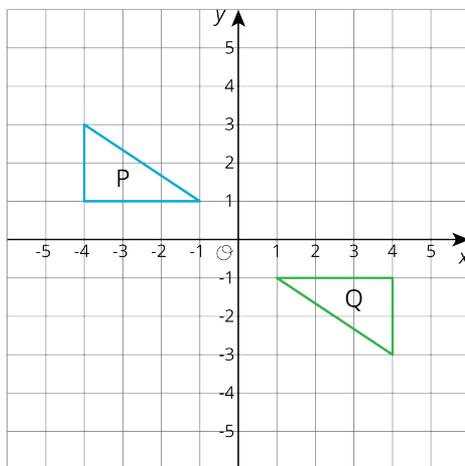
8.G.A.1

Problem 2

Students selecting A may mistakenly think reflections can apply to points, and might have reflected using the origin; this is not a valid reflection. Students failing to select B may need more practice with 180° rotations using the origin. Students selecting C may mistakenly think that aligning one vertex must align all others, or that this is enough to count as a correct transformation. Students selecting D may mistakenly think that aligning one edge must align all others, or that this is enough to count as a correct transformation. Students failing to select E need more practice with reflections across the x - and y -axes.

Statement

Select **all** the sequences of transformations that *could* take Figure P to Figure Q.



- A. a single reflection
- B. a single rotation
- C. a single translation
- D. a translation, then a reflection
- E. a reflection, then a different reflection

Solution

["B", "E"]

Aligned Standards

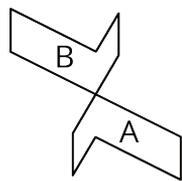
8.G.A.1

Problem 3

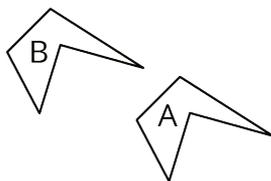
Students selecting B have confused a translation with a rotation. Students selecting C may believe the common point between the two figures is the center of a 180° rotation, but the shapes of the figures do not permit this. Students selecting D have likely confused a reflection with a rotation, but may also think a rotation will produce the same effect; it might be useful to show the result of a rotation of 180° around the point where the figures closely approach.

Statement

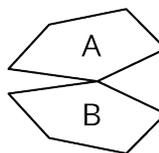
In which pair of figures can Figure A be taken to Figure B by a rotation?



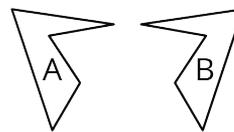
pair 1



pair 2



pair 3



pair 4

- A. Pair 1
- B. Pair 2
- C. Pair 3
- D. Pair 4

Solution

A

Aligned Standards

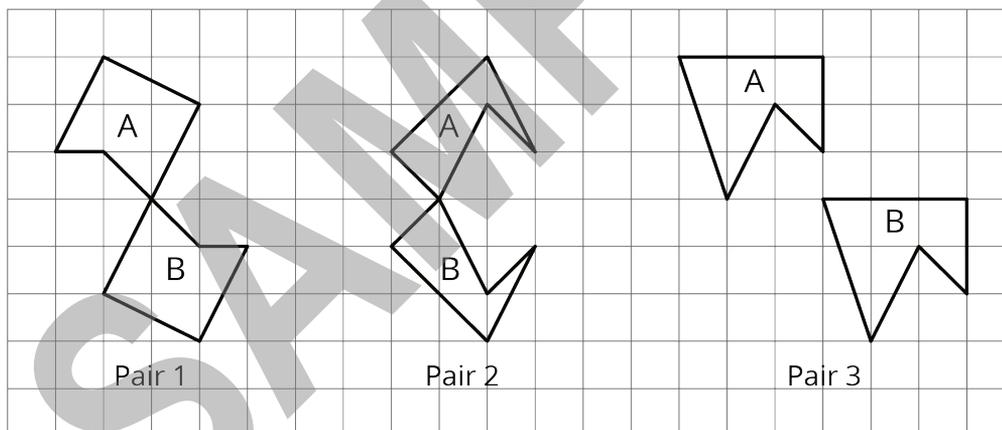
8.G.A.1

Problem 4

Students are presented with an shape and an image that is a translation, rotation, or reflection of the given shape. Students then identify the translation, rotation, or reflection. For this task, the shapes are on a grid.

Statement

Here are three pairs of figures.



1. Which transformation takes Figure A to Figure B in Pair 1: a translation, rotation, or reflection?
2. Which transformation takes Figure A to Figure B in Pair 2: a translation, rotation, or reflection?
3. Which transformation takes Figure A to Figure B in Pair 3: a translation, rotation, or reflection?

Solution

1. rotation (this is a 180° rotation using the vertex shared by Figures A and B)

2. reflection (this is a reflection using the horizontal line through where the vertices shared by Figures A and B meet)
3. translation (this is a translation of four units down and three units to the right)

Aligned Standards

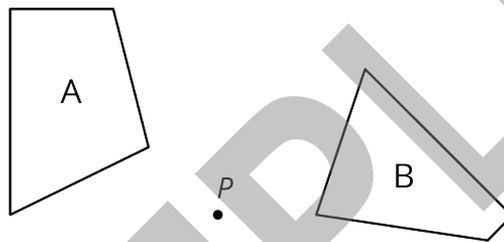
8.G.A.1

Problem 5

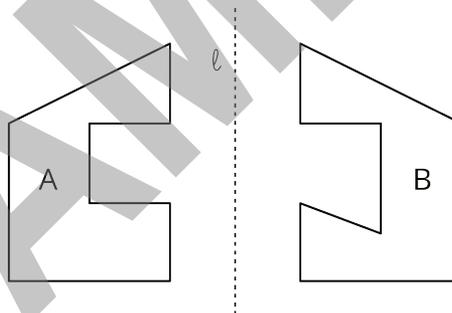
Segment lengths and angle measures are preserved by translations, reflections, and rotations. If two shapes do not have the same side lengths or angles, then one of them cannot be obtained from the other by a rigid transformation.

Statement

1. Explain why Figure B is *not* the image of Figure A after a rotation using center P .



2. Explain why Figure B is *not* the image of Figure A after a reflection using line ℓ .



Solution

1. The shortest side of Figure A is shorter than the shortest side of Figure B. Since rotations do not change lengths, Figure B is not a rotation of Figure A.
2. The angles in Figure B do not match the corresponding angles in Figure A. Some of the angles in Figure A look like right angles while some corresponding angles in Figure B are definitely not right angles.

Minimal Tier 1 response:

- Work is complete and correct.
- Sample:

1. The side lengths of the figures aren't the same.
2. The angle measures of the figures aren't the same.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: generic statement about shape, such as "The two figures have different shapes"; stating that the second shape has different side lengths without being specific (only one of the side lengths is visibly different); stating that the first shape has different angle measures without being specific.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: omitted explanations; stating that Figure B is the image of Figure A.

Aligned Standards

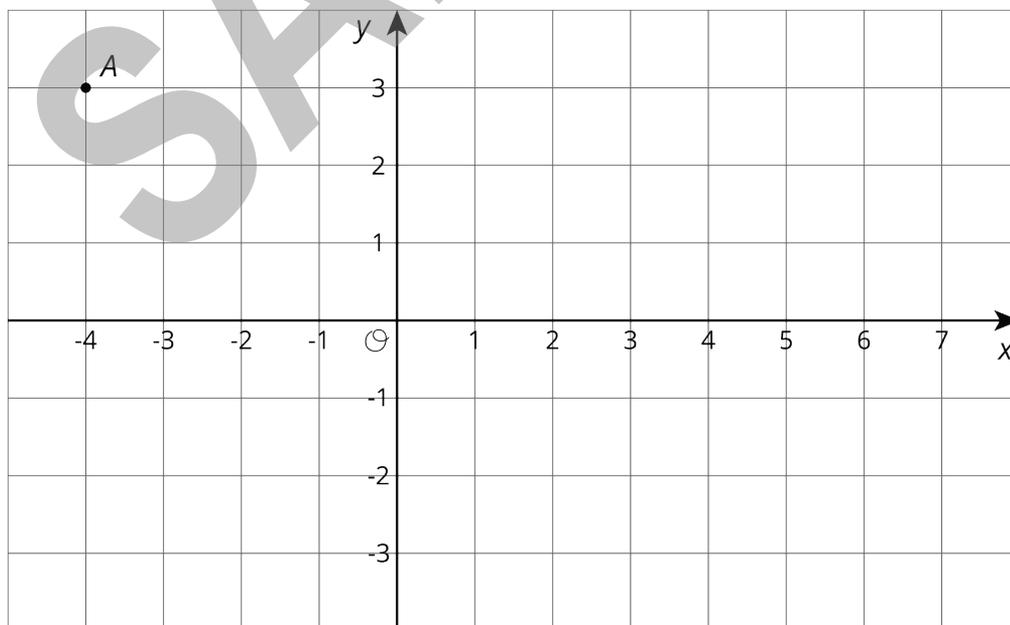
8.G.A.1.a, 8.G.A.1.b

Problem 6

In order to apply rigid transformations to polygons and other shapes, students practice applying them to single points. Lines of reflection are restricted to the x - and y -axes, while rotations are about the origin and are multiples of 90° .

Statement

Point A is located at coordinates $(-4, 3)$.



What are the coordinates of each point?

1. Point B is the image of A after a rotation of 180° using $(0, 0)$ as center.
2. Point C is the image of A after a translation two units to the right, then a reflection using the x -axis.
3. Point D is the image of A after a reflection using the y -axis, then a translation two units to the right.

Solution

1. $B = (4, -3)$
2. $C = (-2, -3)$
3. $D = (6, 3)$

Aligned Standards

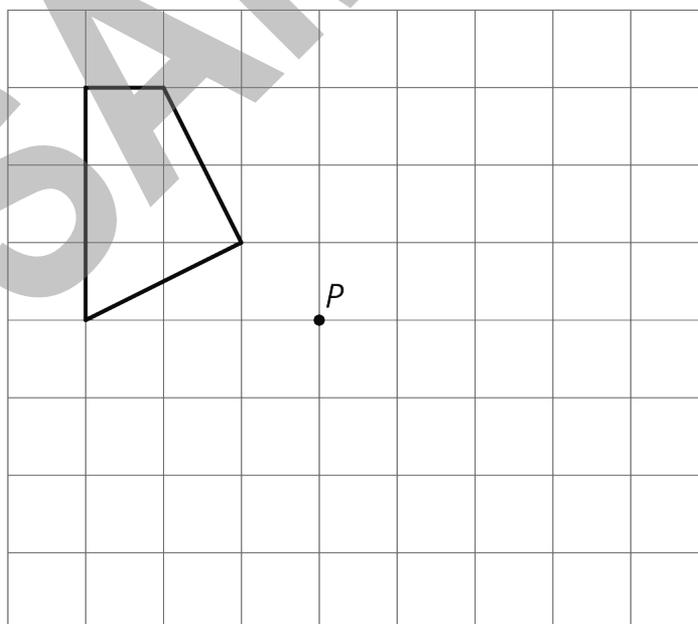
8.G.A.3

Problem 7

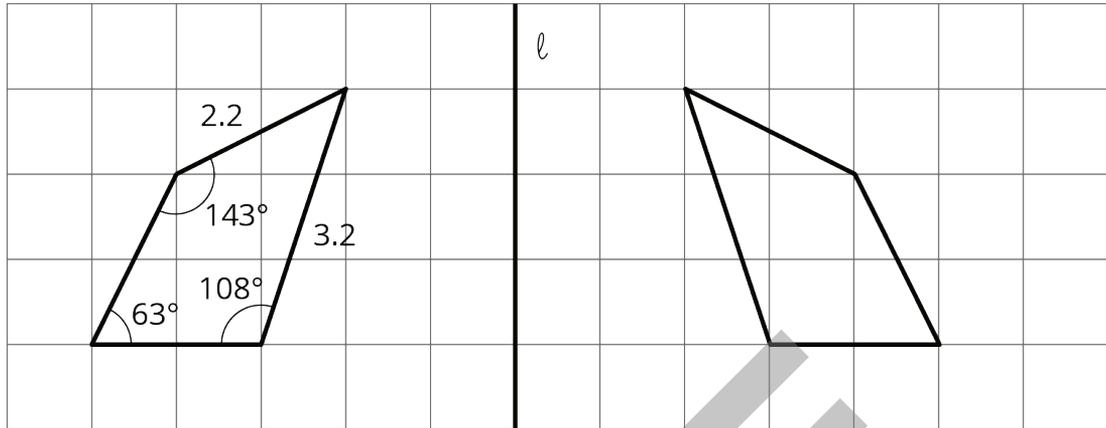
A figure is given on a grid. Students apply a rigid transformation to the shape, drawing the final result. Students also identify and label corresponding parts in an image given information on the original figure.

Statement

1. Draw the image of this figure under a 90° clockwise rotation using center P .

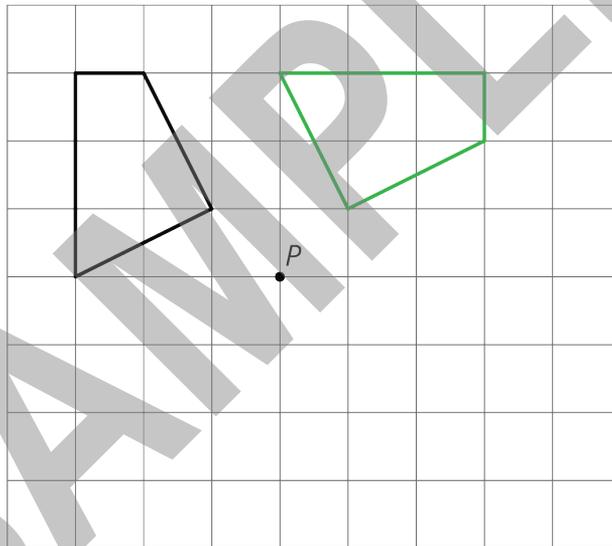


2. The figure on the left is reflected using line ℓ to form the image on the right. Use the information in the original figure to label the corresponding parts in the image.

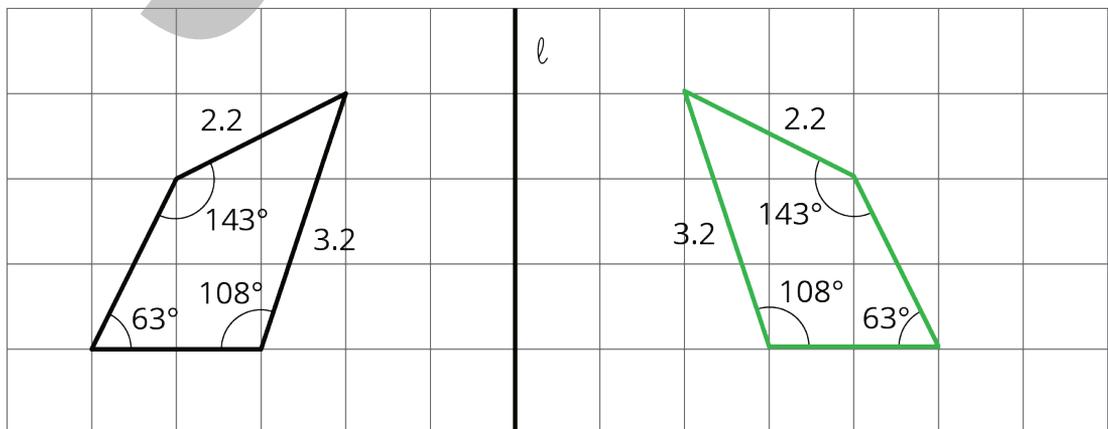


Solution

1.



2.



Minimal Tier 1 response:

- Work is complete and correct, with complete explanation or justification.
- Sample:
 1. See diagram. The image is drawn correctly, with all vertices in correct locations, and all edges drawn.
 2. See diagram.

Tier 2 response:

- Work shows good conceptual understanding and mastery, with either minor errors or correct work with insufficient explanation or justification.
- Sample errors: one point in the image for part a is incorrectly located; the image in part a is drawn with a 90° counterclockwise rotation; one angle or length in part b is incorrect.

Tier 3 response:

- Work shows a developing but incomplete conceptual understanding, with significant errors.
- Sample errors: more than one point in part a is incorrectly located; the image in part a is drawn with a 180° rotation or other incorrect angle of rotation; more than one angle or length in part b is incorrect: this includes the case where all angles and lengths have been drawn in as if the image were a rotation or if it were the original shape, without reflection.

Tier 4 response:

- Work includes major errors or omissions that demonstrate a lack of conceptual understanding and mastery.
- Sample errors: major errors or omissions in one problem part along with errors in the other; two or more error types from Tier 3 response.

Aligned Standards

8.G.A.1

Assessment : End-of-Unit Assessment

Teacher Instructions

Calculators are optional but not required. Provide access to geometry toolkit. In particular, students need access to a protractor, ruler, and tracing paper for this assessment.

Student Instructions

You may use a calculator. A protractor and ruler are required for this assessment.

Problem 1

This problem calls upon students to use their understanding of the meaning of congruence and apply it in an abstract situation in which they must think carefully about the taxonomy of quadrilaterals.

Students selecting C or D may be thinking that all rhombuses (or all parallelograms) have the same shape: they may be envisioning one of the pattern blocks, for instance, and forget that different rhombuses can have different angles. Students failing to select A or B may be forgetting that all squares and rectangles must have four right angles. Students selecting E may not have taken into consideration parallelograms and rhombuses are quadrilaterals that may not have the same shape.

Statement

Select **all** the true statements.

- A. Two squares with the same side lengths are always congruent.
- B. Two rectangles with the same side lengths are always congruent.
- C. Two rhombuses with the same side lengths are always congruent.
- D. Two parallelograms with the same side lengths are always congruent.
- E. Two quadrilaterals with the same side lengths are always congruent.

Solution

["A", "B"]

Aligned Standards

8.G.A.2

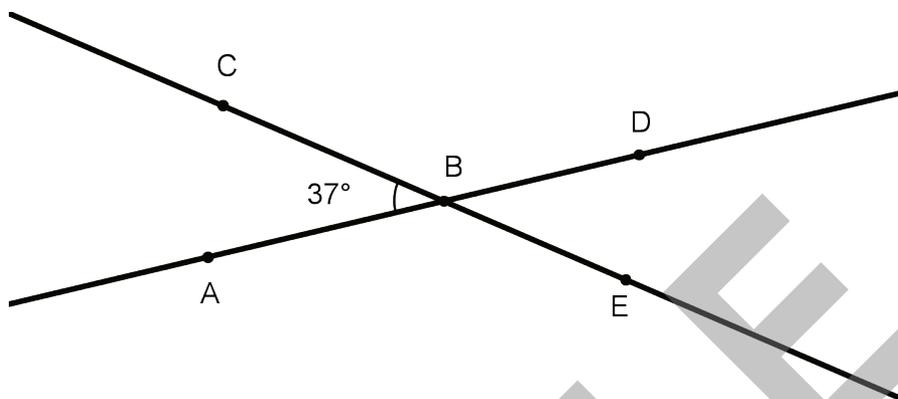
Problem 2

Students identify pairs of angles in a diagram whose measures are equal and whose measures sum to 180 degrees. Students failing to select A or C may not recognize vertical angles in a diagram, or may not understand that the measures of vertical angles are equal. Students selecting B may believe angles CBA and DBE must be supplementary rather than have equal measure. Students failing to select D may not recognize a linear pair in a diagram, or may not understand that these

angles are supplementary. Students selecting E may believe angles CBA and DBE must be complementary rather than have equal measure.

Statement

Lines CE and AD intersect at B .



Select **all** the true statements.

- A. The measure of angle CBA is equal to the measure of angle DBE .
- B. The sum of the measures of angles CBA and DBE is 180 degrees.
- C. The measure of angle CBD is equal to the measure of angle ABE .
- D. The sum of the measures of angles CBD and CBA is 180 degrees.
- E. The sum of the measures of angles CBA and DBE is 90 degrees.

Solution

["A", "C", "D"]

Aligned Standards

8.G.A.1

Problem 3

Students selecting A may need more work on approximate angle measures; the angles are much too large. Students failing to select B, or students selecting C, need a review of the triangle inequality: 4 cm, 5 cm, and 6 cm are fine, and 15 cm is far too long. Students failing to select D or E may have said so because they aren't given the specific locations of the sides and angles, but this is a reason for more than one triangle to exist with the given conditions.

Statement

Select **all** the conditions for which it is possible to construct a triangle.

- A. A triangle with angle measures 60° , 80° , and 80°
- B. A triangle with side lengths 4 cm, 5 cm, and 6 cm
- C. A triangle with side lengths 4 cm, 5 cm, and 15 cm
- D. A triangle with side lengths 4 cm and 5 cm and a 50° angle
- E. A triangle with angle measures 30° and 60° , and a 3 cm side length

Solution

["B", "D", "E"]

Aligned Standards

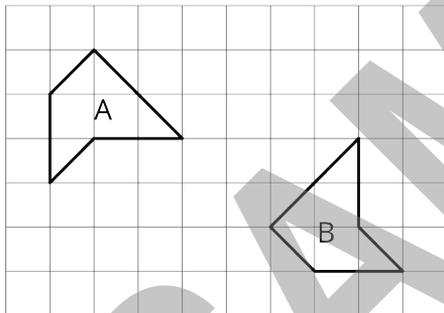
7.G.A.2

Problem 4

Students show multistep congruence on a grid.

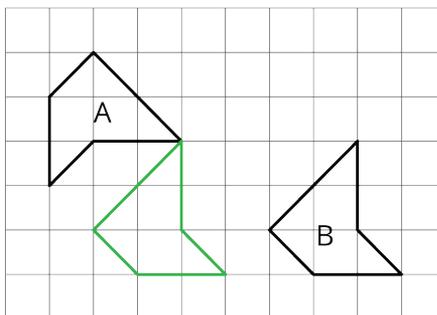
Statement

Describe a sequence of transformations that shows that Polygon A is congruent to Polygon B.



Solution

Answers vary. Sample solution: Polygon A can be rotated 90 degrees counterclockwise around the point shown in the picture and then translated 4 units to the right.



Minimal Tier 1 response:

- Work is complete and correct.

- Sample: Rotate 90 degrees counterclockwise around the rightmost point of Polygon A, translate 4 right.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: a drawing showing the intermediate transformation (the green polygon in the sample response), but no verbal descriptions; incomplete verbal descriptions such as reference to a rotation without specifying a center point; the sequence of transformations contains a small, easily identifiable error (such as saying to rotate clockwise when the counterclockwise direction is the one that works); sequence of transformations is correct but does not use proper vocabulary (“turn” instead of rotate; “move” instead of translate).

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: explanation without reference to rigid transformations, or a sequence of transformations that does not take Polygon A to Polygon B (with no obvious small mistakes responsible for this error); transformations are so poorly described that the intended meaning is not clear.

Aligned Standards

8.G.A.2

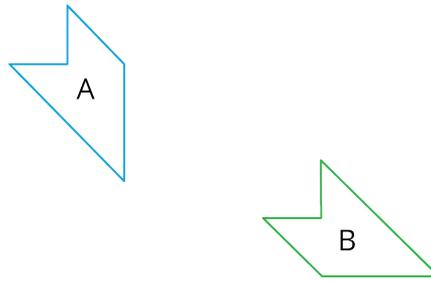
Problem 5

Students determine if two shapes are congruent without the use of a grid. Tracing paper would be useful for this task. The description of the transformations when there is a congruence do not have the same precision as a description aided by a grid. That is, students may talk about translating to the left rather than specifying the exact distance on a grid. Similarly, they may talk about a vertical or horizontal reflection or a rotation without necessarily drawing the line of reflection or providing the measure of the angle of rotation.

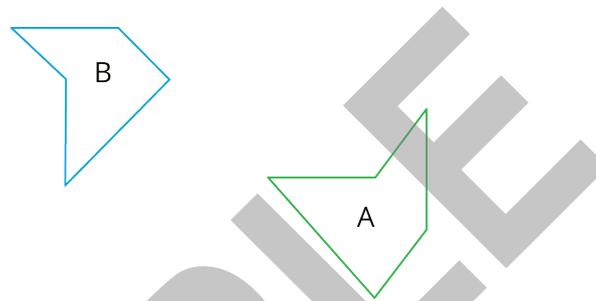
Statement

For each pair of shapes, decide whether or not Shape A is congruent to Shape B. Explain your reasoning.

1. First pair:



2. Second pair:



Solution

1. Congruent. If Shape A is reflected over its right side, then rotated 90 degrees counterclockwise around the lower vertex, it can be placed on top of Shape B with a translation down and to the right.
2. *Not* congruent. The shapes look congruent, but when Shape A is moved on top of Shape B with a 90-degree counterclockwise rotation and a translation, they do not match up.

Minimal Tier 1 response:

- Work is complete and correct.
- Acceptable errors: omitting reference to lines of reflection, centers of rotation, angles of rotation, and distance of translation, provided the visual makes these things clear.

• Sample:

1. (with accompanying accurate drawing) Congruent, because I can reflect Shape A, rotate it, and then translate it onto Shape B.
2. (with accompanying accurate drawing) Not congruent, because when I rotate Shape A and then translate it, it still doesn't match up with Shape B. Alternate response: I measured the angles, and they are not the same in the two shapes.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.

- Sample errors: transformations are shown, but with no written descriptions; in part b, transformations are done mostly correctly but enough accuracy was lost that the shapes appear to coincide.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: work states that shapes are or are not congruent with no justification; transformations are so poorly described that the intended meaning is not clear; vague explanations such as “the shapes look the same.”

Aligned Standards

8.G.A.2

Problem 6

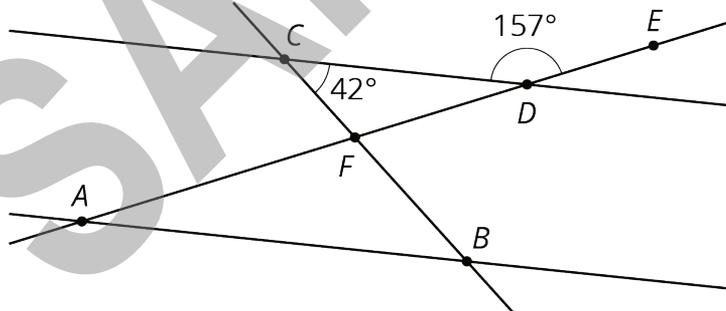
Students can use the following to calculate angles:

- Supplementary angles add to 180° .
- Alternate interior angles made by parallel lines cut by a transverse are congruent.
- The three angles of a triangle add to 180° .

Here they are asked to use this information to find the angles of a triangle.

Statement

Lines AB and CD are parallel. Find the measures of the three angles in triangle ABF .



Solution

$B : 42^\circ, A : 23^\circ, F : 115^\circ$

Aligned Standards

8.G.A.5

Problem 7

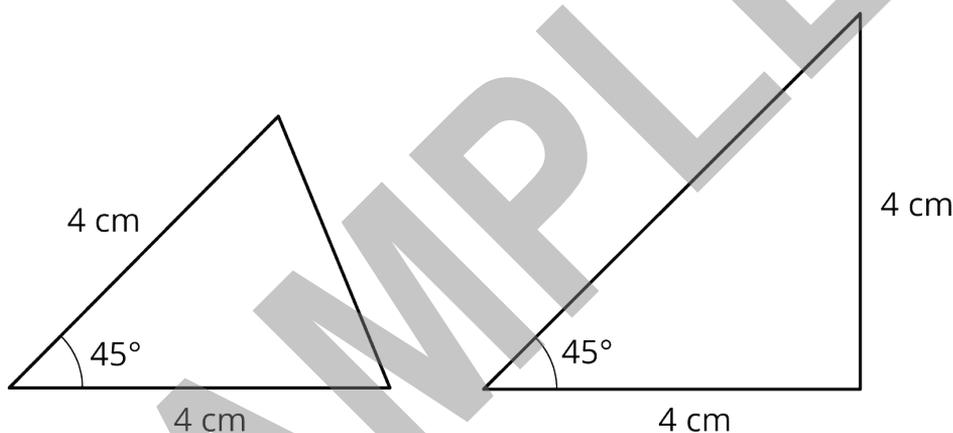
Some students may accidentally draw a third triangle (with the 45° angle in a third position); they should attempt to verify that it is the same as one of the others (generally, it should be another of the triangles, flipped over). The given information forces 2 possible triangles and not 3 because the two given side lengths are equal.

Statement

Draw as many different triangles as possible that have two sides of length 4 cm and a 45° angle. Clearly mark the side lengths and angles given.

Solution

There are 2 different triangles.



Minimal Tier 1 response:

- Work is complete and correct.
- Sample: exactly 2 triangles are drawn; lengths and angles marked and reasonably accurate; no other lengths or angles are marked.
- Acceptable errors: other lengths and angles, besides the ones given, may be marked with reasonable approximations of their measures; sum of marked measures of three angles close to, but not equal to, 180 degrees.

Tier 2 response:

- Work shows general conceptual understanding and mastery, with some errors.
- Sample errors: 3 triangles instead of 2; 45° angle drawn with significant inaccuracy; sides of length 4 cm drawn with significant inaccuracy, notably if they are significantly different in length; other lengths and angles, besides the ones given, measured or marked with significant inaccuracy.

Tier 3 response:

- Significant errors in work demonstrate lack of conceptual understanding or mastery.
- Sample errors: 0 triangles, 1 triangle, or more than 3 triangles drawn; two or more error types from Tier 2 response; an explanation of why the triangle cannot be drawn.

Aligned Standards

7.G.A.2

SAMPLE

Section: Rigid Transformations

Lesson 1: Moving in the Plane

Goals

- Describe (orally and in writing) a translation or rotation of a shape using informal language, e.g., “slide,” “turn left,” etc.
- Identify angles and rays that do not belong in a group and justify (orally) why the object does not belong.

Learning Targets

- I can describe how a figure moves and turns to get from one position to another.

Lesson Narrative

The purpose of this lesson is to introduce students to translations and rotations of plane figures and to have them describe these movements in everyday language. Expect students to use words like “slide” and “turn.” In the next lesson, they will be introduced to the mathematical terms. The term “transformation” is not yet used and will be introduced later in a later lesson.

In all of the lessons in this unit, students should have access to their geometry toolkits, which should contain tracing paper, graph paper, colored pencils, scissors, ruler, protractor, and an index card. For this unit, access to tracing paper and a straight edge are particularly important. Students may not need all (or even any) of these tools to solve a particular problem. However, to make strategic choices about when to use which tools (MP5), students need to have opportunities to make those choices. Apps and simulations should supplement rather than replace physical tools.

Alignments

Building On

- 4.MD.C.5: Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

Building Towards

- 8.G.A.1: Verify experimentally the properties of rotations, reflections, and translations:

Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share
- Which One Doesn't Belong?

Required Materials

Copies of blackline master

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Required Preparation

You will need the Triangle Square Dance blackline master for this lesson. Make 1 copy of all 3 pages for every 2 students.

Assemble geometry toolkits. It would be best if students had access to these toolkits at all times throughout the unit. Toolkits include tracing paper, graph paper, colored pencils, scissors, ruler, protractor, and an index card to use as a straightedge or to mark right angles. Access to tracing paper is particularly important in this unit. Tracing paper cut to a small-ish size (roughly 5" by 5") is best—commercially available "patty paper" is ideal for this. If using larger sheets of tracing paper, such as 8.5" by 11", cut each sheet into fourths.

Student Learning Goals

Let's describe ways figures can move in the plane.

1.1 Which One Doesn't Belong: Diagrams

Warm Up: 10 minutes

This warm-up prompts students to compare four images. It encourages students to explain their reasoning and hold mathematical conversations. It gives you the opportunity to hear how they use terminology and talk about characteristics of the images in comparison to one another. To allow all students to access the activity, each image has one obvious reason it does not belong. Encourage students to find reasons based on mathematical properties (e.g., Figure B is the only right angle). During the discussion, listen for important ideas and terminology that will be helpful in upcoming work of the unit. The activity also gives students an opportunity to find useful tools in their geometry toolkit.

Before students begin, consider establishing a small, discreet hand signal students can display to indicate they have an answer they can support with reasoning. This signal could be a thumbs up, or students could show the number of fingers that indicate the number of responses they have for the problem. This is a quick way to see if students have had enough time to think about the problem and keeps them from being distracted or rushed by hands being raised around the class.

As students share their responses, listen for important ideas and terminology that will be helpful in upcoming work of the unit, such as reference to angles and their measures.

Building On

- 4.MD.C.5

Instructional Routines

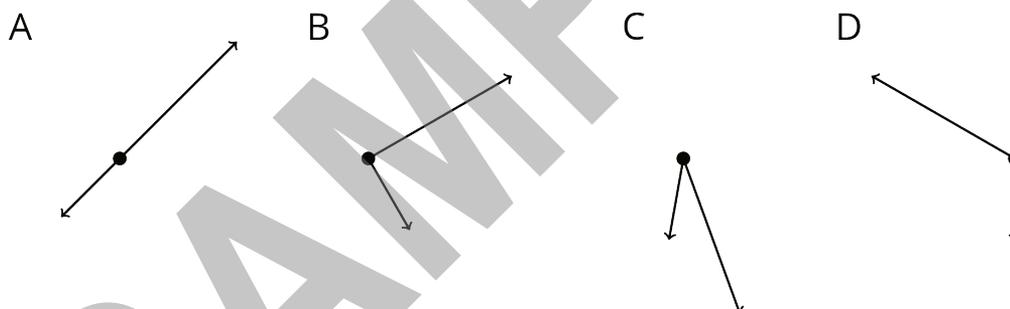
- Think Pair Share
- Which One Doesn't Belong?

Launch

Arrange students in groups of 2–4, and provide access to geometry toolkits. Display the figures for all to see. Ask students to indicate when they have noticed one that does not belong and can explain why. Give students 1 minute of quiet think time and then time to share their thinking with their small group. In their small groups, tell each student to share their reasoning why a particular question does not belong. Together, find at least one reason each question doesn't belong.

Student Task Statement

Which one doesn't belong?



Student Response

Answers vary. Sample responses:

A doesn't belong because:

- The rays point in opposite directions.
- It is not possible to make a triangle by joining points on the rays.

B doesn't belong because:

- They make a right angle.
- Both rays are to the right of the vertex.

C doesn't belong because:

- It is an acute angle.

- Both rays point downward.

D doesn't belong because:

- It is an obtuse angle (measuring less than 180 degrees).
- The long ray points to the left of the short ray.

Activity Synthesis

Ask each group to share one reason why a particular image does not belong. Record and display the responses for all to see. After each response, ask the class if they agree or disagree. Since there is no single correct answer to the question of which one does not belong, attend to students' explanations and ensure the reasons given are correct.

During the discussion, prompt students to explain the meaning of any terminology they use, such as ray, degree, or acute angle. Also, press students on unsubstantiated claims. For example, a student may make claims about the angle measures. Ask how they know the measure and demonstrate how the tracing paper or the ruler from the toolkit could be used to check.

1.2 Triangle Square Dance

25 minutes (there is a digital version of this activity)

The purpose of this activity is for students to begin to observe and describe translations and rotations. In groups of 2, they describe one of 3 possible dances, presented in cartoon form, and the partner identifies which dance is being described. Identify students who use specific and detailed language to describe the dance and select them to share during class discussion.

While students are not expected to use precise language yet, this activity both provides the intellectual need for agreeing upon common language and give students a chance to experiment with different ways of describing some moves in the plane (MP6).

Building Towards

- 8.G.A.1

Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students in groups of 2, and give a copy of all 3 blackline masters to each group. Explain that each sheet is a cartoon with 6 frames showing the moves made by the dancing figures. Instruct students to place all three sheets face up, and tell them to take turns selecting a dance and describing it to their partner, without revealing which dance they have selected. The other student identifies which dance is being described. On a display, record language students use to describe the movement of shapes to later be grouped and connected to more formal language such as

“rotation,” and “translation.” Give students 15 minutes to work in their groups followed by a whole-class discussion.

If using the digital activity, ask students to close their devices. Distribute the blackline masters and review the rules of the game to make sure students understand the task. Give students around 10 minutes to play the game. After 10 minutes, invite students to open their devices and notice how the applets correspond to the three dances. Give students an additional 5 minutes to come to consensus about how to best describe the moves in their own words before a whole-class discussion.

Support for Students with Disabilities

Representation: Provide Access for Perception. Display or provide students with a physical copy of the written directions and read them aloud. Check for understanding by inviting students to rephrase directions in their own words. Consider keeping the display of directions visible throughout the activity.

Supports accessibility for: Language; Memory

Anticipated Misconceptions

Some students may interpret directions like “left” and “right” different from how their partner intended it, depending on whether they are thinking from the point of view of an observer watching the dance or putting themselves in the dance and describing things in terms of the triangle’s left, right, up, and down. Watch for miscommunications like this, point out that neither perspective is wrong, and encourage students to be more precise in their language.

Students often confuse or are unsure about the meaning of the terms clockwise and counterclockwise. Discuss with them (and demonstrate, if possible) how the hands on a clock rotate, emphasizing the direction of the rotation. Students may also be unsure of how to describe the amount of rotation. Consider asking a student who expresses angle measures in terms of degrees to explain how they see it.

Student Task Statement

Your teacher will give you three pictures. Each shows a different set of dance moves.

1. Arrange the three pictures so you and your partner can both see them right way up. Choose who will start the game.
 - The starting player mentally chooses A, B, or C and describes the dance to the other player.
 - The other player identifies which dance is being talked about: A, B, or C.
2. After one round, trade roles. When you have described all three dances, come to an agreement on the words you use to describe the moves in each dance.

3. With your partner, write a description of the moves in each dance.

Student Response

Answers vary. Sample response:

A: Move right, turn 90° clockwise, move up, move left, and turn 90° counterclockwise.

B: Move right, turn 90° clockwise, move left, move up, and turn 90° counterclockwise.

C: Move right, turn 90° counterclockwise, move left, move up, and turn 90° clockwise.

The terms left, right, and up in this answer are from the point of view of an observer watching the dance. Alternatively students might put themselves in the shoes of the triangles and describe things in terms of the triangle's left, right, up, and down. Students might use other words, such as "shift" and "step" for translations, and "spin" and "rotate" for the turns. They might describe the 90° turns as "quarter turns."

Are You Ready for More?

We could think of each dance as a new dance by running it in reverse, starting in the 6th frame and working backwards to the first.

1. Pick a dance and describe in words one of these reversed dances.
2. How do the directions for running your dance in the forward direction and the reverse direction compare?

Student Response

1. Answers vary. Sample response:

A: turn 90° clockwise, move right, move down, turn 90° counterclockwise, move left, B: turn 90° clockwise, move down, move right, turn 90° counterclockwise, move left, C: turn 90° counterclockwise, move down, move right, turn 90° clockwise, move left

2. The steps are listed in reverse order. Right gets replaced by left and left with right and clockwise gets replaced with counterclockwise and vice versa.

Activity Synthesis

Select one student to share their description for each of the pages, and display the language you observed and recorded during the activity for the different types of moves. Arrange the words in two groups, those that describe translations and those that describe rotations (but do not use these terms). Come to agreement on a word for each type, and discuss what extra words are needed to specify the transformation exactly (e.g., move right, turn clockwise 90°).

Consider asking students what they found most challenging about describing the dances: expected responses include being as precise as possible about the different motions (for example, describing whether the shape is rotating clockwise or counterclockwise). Also consider asking students if they were sometimes able to identify the dance before their partner finished describing *all* of the moves.

All three dances begin by moving to the right, but in the second step, Dances A and B rotate 90 degrees clockwise while Dance C rotates 90 degrees counterclockwise. (So if the second move was to rotate 90 degrees counterclockwise, this must be Dance C.) Dances A and B diverge at slide 4.

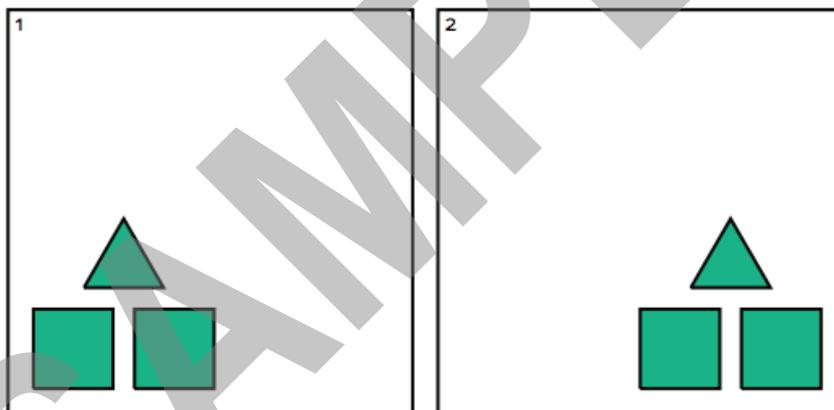
Support for English Language Learners

Representing: MLR8 Discussion Supports. To support student understanding of the language on the display, invite students to "act out" each of the different types of moves. Include diagrams or pictures on the display to provide students with a visual reminder of the meaning of each term. For example, draw arrows to help illustrate direction. Remind students to borrow language from the display as needed.

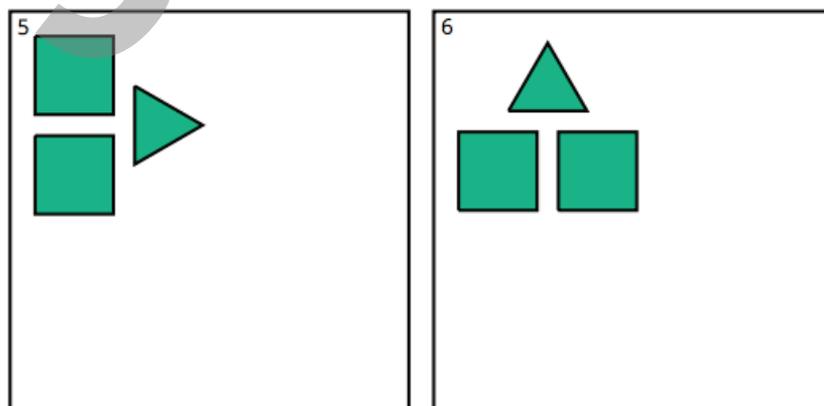
Design Principle(s): Support sense-making

Lesson Synthesis

We have started to reason about what it means to move a figure in the plane. Display two figures that clearly show a slide and two figures that clearly show a turn. Example of a slide:



Example of a turn:



Invite students to share the language they would use to describe them: for example “moving” or “sliding” for translations and “turning” for rotations. Consider asking students how they might *quantify* each move, for example with a distance and direction for the slides and a center and angle of rotation for the turns. Tell them that we will continue to look at these moves in more detail in future lessons.

1.3 Frame to Frame

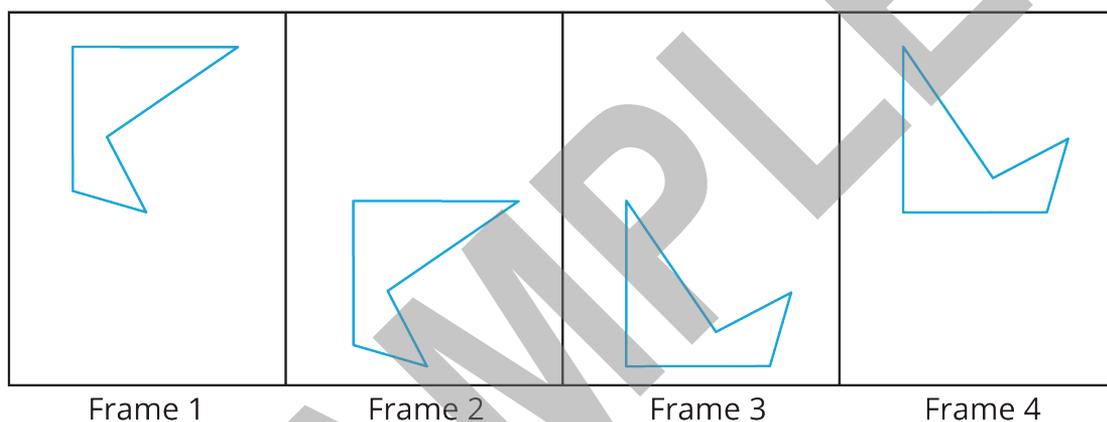
Cool Down: 5 minutes

Building Towards

- 8.G.A.1

Student Task Statement

Here are successive positions of a shape:



Describe how the shape moves from:

1. Frame 1 to Frame 2.
2. Frame 2 to Frame 3.
3. Frame 3 to Frame 4.

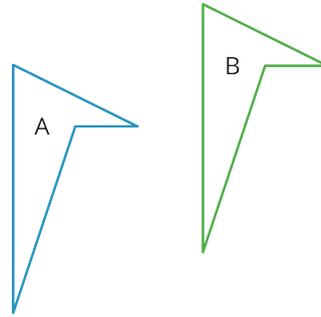
Student Response

1. Slide down.
2. Turn counterclockwise 90 degrees (or one quarter of a full turn).
3. Slide up.

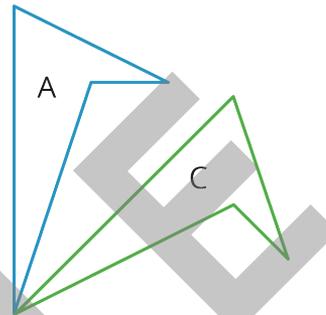
Student Lesson Summary

Here are two ways for changing the position of a figure in a plane without changing its shape or size:

- Sliding or shifting the figure without turning it. Shifting Figure A to the right and up puts it in the position of Figure B.



- Turning or rotating the figure around a point. Figure A is rotated around the bottom vertex to create Figure C.



Glossary

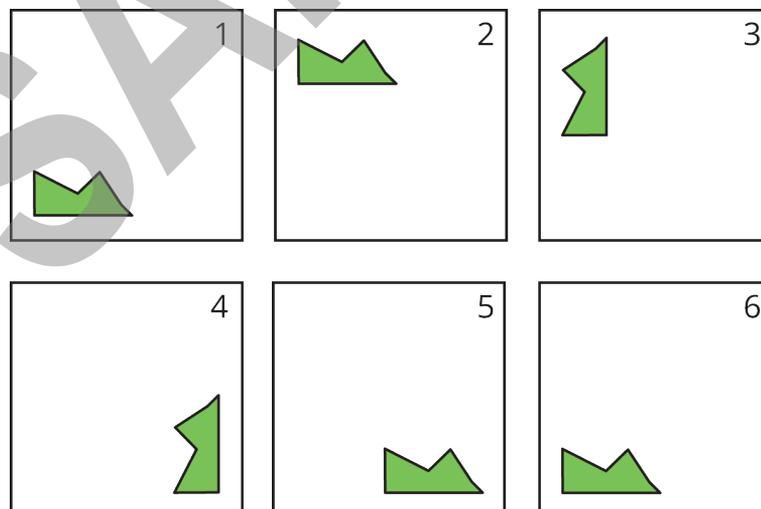
- vertex

Lesson 1 Practice Problems

Problem 1

Statement

The six frames show a shape's different positions.



Describe how the shape moves to get from its position in each frame to the next.

Solution

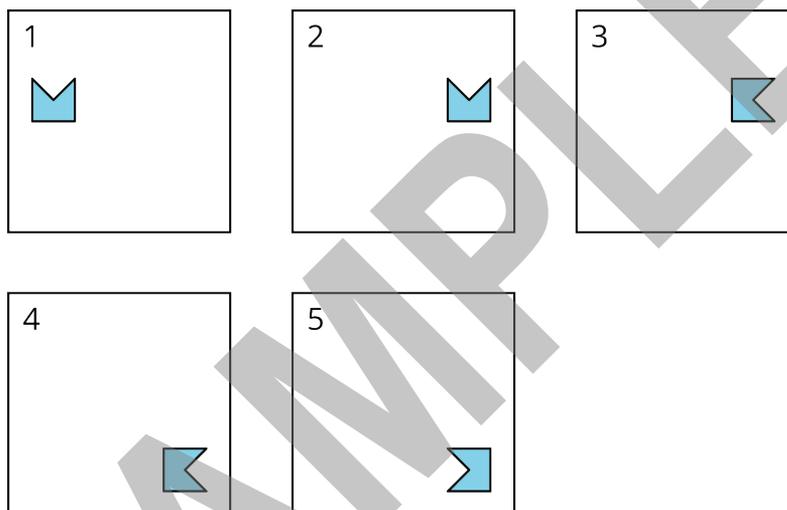
To get from Position 1 to Position 2, the shape moves up. To get from Position 2 to Position 3, the shape rotates 90 degrees counterclockwise. To get from Position 3 to Position 4, the shape moves down and to the right. To get from Position 4 to Position 5 the shape rotates 90 degrees clockwise. To get from Position 5 to Position 6, the shape moves to the left.

Note: 90 degrees counterclockwise is the same as 270 degrees clockwise, and similarly 90 degrees clockwise is the same as 270 degrees counterclockwise.

Problem 2

Statement

These five frames show a shape's different positions.



Describe how the shape moves to get from its position in each frame to the next.

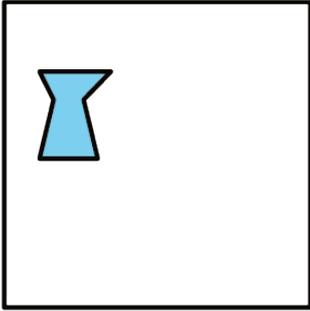
Solution

To get from Position 1 to Position 2, the shape moves to the right. To get from Position 2 to Position 3, the shape rotates 90 degrees clockwise. To get from Position 3 to Position 4, the shape moves down. To get from Position 4 to Position 5, the shape rotates 180 degrees.

Problem 3

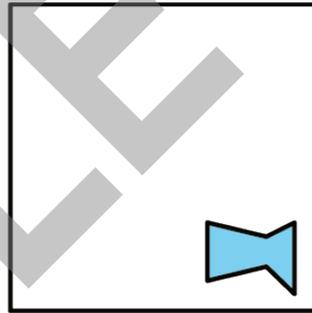
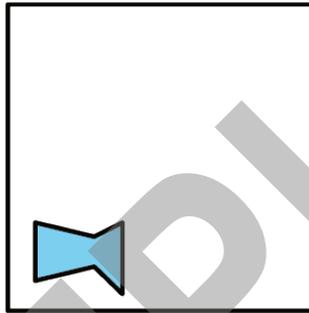
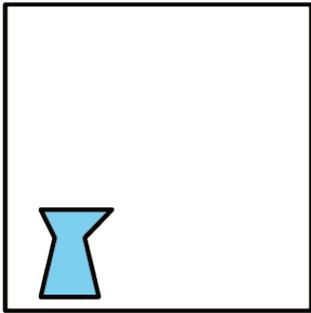
Statement

Diego started with this shape.



Diego moves the shape down, turns it 90 degrees clockwise, then moves the shape to the right. Draw the location of the shape after each move.

Solution



Lesson 2: Naming the Moves

Goals

- Describe (orally and in writing) the movement of shapes informally and formally using the terms “clockwise,” “counterclockwise,” “translations,” “rotations,” and “reflections” of figures.

Learning Targets

- I can identify corresponding points before and after a transformation.
- I know the difference between translations, rotations, and reflections.

Lesson Narrative

In this lesson, students begin to describe a given translation, rotation, or reflection with greater precision and are introduced to the terms **translation**, **rotation**, and **reflection**. The collective terms “transformation” and “rigid transformation” are not used until later lessons. Students are introduced to the terms **clockwise** and **counterclockwise**. Students then use this language to identify the individual moves on various figures.

Students engage in MP6 as they experiment with ways to describe moves precisely enough for another to understand their meaning.

Alignments

Building On

- 4.MD.C.5: Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

Addressing

- 8.G.A.1: Verify experimentally the properties of rotations, reflections, and translations:

Instructional Routines

- MLR8: Discussion Supports
- Take Turns
- Think Pair Share

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Pre-printed cards, cut from copies of the blackline master

Required Preparation

Print and cut up cards from the Translations, Rotations, and Reflections blackline master. Prepare 1 copy for every 3 students.

Make sure students have access to items in their geometry toolkits: tracing paper, graph paper, colored pencils, scissors, ruler, protractor, and an index card to use as a straightedge or to mark right angles.

Access to tracing paper is particularly important. Each student will need about 10 small sheets of tracing paper (commercially available "patty paper" is ideal). If using large sheets of tracing paper, such as 8.5 inches by 11 inches, cut each sheet into fourths.

Student Learning Goals

Let's be more precise about describing moves of figures in the plane.

2.1 A Pair of Quadrilaterals

Warm Up: 10 minutes

Students estimate an angle of rotation. While they do not need to use a protractor, a protractor is an ideal tool and allows them to estimate the angle measure more accurately. Monitor for how students report the measure of the angle: do they round to the nearest degree, to the nearest 5 degrees?

Building On

- 4.MD.C.5

Addressing

- 8.G.A.1

Instructional Routines

- Think Pair Share

Launch

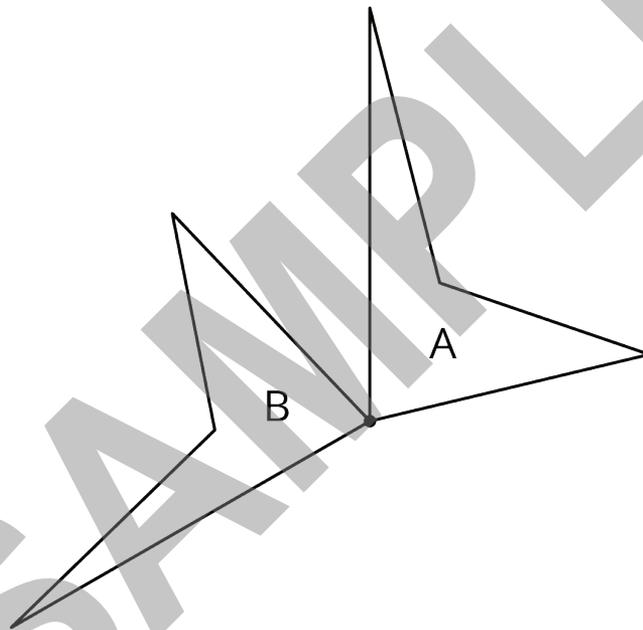
Arrange students in groups of 2–4. Provide access to geometry toolkits. Display the two quadrilateral figures for all to see. (They should also look at the task statement in their workbooks.) Ask students to give a discreet hand signal when they have an estimate for the angle of rotation. Give students 2 minutes of quiet think time and then time to share their thinking with their group before a whole-class discussion.

Anticipated Misconceptions

Students may not be sure which angle to measure. They may measure the acute angle between Shape A and Shape B. Ask these students to trace Shape A on tracing paper and rotate it by that angle to see that this does not give Shape B.

Student Task Statement

Quadrilateral A can be rotated into the position of Quadrilateral B.

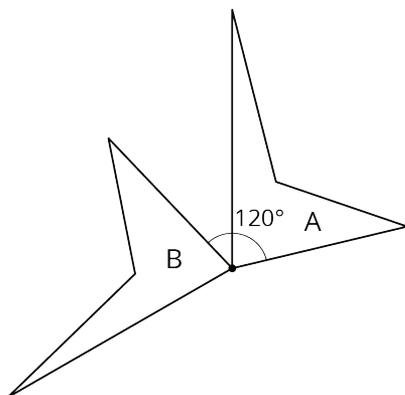


Estimate the angle of rotation.

Student Response

Answers vary. Sample response: About 120 degrees (counterclockwise)

This figure doesn't need to be part of students' responses but is provided as an example of an angle between two segments that could be measured to find the angle of rotation from A to B.



Activity Synthesis

Invite students to share their estimates for the angle of rotation. Ask students how they knew, for example, that the angle is *more* than 90 degrees (because the angle is obtuse) but *less* than 180 degrees (because the angle is less than a straight line).

Introduce or reiterate the language of **clockwise** (for rotating in the direction the hands on a clock move) and **counterclockwise** (for rotating in the opposite direction). In this case, the direction of rotation is not specified but it is natural to view Figure A being rotated counterclockwise onto Figure B. Make sure to introduce the language of the *center* of rotation (the vertex shared by A and B is the center of rotation).

It may be helpful to display the picture from the task statement to support this discussion, and if possible, show the 120° counterclockwise turn dynamically.

2.2 How Did You Make That Move?

10 minutes

This activity informally introduces reflections, which appear in addition to some translations and rotations (that were introduced informally in the previous lesson). Students are given a 6-frame cartoon showing the change in position of a polygon. As in the previous lesson, they describe the moves, but this time there are reflections, which may seem impossible as physical moves unless you allow the shape to leave the plane. Students identify the new moves and try to describe them.

After the end of this activity, the three basic moves have been introduced and the next activity will introduce their names (translations, rotations, and reflections).

Addressing

- 8.G.A.1

Instructional Routines

- Think Pair Share

Launch

Keep students in the same groups, and maintain access to geometry toolkits. Give students 5 minutes of quiet work time, and then invite them to share their responses with their group. Follow with a whole-class discussion. Tell students that they will be describing moves as they did in the previous lesson, but this time there is a new move to look out for. Recall the words the class used to describe slides and turns.

Support for Students with Disabilities

Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. During the launch, take time to review terms that students will need to access for this activity. Invite students to suggest language or diagrams to include that will support their understanding of moving, sliding, and turning.

Supports accessibility for: Conceptual processing; Language

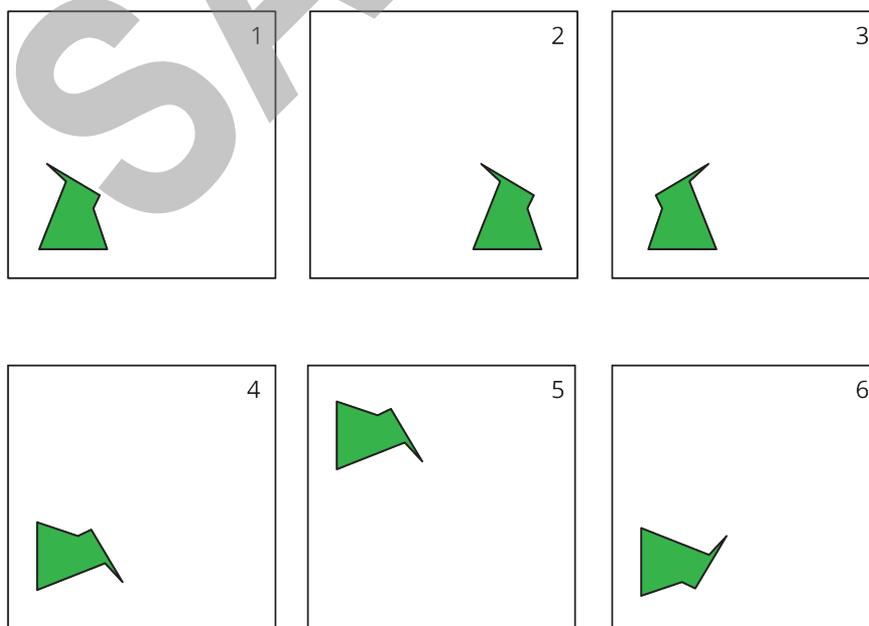
Anticipated Misconceptions

Students may see a reflection as a translation especially since the figures are not on the same frame. Ask these students to trace Frame 2 on tracing paper. Is there any way to turn it into Frame 3 by sliding it? What do they have to do to turn it into Frame 3? (They have to flip the tracing paper over, so, this is a new kind of move.)

In describing reflections, students may confuse the terms horizontal and vertical. Consider posting the terms horizontal and vertical with examples in your room.

Student Task Statement

Here is another set of dance moves.



1. Describe each move or say if it is a new move.

- a. Frame 1 to Frame 2.
- b. Frame 2 to Frame 3.
- c. Frame 3 to Frame 4.
- d. Frame 4 to Frame 5.
- e. Frame 5 to Frame 6.

2. How would you describe the new move?

Student Response

Answers vary. Sample response:

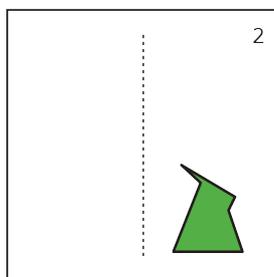
1.
 - a. Frame 1 to Frame 2: Shift to the right
 - b. Frame 2 to Frame 3: New move
 - c. Frame 3 to Frame 4: Turn 90° clockwise
 - d. Frame 4 to Frame 5: Shift up
 - e. Frame 5 to Frame 6: New move
2. The new move is like becoming your mirror image through a mirror placed at the center of the frame. For the second move, the mirror is vertical and for the last move it is horizontal.

Activity Synthesis

The purpose of this discussion is an initial understanding that there is a third type of move that is fundamentally different from the moves encountered in the previous lesson, because it reverses directions. Some possible discussion questions to help them identify these are:

- “How is the motion from panel 2 to panel 3 different than the ones we discussed yesterday?”
- “Is there anywhere else that happens in this cartoon?”
- “What features of the image help us to see that this move is happening?”

To help answer these questions, tell students to pay attention to the direction that the “beak” of the polygon is pointing, left or right. Draw a dotted vertical line in the middle of Frame 2, and say, “Here is a mirror. The polygon in Frame 3 is what the polygon in Frame 2 sees when it looks in the mirror.”



Demonstrate using tracing paper or transparencies to show they are mirror images. Then ask students if there are any other mirror lines in other frames. For the second reflection, from Frame 5 to Frame 6, point out that again left and right is reversed: In Frame 5 the head is on the right of the body, and in Frame 6 it is on the left, with the beak pointing down in both cases. Contrast this with a rotation through 180° , which would leave the beak pointing up. Demonstrate with tracing paper or transparencies.

2.3 Card Sort: Move

15 minutes (there is a digital version of this activity)

The purpose of this card sort activity is to give students further practice identifying translations, rotations, and reflections, and in the discussion after they have completed the task, introduce those terms. In groups of 3 they sort 9 cards into categories. There are 3 translations, 3 rotations, and 3 reflections. Students explain their categories and come to agreement on them.

On the blackline master, there are actually 12 cards. The last three show slightly more complicated moves than the first 9. These can be withheld, at first, and used if time permits.

Students might identify only 2 categories, putting the reflections with the translations (in the case of Card 3) or the rotations (in the case of Card 5). As students work, monitor for groups who have sorted the cards into translations, rotations, and reflections (though not necessarily using those words). Also monitor for descriptions of corresponding points such as “these points go together” or “here are before and after points.”

Teacher Notes for IM 6–8 Math Accelerated

When distributing the cards to groups, include all 12 cards instead of reserving the last 3.

Addressing

- 8.G.A.1

Instructional Routines

- MLR8: Discussion Supports
- Take Turns

Launch

Arrange students into groups of 3, and provide access to geometry toolkits. Give each group the first 9 cards. Reserve the last 3 cards for use if time permits.

Tell students that their job is to sort the cards into categories by the type of move that they show. After they come to consensus about which categories to use, they take turns placing a card into a category and explaining why they think their card goes in that category. When it is not their turn, their job is to listen to their partner's reasoning and make sure they understand. Consider conducting a short demonstration with a student of productive ways to communicate during this activity. For example, show what it looks like to take turns, explain your thinking, and listen to your partner's thinking.

Give students about 10 minutes to sort the cards. Do *not* explicitly instruct students at the beginning to use the words translations, rotations, and reflections. Monitor for a group who uses these categories, even if they use different names for them. If time permits, distribute the remaining 3 cards. Follow with whole-class discussion.

If using the digital activity, ask the students to close their devices, at first. After they have come to agreement about how their cards should be sorted, they can open their devices and use the applets to help them refine the way they describe the moves.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Display sentence frames to support students when they explain their strategy. For example, "This card belongs in ____ category because . . ." or "I noticed that this image ____ so I . . ."

Supports accessibility for: Language; Social-emotional skills

Anticipated Misconceptions

Students may struggle to differentiate between the three moves, confusing reflections with either translations or rotations. After they make their best decision, encourage these students to use tracing paper to justify their response. In Card 10, students may be confused when the translated figure overlaps the original. For Card 4, students may first think that this is a rotation (much like Cards 6 and 9). Encourage these students here to use tracing paper to check their answers.

Student Task Statement

Your teacher will give you a set of cards. Sort the cards into categories according to the type of move they show. Be prepared to describe each category and why it is different from the others.

Student Response

Translations: 1, 7, 8, 10

Rotations: 2, 6, 9, 12

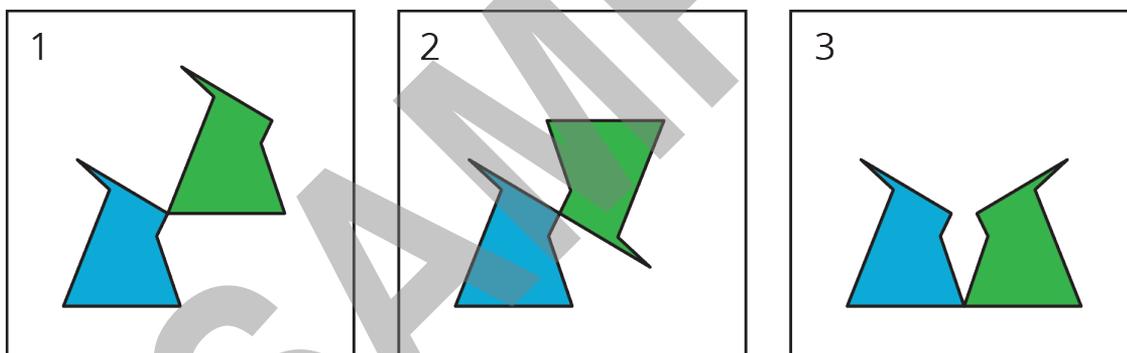
Reflections: 3, 4, 5, 11

To detect if one figure is a translation of another, look to see if it is still sitting in exactly the same way, e.g., the two figures have the same orientation and are sitting on the same base. To detect if one figure is a rotation of another, look to see if one figure is not standing up in exactly the same way as the other but appears to be turned. Reflections can be confused with both translations (if the two figures are still on the same base) and rotations (if they appear to be turned). The way to detect a reflection in these examples is to choose a feature of the figure that exists on one side of it but not the other (e.g., the sharp “rabbit ears” in this activity) and see if it is pointing to the left in one figure and to the right in the other. (Alternatively, up and down if the line of reflection is horizontal.)

Activity Synthesis

Select one or more groups to share the names of their categories. Select one or more groups to share how they sorted the cards into the categories. Ask the class if they disagree with any of the choices, and give students opportunities to justify their reasoning (MP3).

Introduce the terms **translation**, **rotation**, and **reflection**. It may be helpful to display an example of each to facilitate discussion:



Alternatively, you may wish to display the geogebra applets used in the digital version of the student materials to facilitate discussion:

- Translation: ggbm.at/wYYvZH7A
- Rotation: ggbm.at/RUtdpQmN
- Reflection: ggbm.at/nKQmSnDW

Point out ways to identify which type of move it is. Translations are a slide with no turning. Rotations are a turn. Reflections face the opposite direction. If desired, introduce the terms *image* and *corresponding points*. If we see the figures as rabbits, then the ear tips in the original figure and the ear tips in its image are *corresponding points*, for example. The *image* is the figure after a transformation is applied: for each of the cards, one figure is the image of the other figure after a translation, rotation, or reflection has been applied.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to support the introduction of new terms. As groups share how they categorized and sorted the shapes, revoice their ideas using the terms translation, rotation, reflection, image, and corresponding points. Some students may benefit from practicing words or phrases or words in context through choral repetition.

Design Principle(s): Optimize output (for explanation)

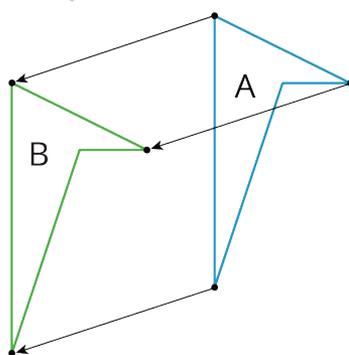
Lesson Synthesis

Questions for discussion:

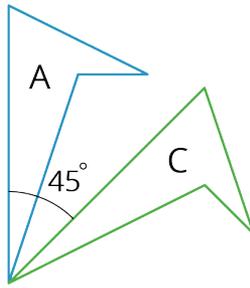
- “We encountered a new type of move that was different from yesterday. What can you tell me about it?” (It’s like a mirror image, you can’t make the move by sliding or turning, the figure faces the opposite direction.)
- “We gave mathematical names to the three types of moves we have seen. What are they called?” (The “slide” is called a translation, the “turn” is called a rotation, and the mirror image is called a reflection.)

Consider creating a semi-permanent display that shows these three terms and their definitions for reference throughout the unit.

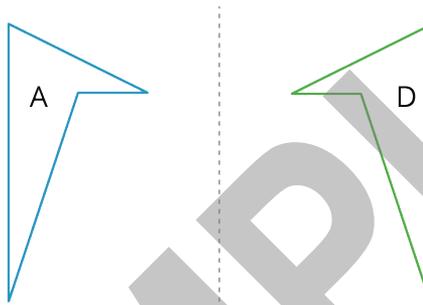
A **translation** slides a figure without turning it. Every point in the figure goes the same distance in the same direction. For example, Figure A was translated down and to the left, as shown by the arrows. Figure B is a translation of Figure A.



A **rotation** turns a figure about a point, called the center of the rotation. Every point on the figure goes in a circle around the center and makes the same angle. The rotation can be **clockwise**, going in the same direction as the hands of a clock, or **counterclockwise**, going in the other direction. For example, Figure A was rotated 45° clockwise around its bottom vertex. Figure C is a rotation of Figure A.



A **reflection** places points on the opposite side of a reflection line. The mirror image is a backwards copy of the original figure. The reflection line shows where the mirror should stand. For example, Figure A was reflected across the dotted line. Figure D is a reflection of Figure A.



2.4 Is It a Reflection?

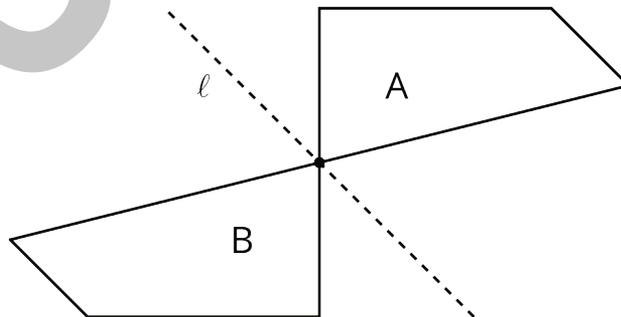
Cool Down: 5 minutes

Addressing

- 8.G.A.1

Student Task Statement

What type of move takes Figure A to Figure B?



Explain your reasoning.

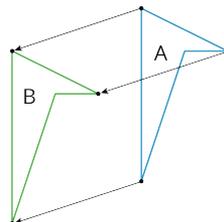
Student Response

Answers vary. Sample response: It is a rotation. If Figure A is turned around the point shared by Figures A and B, it can land on Figure B.

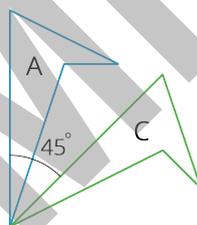
Student Lesson Summary

Here are the moves we have learned about so far:

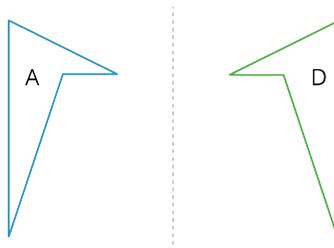
- A **translation** slides a figure without turning it. Every point in the figure goes the same distance in the same direction. For example, Figure A was translated down and to the left, as shown by the arrows. Figure B is a translation of Figure A.



- A **rotation** turns a figure about a point, called the center of the rotation. Every point on the figure goes in a circle around the center and makes the same angle. The rotation can be **clockwise**, going in the same direction as the hands of a clock, or **counterclockwise**, going in the other direction. For example, Figure A was rotated 45° clockwise around its bottom vertex. Figure C is a rotation of Figure A.



- A **reflection** places points on the opposite side of a reflection line. The mirror image is a backwards copy of the original figure. The reflection line shows where the mirror should stand. For example, Figure A was reflected across the dotted line. Figure D is a reflection of Figure A.



We use the word *image* to describe the new figure created by moving the original figure. If one point on the original figure moves to another point on the new figure, we call them *corresponding* points.

Glossary

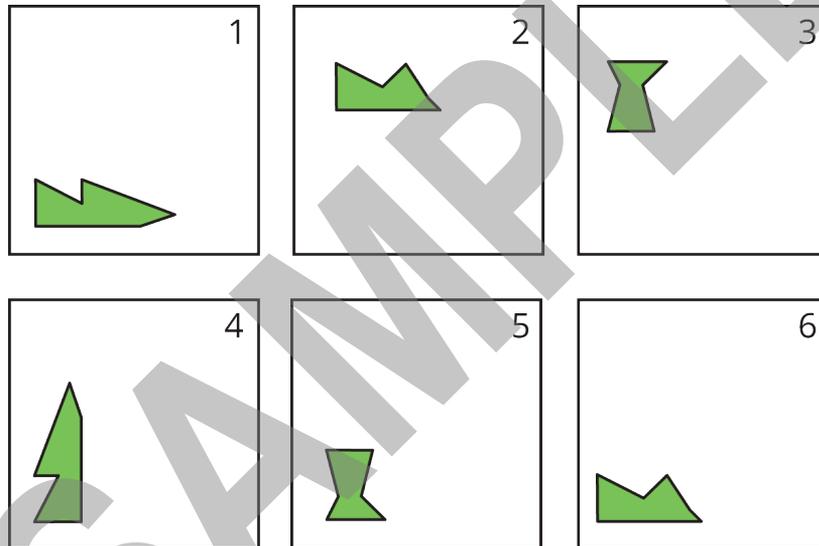
- clockwise
- counterclockwise
- reflection
- rotation
- translation

Lesson 2 Practice Problems

Problem 1

Statement

Each of the six cards shows a shape.



- Which pair of cards shows a shape and its image after a rotation?
- Which pair of cards shows a shape and its image after a reflection?

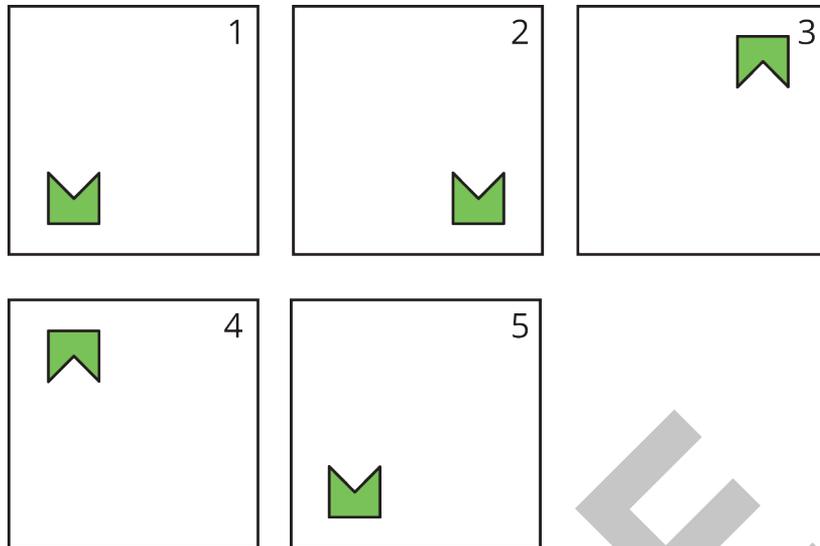
Solution

- Cards 1 and 4
- Cards 3 and 5

Problem 2

Statement

The five frames show a shape's different positions.



Describe how the shape moves to get from its position in each frame to the next.

Solution

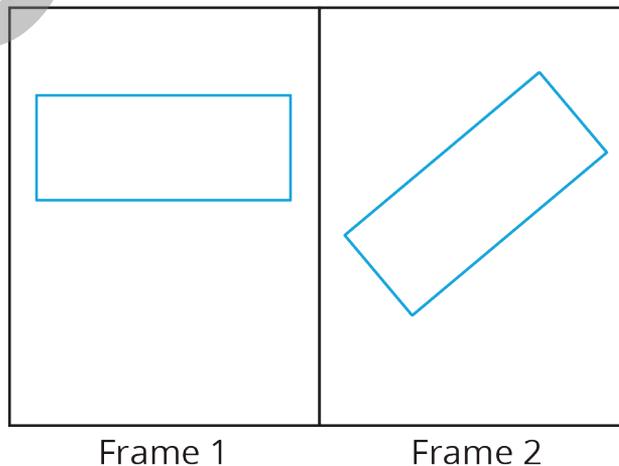
To get from Position 1 to Position 2, the shape moves to the right. To get from Position 2 to Position 3, the shape flips over a horizontal line. To get from Position 3 to Position 4, the shape moves to the left. To get from Position 4 to Position 5, the shape flips over a horizontal line again. The shape has then returned to its original position in Position 1.

Alternatively, to get from Position 1 to Position 2 or from Position 3 to Position 4, the shape may flip over a vertical line. Since the shape is symmetric, a flip looks the same as a shift here. To get from Position 2 to Position 3 or from Position 4 to Position 5, the shape may be rotated 180 degrees about a point not on the polygon.

Problem 3

Statement

The rectangle seen in Frame 1 is rotated to a new position, seen in Frame 2.



Select **all** the ways the rectangle could have been rotated to get from Frame 1 to Frame 2.

- A. 40 degrees clockwise
- B. 40 degrees counterclockwise
- C. 90 degrees clockwise
- D. 90 degrees counterclockwise
- E. 140 degrees clockwise
- F. 140 degrees counterclockwise

Solution

["B", "E"]

(From Unit 1, Lesson 1.)

SAMPLE

Lesson 3: Making the Moves

Goals

- Comprehend that a “transformation” is a translation, rotation, reflection, or a combination of these.
- Draw the “image” of a figure that results from a translation, rotation, and reflection in square and isometric grids and justify (orally) that the image is a transformation of the original figure.
- Explain (orally) the “sequence of transformations” that “takes” one figure to its image.
- Identify (orally and in writing) the features that determine a translation, rotation, or reflection.

Learning Targets

- I can use grids to carry out transformations of figures.
- I can use the terms translation, rotation, and reflection to precisely describe transformations.

Lesson Narrative

Prior to this lesson, students have learned the names for the basic moves (translation, rotation, and reflection) and have learned how to identify them in pictures. In this lesson, they apply these moves to figures and understand that:

- A translation is determined by two points that specify the distance and direction of the translation.
- A rotation is determined by a point, angle of rotation, and a direction of rotation.
- A reflection is determined by a line.

These moves are called **transformations** for the first time and students draw images of figures under these transformations. They also study where shapes go under sequences of these transformations and identify the steps in a **sequence of transformations** that takes one figure to another. Note the subtle shift in language. In the previous lesson, one shape “moves” to the other shape—it is as if the original shape has agency and does the moving. In this lesson, the transformation “takes” one shape to the other shape—this language choice centers the transformation itself as an object of study. Also, in this lesson students label the image of a point P as P' . While not essential, this practice helps show the structural relationship (MP7) between a figure and its image.

Students practice performing translations, rotations, and reflections both on a square grid and on an isometric grid (one made of equilateral triangles with 6 meeting at each vertex) in this lesson for the first time. Expect a variety of approaches, mainly making use of tracing paper (MP5) but students may also begin to notice how the structure of the different grids helps draw images resulting from certain moves (MP7).

Students using the print version may make use of tracing paper to experiment moving shapes. Students using the digital version have access to Geogebra applets with which to perform transformations. This is the lesson where students learn to use the transformation tools in Geogebra, that will help them select appropriate tools in their future work (MP5). Students are also likely starting to begin thinking strategically about which transformations will take one figure to another, identifying properties of the shapes that indicate whether a translation, rotation, reflection, or sequence of these will achieve this goal (MP7).

Alignments

Building On

- 7.G.A: Draw, construct, and describe geometrical figures and describe the relationships between them.

Addressing

- 8.G.A.1: Verify experimentally the properties of rotations, reflections, and translations:

Building Towards

- 8.G.A.1: Verify experimentally the properties of rotations, reflections, and translations:

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect
- MLR8: Discussion Supports
- Notice and Wonder

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Required Preparation

Make sure students have access to items in their geometry toolkits: tracing paper, graph paper, colored pencils, scissors, ruler, protractor, and an index card to use as a straightedge or to mark right angles.

For classrooms using the print version of the materials: Access to tracing paper is particularly important. Each student will need about 10 small sheets of tracing paper (commercially available

"patty paper" is ideal). If using large sheets of tracing paper, such as 8.5 inches by 11 inches, cut each sheet into fourths.

For classrooms using the digital version of the materials: If you have access to extra help from a tech-savvy person, this would be a good day to request their presence in your class.

Student Learning Goals

Let's draw and describe translations, rotations, and reflections.

3.1 Notice and Wonder: The Isometric Grid

Warm Up: 10 minutes

The purpose of this warm-up is to familiarize students with an isometric grid. While students may notice and wonder many things, characteristics such as the measures of the angles in the grid and the diagonal parallel lines will be important properties for students to notice in their future work performing transformations on the isometric grid. Students are not expected to know each angle in an equilateral triangle is 60 degrees, but after previous experience with supplementary angles, circles and rotations, they may be able to explain why each smaller angle is 60 degrees. Many things they notice may be in comparison to the square grid paper which is likely more familiar.

Teacher Notes for IM 6–8 Math Accelerated

Adjust the timing of this activity to 5 minutes.

Building On

- 7.G.A

Building Towards

- 8.G.A.1

Instructional Routines

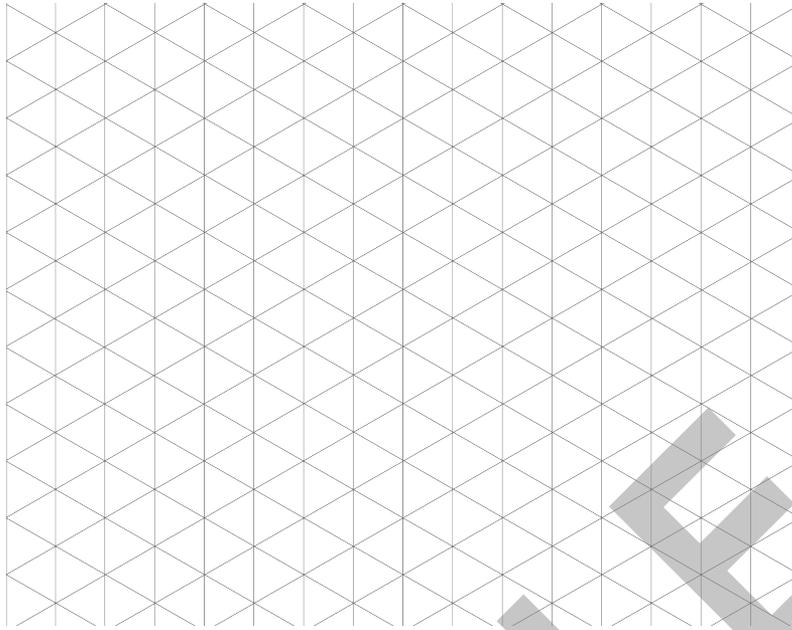
- Notice and Wonder

Launch

Arrange students in groups of 2. Tell students that they will look at an image. Their job is to think of at least one thing they notice and at least one thing they wonder. Display the image for all to see. Ask students to give a signal when they have noticed or wondered about something. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner, followed by a whole-class discussion.

Student Task Statement

What do you notice? What do you wonder?



Student Response

Things students may notice:

- There are three sets of parallel grid lines.
- The line segments form equilateral triangles.
- The individual angles in the equilateral triangles are 60 degrees.
- There are vertical lines but no horizontal lines.
- There are no 90 degree angles made by the grid lines.
- Each vertex has 6 line segments coming from it.
- The grid is made out of equilateral triangles instead of squares.

Things students may wonder:

- Why are there no 90 degree angles?
- Why are there no squares?
- Are we going to use this kind of grid?
- Why would we use this grid instead of the square grid?
- Why are there no horizontal lines?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image, and show where each of the features students notice is located on the actual grid itself, such as triangles, angles, and line segments. After each response, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to the images each time. If angle measures do not come up during the conversation, ask them to think about how they could figure out the measure of each angle. Some may measure with a protractor, and some may argue that since 6 angles share a vertex where each angle is identical, each angle measures 60° because $360 \div 6 = 60$. Establish that each angle measures 60° .

3.2 Transformation Information

25 minutes (there is a digital version of this activity)

The purpose of this activity is for students to interpret the information needed to perform a transformation and draw an image resulting from the transformation.

For digital classrooms, an additional purpose of this activity is for students to learn how to use the transformation tools available in geogebra. These tools will be used throughout the unit. As they become familiar with the dynamic tools, they see that geogebra places the image of a figure based on the instructions given by the user.

Through hands-on experience with transformations, students prepare for the more precise definitions they will learn in later grades. This activity is the first time students start to use A' , B' , etc. to denote points in the image that correspond to A , B , etc. in the original figure. This is also a good activity to use the word "image" to describe the transformed figure—this can happen before, as, or after students work.

If students exploit the mathematical properties of the grid lines to draw transformed figures, they are making use of structure (MP7). In order to draw the transformed figures correctly, students must attend to the details of the given information (MP6).

Watch for students who use tracing paper and those who use properties of the grids to help decide where to place the transformed figures. Tracing paper may be particularly useful for the isometric grid which may be unfamiliar to some students.

Teacher Notes for IM 6–8 Math Accelerated

Adjust the timing of this activity to 15 minutes. Arrange students in groups of 2. For each set of figures (1–4 and 5–8), each student should complete 2 transformations while their partner completes the other 2. Partners should share their results and reasoning after completing each set.

Addressing

- 8.G.A.1

Instructional Routines

- MLR8: Discussion Supports

Launch

Point out A' in the first question. Tell students we call point A' "A prime" and that, after a transformation, it corresponds to A in the original.

For students using print materials: Optionally, before students start working, demonstrate the mechanics of performing each type of transformation using tracing paper. Distribute about 10 small sheets of tracing paper to each student (or ensure they can find it in their geometry toolkits). Give students about 10 minutes of quiet work time followed by whole-class discussion.

For students using digital materials: Depending on the needs of your class, either demonstrate how each transformation tool works in the applet, or instruct students to read and follow the instructions for working the applets. It would work well to demonstrate the first, third, and fourth items and allow students to complete the other items independently.

Support for Students with Disabilities

Representation: Internalize Comprehension. Begin with a physical demonstration of using tracing paper to perform each type of transformation to support connections between new situations and prior understandings. Ask students, "What does this demonstration have in common with previous activities where both images were given?" or "How does the point A' correspond to the point A ?"

Supports accessibility for: Conceptual processing; Visual-spatial processing

Support for English Language Learners

Listening, Representing: MLR8 Discussion Supports. To help develop students' meta-awareness and understanding of the task expectations, think aloud as you transform the quadrilateral in the question about rotating quadrilateral $ABCD$ 60° counterclockwise using center B . As you talk, model mathematical language use and highlight the relationship between quadrilateral $ABCD$, the image (i.e., quadrilateral $A'B'C'D'$), and the steps taken to rotate quadrilateral $ABCD$.

Design Principle(s): Maximize meta-awareness

Anticipated Misconceptions

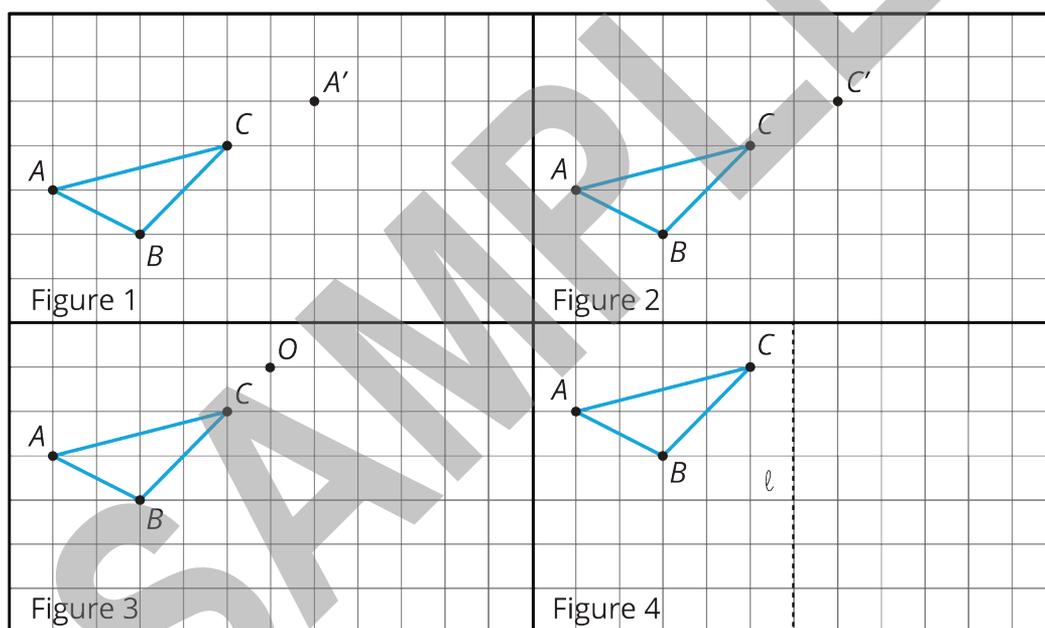
Students may struggle to understand the descriptions of the transformations to carry out. For these students, explain the transformations using the words they used in earlier activities, such as "slide," "turn," and "mirror image" to help them get started. Students may also struggle with reflections that are not over horizontal or vertical lines.

Some students may need to see an actual mirror to understand what reflections do, and the role of the reflection line. If you have access to rectangular plastic mirrors, you may want to have students check their work by placing the mirror along the proposed mirror line.

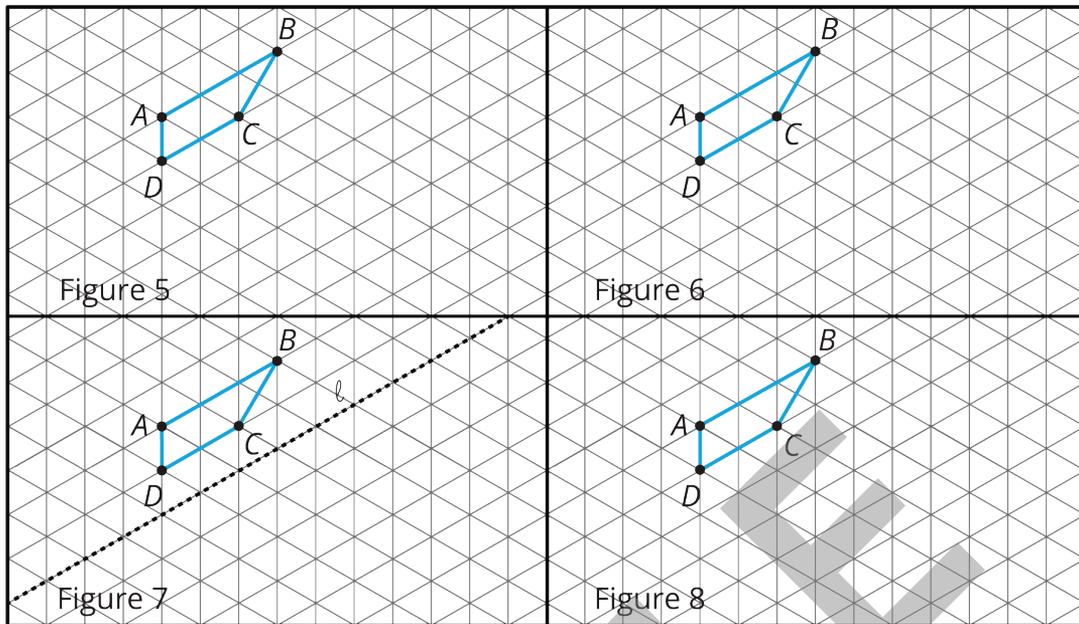
Working with the isometric grid may be challenging, especially rotations and reflections across lines that are not horizontal or vertical. For the rotations, you may want to ask students what they know about the angle measures in an equilateral triangle. For reflections, the approach of using a mirror can work or students can look at individual triangles in the grid, especially those with a side on the line of reflection, and see what happens to them. After checking several triangles, they develop a sense of how these reflections behave.

Student Task Statement

Your teacher will give you tracing paper to carry out the moves specified. Use A' , B' , C' , and D' to indicate vertices in the new figure that correspond to the points A , B , C , and D in the original figure.

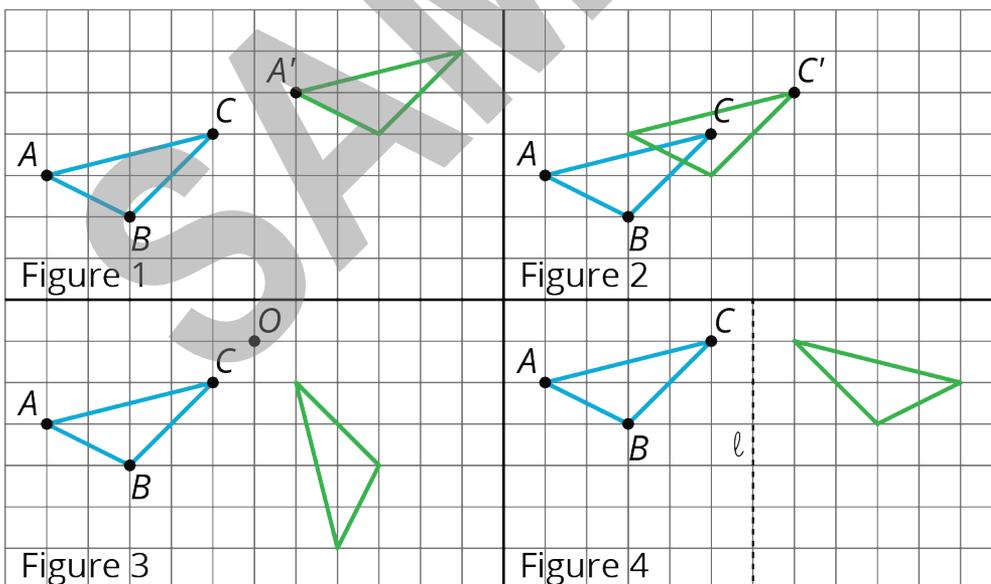


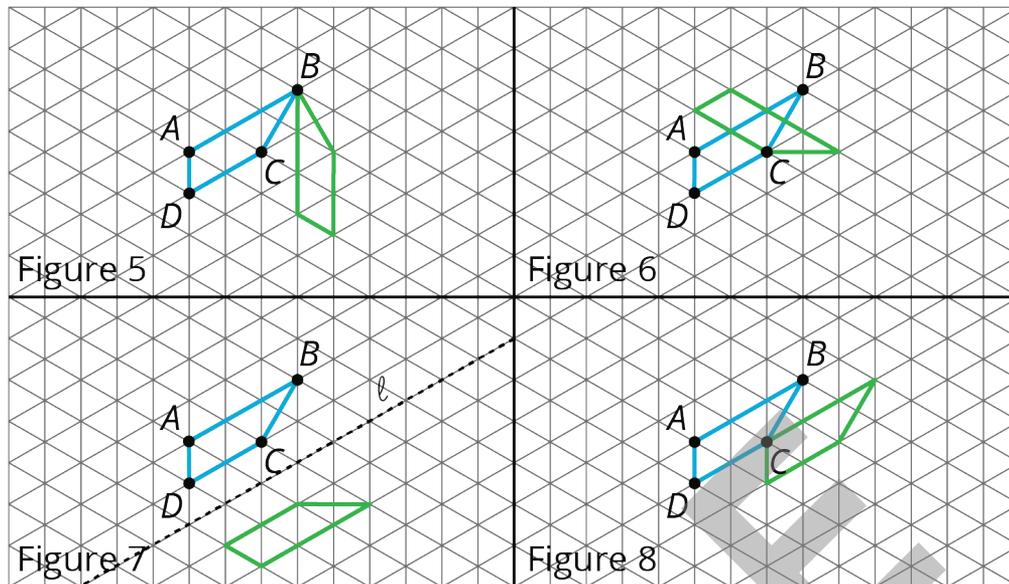
1. In Figure 1, translate triangle ABC so that A goes to A' .
2. In Figure 2, translate triangle ABC so that C goes to C' .
3. In Figure 3, rotate triangle ABC 90° counterclockwise using center O .
4. In Figure 4, reflect triangle ABC using line ℓ .



5. In Figure 5, rotate quadrilateral $ABCD$ 60° counterclockwise using center B .
6. In Figure 6, rotate quadrilateral $ABCD$ 60° clockwise using center C .
7. In Figure 7, reflect quadrilateral $ABCD$ using line ℓ .
8. In Figure 8, translate quadrilateral $ABCD$ so that A goes to C .

Student Response





Sample strategy:

- Trace the figure onto tracing paper, and then move the tracing paper according to the description of each move. Observe where the tracing paper ends up, and draw a copy of the figure at that location.
- Use the structure of the grid to move each vertex of the original figure according to the description of each move. For example in Figure 1, point A moves up 2 and right 6 to A' . A translation is a slide, so each vertex makes this same move along the grid from its original location.

Are You Ready for More?

The effects of each move can be “undone” by using another move. For example, to undo the effect of translating 3 units to the right, we could translate 3 units to the left. What move undoes each of the following moves?

1. Translate 3 units up
2. Translate 1 unit up and 1 unit to the left
3. Rotate 30 degrees clockwise around a point P
4. Reflect across a line ℓ

Student Response

1. Translate 3 units down.
2. Translate 1 unit down and 1 unit to the right.
3. Rotate 30 degrees counterclockwise around P .
4. Reflect again across ℓ .

Activity Synthesis

Ask students to share how they found the images, and highlight the information they needed in each to perform the transformation. Invite students who used tracing paper to share how they found the images and also ask students what mathematical patterns they found. For example, for the reflection in Figure 4, ask where some intersections of grid lines go (they stay on the same horizontal line and go to the other side of ℓ , the same distance away). How can this be used to identify the image of $\triangle ABC$?

Ask students how working on the isometric grid is similar to working on a regular grid and how it is different. Possible responses include:

- Translations work the same way, identifying how far and in which direction to move the shape
- Rotations also work the same way but the isometric grid works well for multiples of 60 degrees (with center at a grid point), while the regular grid works well for multiples of 90 degrees (also with center at a grid point).
- Reflections on the isometric grid require looking carefully at the triangular pattern to place the reflection in the right place. Like for the regular grid, these reflections are difficult to visualize if the line of reflection is not a grid line.

3.3 A to B to C

15 minutes (there is a digital version of this activity)

Students have seen images showing a sequence of transformations in the first lesson of this unit, however they have not heard the term sequence of transformations. They have also not been asked to describe the moves in the sequence using precise language. The launch of this activity introduces this term and gives students an opportunity to describe the sequence of more than one transformation.

For the second problem, encourage students to find a different sequence of transformations than the one shown in the image. Each time a reflection is mentioned, ask students where the line of reflection is located and when a rotation is mentioned, ask for the center of the rotation and the number of degrees. Monitor for students who apply different transformations (or apply transformations in a different order).

Teacher Notes for IM 6–8 Math Accelerated

In the launch, after collecting a few different responses for how to describe the moves that take one bird to another, introduce the word **transformation**: A transformation is a move (such as a reflection, a translation, or a rotation) or a sequence of moves. It's a mathematical word which will be used in place of "move," which has been used up to this point. Then, introduce a **sequence of transformations** as a set of translations, rotations, reflections, and dilations performed in a particular order on a geometric figure, resulting in a final figure.

Addressing

- 8.G.A.1

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect

Launch

Arrange students in groups of 2, and provide access to their geometry toolkits. Display the image for all to see. Ask students if they can imagine a single translation, rotation, or reflection that would take one bird to another? After a minute, verify that this is not possible.

Ask students to describe how we could use translations, rotations, and reflections to take one bird to another. Collect a few different responses. (One way would be to take the bird on the left, translate it up, and then reflect it over a vertical line.) Tell students when we do one or more transformations in a row to take one figure to another, it is called a **sequence of transformations**.



If using the digital activity, you may want to review the transformation tools in the applet. (The instructions are repeated in the activity for students' reference.)

Give students 2 minutes of quiet work time to engage in the task followed by 3 minutes to discuss their responses with a partner and complete any unfinished questions. Follow with a whole-class discussion.

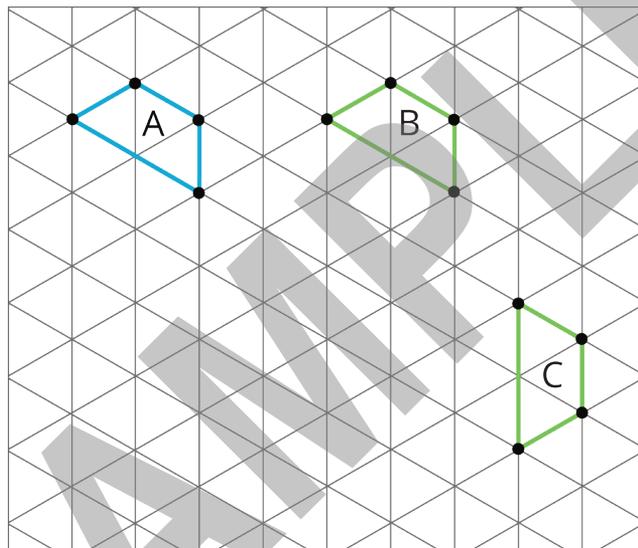
Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as “To take Figure *A* to Figure *B*, I ____ because...”, “I noticed ____ so I...”, “Why did you...?”, or “I agree/disagree because...”

Supports accessibility for: Language; Social-emotional skills

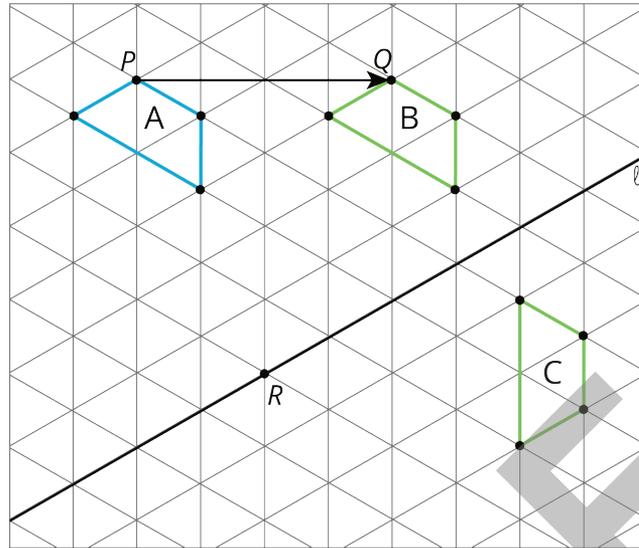
Student Task Statement

Here are some figures on an isometric grid.



1. Name a transformation that takes Figure *A* to Figure *B*. Name a transformation that takes Figure *B* to Figure *C*.
2. What is one **sequence of transformations** that takes Figure *A* to Figure *C*? Explain how you know.

Student Response



1. There are many of ways to describe the translation that takes A to B: Any pair of corresponding points works. In the figure, two corresponding points P and Q are shown. There are two ways to take B to C with a single transformation. One is a reflection with line of reflection ℓ (shown). The other is a rotation 60° clockwise around point R on line ℓ .
2. Answers vary. Sample response: Using the transformations from problem 1, first apply a translation so that A goes to B and then a reflection taking B to C.

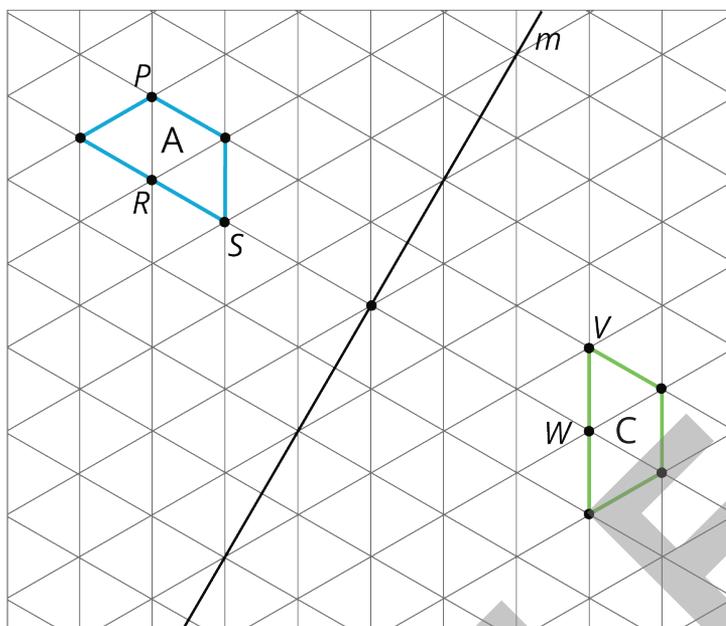
Are You Ready for More?

Experiment with some other ways to take Figure A to Figure C. For example, can you do it with. . .

- No rotations?
- No reflections?
- No translations?

Student Response

Answers vary. Sample response with no reflection: First translate P to S and then rotate 60 degrees clockwise.



Sample response with no rotation: Translate S to V and then reflect over a gridline to make the figures align.

Sample response with no reflection: Translate R to W and then rotate around W to make the figures align.

Sample response with no translation: Reflect across the line m , taking R to W , and then rotate around W to make the figures align.

Activity Synthesis

Select students with different correct responses to show their solutions. Be sure to highlight at least one rotation. If no students mention that, demonstrate a way to take A to C that involves a rotation. Whether or not students use the geogebra applet, it may be helpful to display the applet to facilitate discussion: ggbm.at/jqvTEgsj

- Emphasize that there are many ways to describe the translation that takes figure A to figure B . All one needs is to identify a pair of corresponding points and name them in the correct order (and to use the word “translate”).
- For students who used a reflection to take B to C , emphasize that reflections are determined by lines and we should name the line when we want to communicate about it.
- After a student or the teacher uses a rotation, emphasize that a rotation is defined by a center point and an angle (with a direction). The center point needs to be named and the angle measure or an angle with the correct measure needs to be named as well (as does the direction). Reinforce to students that when we do more than one transformation in a row, we call this a sequence of transformations.

Support for English Language Learners

Speaking: Math Language Routine 7 Compare and Connect. This is the first time Math Language Routine 7 is suggested as a support in this course. In this routine, students are given a problem that can be approached using multiple strategies or representations and are asked to prepare a visual display of their method. Students then engage in investigating the strategies (by means of a teacher-led gallery walk, partner exchange, group presentation, etc.), comparing approaches, and identifying correspondences between different representations. A typical discussion prompt is: “What is the same and what is different?” regarding their own strategy and that of the others. The purpose of this routine is to allow students to make sense of mathematical strategies by identifying, comparing, contrasting, and connecting other approaches to their own, and to develop students’ awareness of the language used through constructive conversations.

Design Principle(s): Maximize meta-awareness: Support sense-making

How It Happens:

1. Use this routine to compare and contrast different strategies for transforming Figure A to Figure C. Invite students to create a visual display showing how they made sense of the problem and why their solution makes sense for transforming Figure A to Figure C.

Students should include these features on their display:

- a sketch of the figures (not necessary if using the applet)
 - a sketch of the figures after each transformation (not necessary if using the applet)
 - a written sequence of transformations with an explanation
2. Before selecting students to show their solutions to the class, first give students an opportunity to do this in a group of 3–4. Ask students to exchange and investigate each other’s work. Allow 1 minute for each display and signal when it is time to switch.

While investigating each other’s work, ask students to consider what is the same and what is different about each approach. Next, give each student the opportunity to add detail to their own display for 1–2 minutes.

3. As groups are presenting, circulate the room and select 2–3 students to share their sequence of transformations taking Figure A to Figure C. Be sure to select a variety of approaches, including one that involves a rotation.

Draw students’ attention to the different ways the figures were transformed (e.g., rotations, reflections, and translations) and how the sequence of transformation is expressed in their explanation. Also, use the bullet points in the Activity Synthesis to emphasize specific features of translations, reflections, and rotations.

-
4. After the selected students have finished sharing with the whole class, lead a discussion comparing, contrasting, and connecting the different approaches.

Consider using these prompts to amplify student language while comparing and contrasting the different approaches: “Why did the approaches lead to the same outcome?”, “What worked well in __’s approach? What did not work well?”, and “What would make __’s strategy more complete or easy to understand?”

Consider using these prompts to amplify student language while connecting the different approaches: “What role does a translation play in each approach?” “Is it possible to use the all three types of transformations?”, and “What transformation do you see present in all the strategies?”

5. Close the discussion by inviting 3 students to revoice the strategies used in the presentations, and then transition back to the Lesson Synthesis and Cool-Down.
-

Lesson Synthesis

The goal for this lesson is for students to begin to identify the features that determine a translation, rotation, or reflection. Refer to the permanent display produced in a previous lesson as you discuss. To highlight the features specific to each type of transformation, consider asking:

- “Describe the two different kinds of grids in this lesson. What is the same and what is different about them?” (A square grid is all right angles and following the grid lines means moving left-right or up-down. This grid makes showing rotations that are multiples of 90 degrees simpler. An isometric grid is made up of equilateral triangles, so following grid lines from an intersection means you can go in 6 different directions instead of just 4. This grid makes showing rotations that are multiples of 60 degrees simpler.)
- “If you want to describe a translation, what important information do you need to include?” (A translation is determined by two points that specify the distance and direction of the translation.)
- “If you want to describe a rotation, what important information do you need to include?” (A rotation is determined by a center point and an angle with a direction.)
- “If you want to describe a reflection, what important information do you need to include?” (A reflection is determined by a line.)
- “What does the word *transformation* mean?” (Translations, rotations, and reflections, or any combination of these, that take a figure to its image.)
- “What does it mean to use a *sequence of transformations*?” (Using more than one basic transformation applied one after the other in a particular order.)

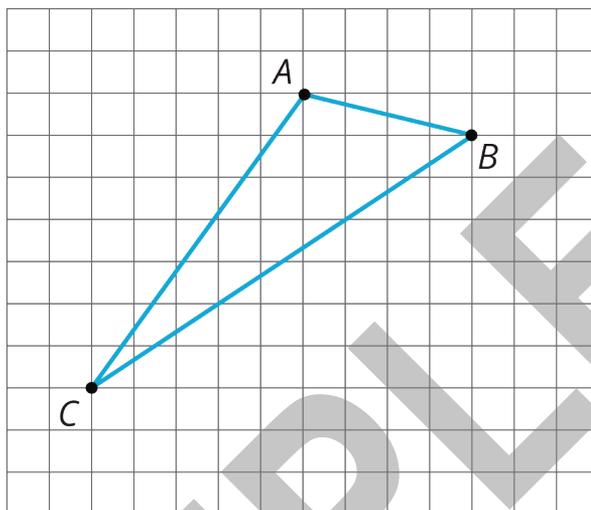
3.4 What Does It Take?

Cool Down: 5 minutes

Addressing

- 8.G.A.1

Student Task Statement



1. If you were to describe a translation of triangle ABC , what information would you need to include in your description?
2. If you were to describe a rotation of triangle ABC , what information would you need to include in your description?
3. If you were to describe a reflection of triangle ABC , what information would you need to include in your description?

Student Response

1. The distance and direction of the translation. One way to do this would be by picking a point on the triangle (A , for example) and then showing where this point goes (A' on the translated triangle).
2. A center point, an angle, and a direction (clockwise or counterclockwise).
3. A line.

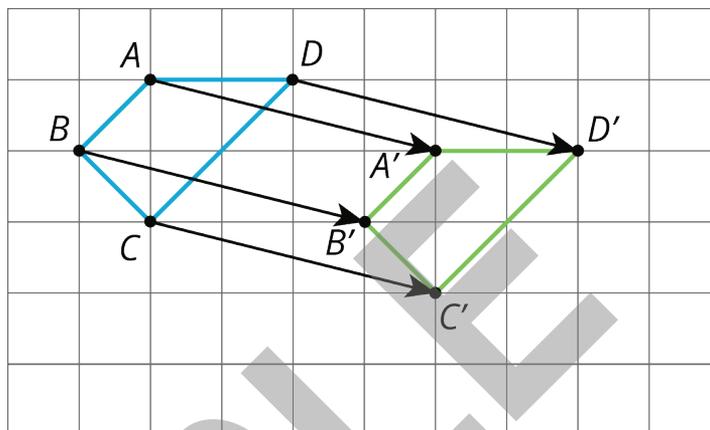
Student Lesson Summary

A move or combination of moves is called a **transformation**. When we do 1 or more moves in a row, we often call that a **sequence of transformations**. When a figure is on a grid, we can use the grid to describe a transformation. We use the word **image** to describe the figure after a transformation. To distinguish the original figure from its image, points in the image are

sometimes labeled with the same letters as the original figure, but with the symbol ' attached, as in A' (pronounced "A prime") is the image of A after a transformation.

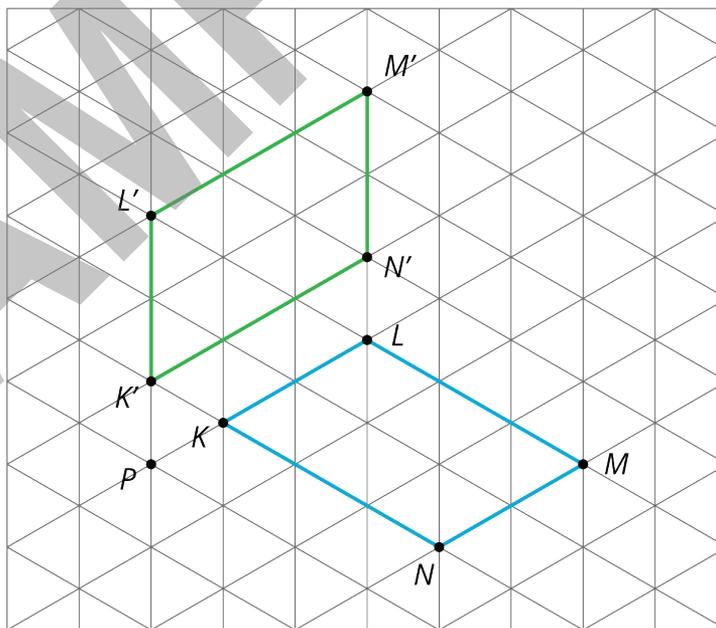
- A translation can be described by two points. If a translation moves point A to point A' , it moves the entire figure the same distance and direction as the distance and direction from A to A' . The distance and direction of a translation can be shown by an arrow.

For example, here is a translation of quadrilateral $ABCD$ that moves A to A' .



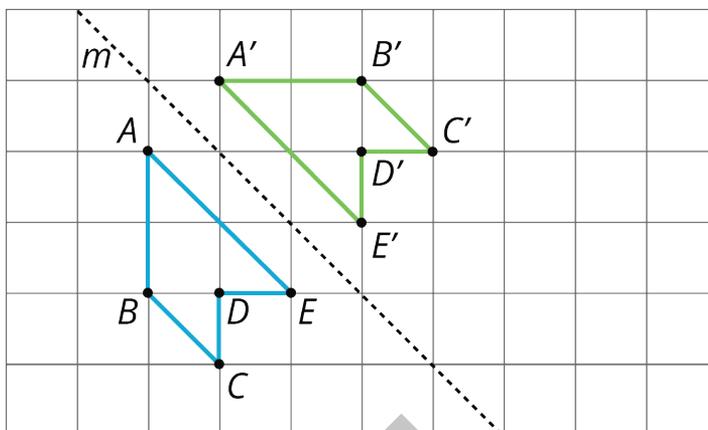
- A rotation can be described by an angle and a center. The direction of the angle can be clockwise or counterclockwise.

For example, quadrilateral $KLMN$ is rotated 60 degrees counterclockwise using center P . This type of grid is called an **isometric grid**. The isometric grid is made up of equilateral triangles. The angles in the triangles each measure 60 degrees, making the isometric grid convenient for showing rotations of 60 degrees.



- A reflection can be described by a line of reflection (the "mirror"). Each point is reflected directly across the line so that it is just as far from the mirror line, but is on the opposite side.

For example, pentagon $ABCDE$ is reflected across line m .



Glossary

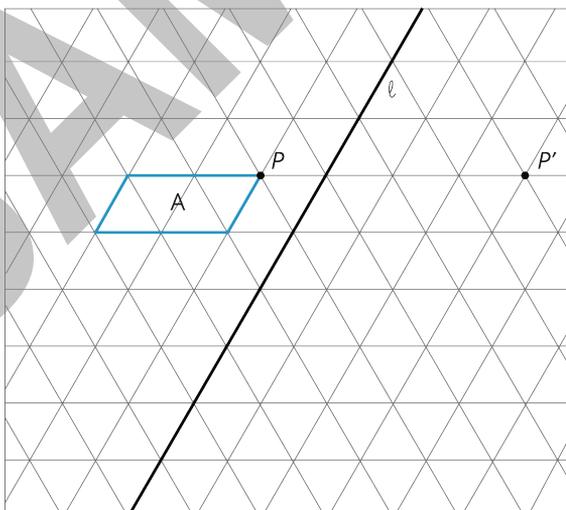
- image
- sequence of transformations
- transformation

Lesson 3 Practice Problems

Problem 1

Statement

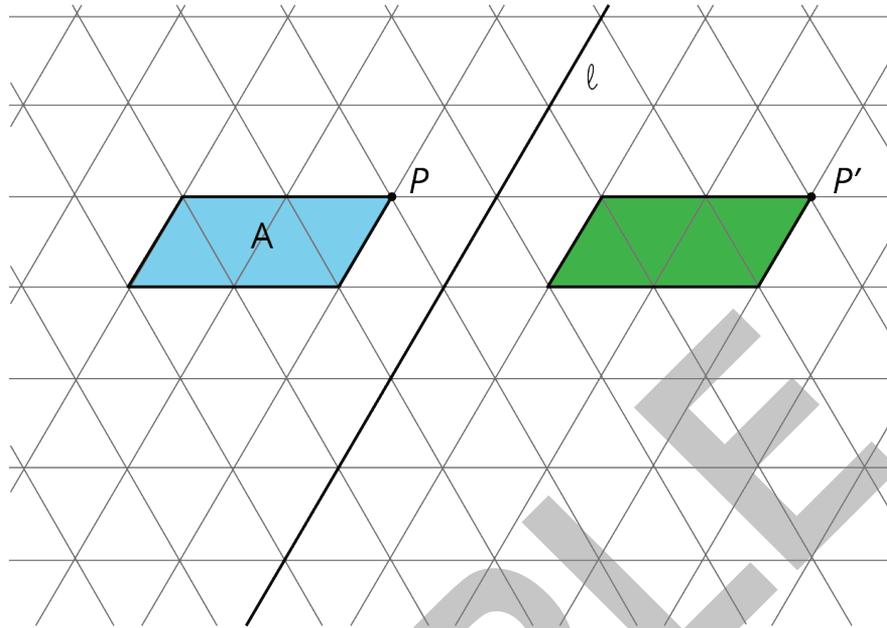
Apply each transformation described to Figure A. If you get stuck, try using tracing paper.



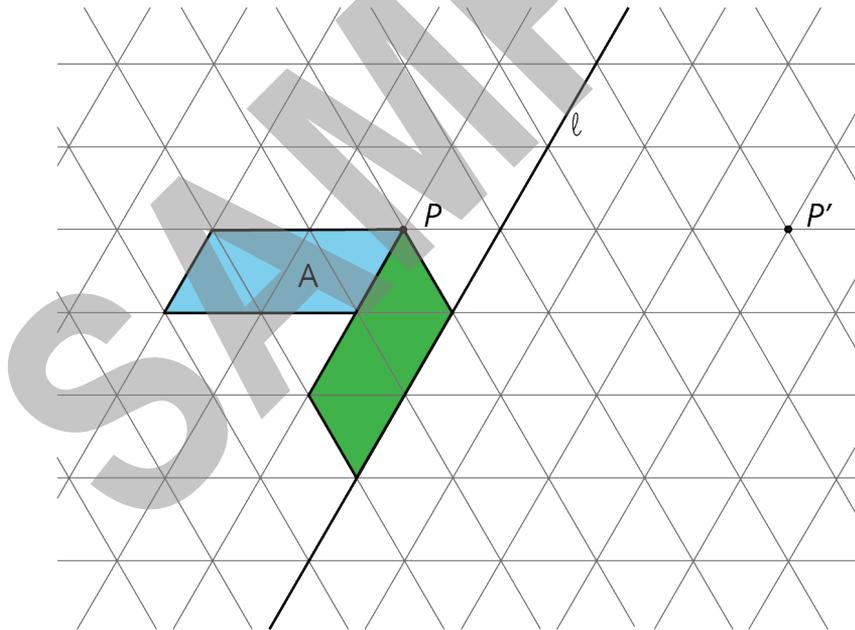
- A translation which takes P to P'
- A counterclockwise rotation of A , using center P , of 60 degrees
- A reflection of A across line ℓ

Solution

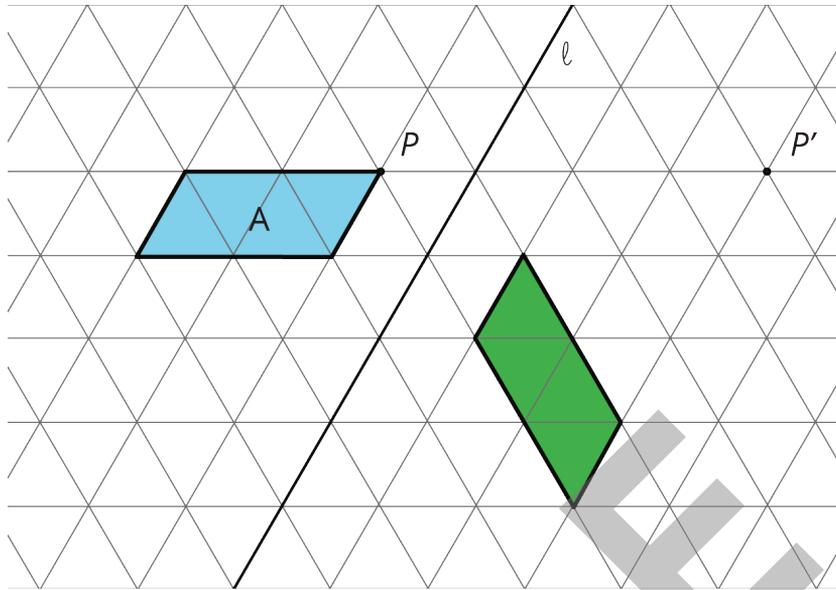
a.



b.



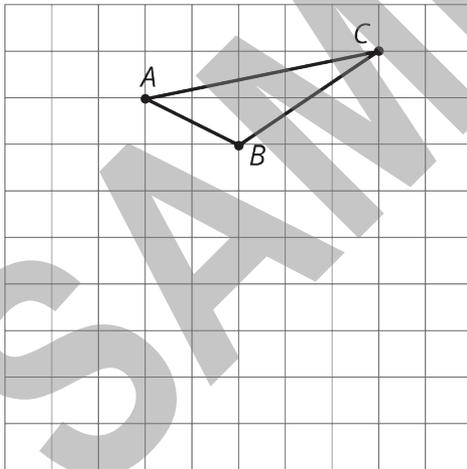
c.



Problem 2

Statement

Here is triangle ABC drawn on a grid.



On the grid, draw a rotation of triangle ABC , a translation of triangle ABC , and a reflection of triangle ABC . Describe clearly how each was done.

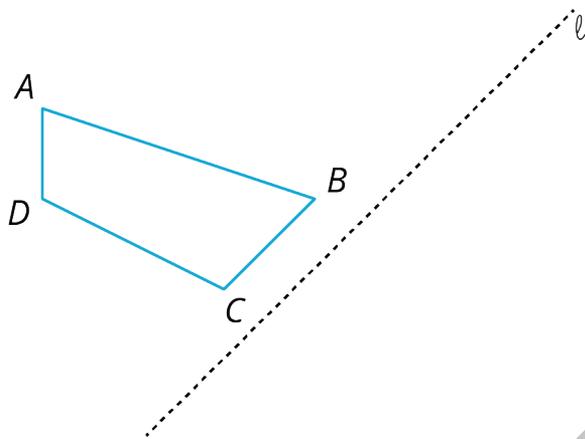
Solution

Answers vary. Sample answers: The rotation is a 90-degree counterclockwise rotation using center A . The translation is 4 units down and 3 to the left. The reflection is across a horizontal line through point B .

Problem 3

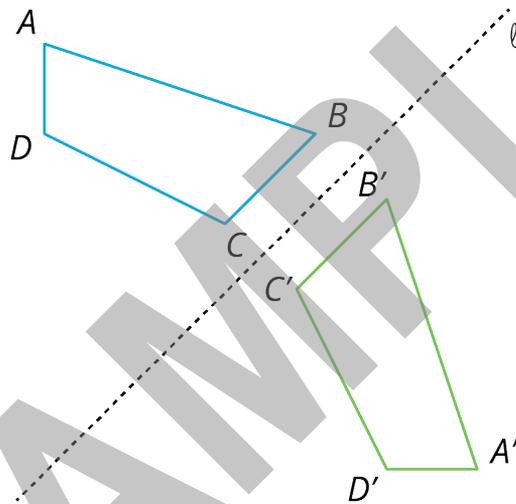
Statement

Here is quadrilateral $ABCD$ and line ℓ .



Draw the image of quadrilateral $ABCD$ after reflecting it across line l .

Solution



(From Unit 1, Lesson 2.)

Lesson 4: Coordinate Moves

Goals

- Draw and label a diagram of a line segment rotated 90 degrees clockwise or counterclockwise about a given center.
- Generalize (orally and in writing) the process to reflect any point in the coordinate plane.
- Identify (orally and in writing) coordinates that represent a transformation of one figure to another.

Learning Targets

- I can apply transformations to points on a grid if I know their coordinates.

Lesson Narrative

Students continue to investigate the effects of transformations. The new feature of this lesson is the coordinate plane. In this lesson, students use coordinates to describe figures and their images under transformations in the coordinate plane. Reflections over the x -axis and y -axis have a very nice structure captured by coordinates. When we reflect a point like $(2, 5)$ over the x -axis, the distance from the x -axis stays the same but instead of lying 5 units *above* the x -axis the image lies 5 units *below* the x -axis. That means the image of $(2, 5)$ when reflected over the x -axis is $(2, -5)$. Similarly, when reflected over the y -axis, $(2, 5)$ goes to $(-2, 5)$, the point 2 units to the left of the y -axis.

Using the coordinates to help understand transformations involves MP7 (discovering the patterns coordinates obey when transformations are applied).

Alignments

Building On

- 8.G.A.1: Verify experimentally the properties of rotations, reflections, and translations:

Addressing

- 8.G.A.3: Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

Building Towards

- 8.G.A.3: Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

Instructional Routines

- MLR7: Compare and Connect
- MLR8: Discussion Supports

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's transform some figures and see what happens to the coordinates of points.

4.1 Translating Coordinates

Warm Up: 5 minutes

The purpose of this warm-up is to remind students how the coordinate plane works and to give them an opportunity to see how one might describe a translation when the figure is plotted on the coordinate plane.

There are many ways to express a translation because a translation is determined by two points P and Q once we know that P is translated to Q . There are many pairs of points that express the *same* translation. This is different from reflections which are determined by a unique line and rotations which have a unique center and a specific angle of rotation.

Building On

- 8.G.A.1

Building Towards

- 8.G.A.3

Launch

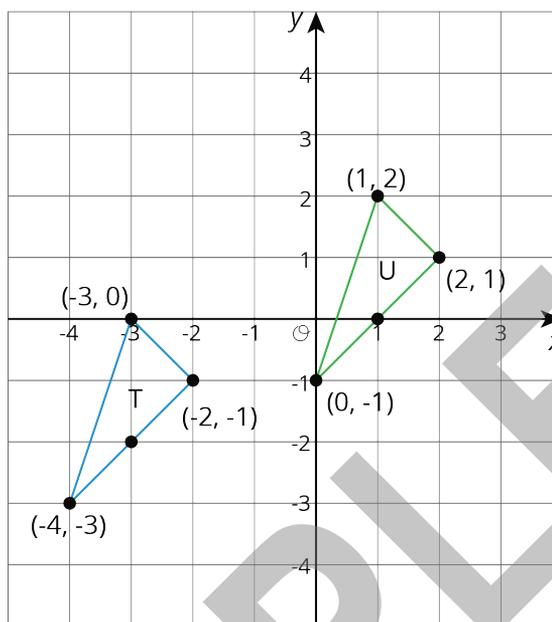
Ask students how they describe a translation. Is there more than one way to describe the same translation? After they have thought about this for a minute, give them 2 minutes of quiet work time followed by a whole-class discussion.

Anticipated Misconceptions

Students may think that they need more information to determine the translation. Remind them that specifying one point tells you the distance and direction all of the other points move in a translation.

Student Task Statement

Select all of the translations that take Triangle T to Triangle U. There may be more than one correct answer.



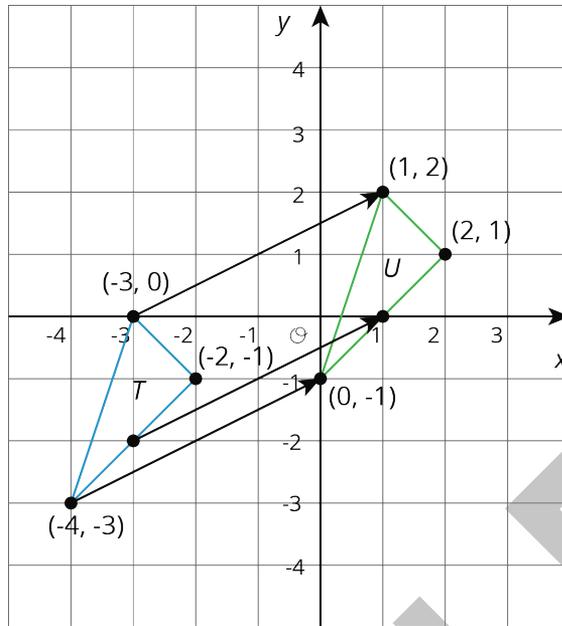
1. Translate $(-3, 0)$ to $(1, 2)$.
2. Translate $(2, 1)$ to $(-2, -1)$.
3. Translate $(-4, -3)$ to $(0, -1)$.
4. Translate $(1, 2)$ to $(2, 1)$.

Student Response

These are both correct: $(-3, 0)$ to $(1, 2)$ and $(-4, -3)$ to $(0, -1)$

Activity Synthesis

Remind students that once you name a starting point and an ending point, that completely determines a translation because it specifies a distance and direction for *all* points in the plane. Appealing to their experiences with tracing paper may help. In this case, we might describe that distance and direction by saying "all points go up 2 units and to the right 4 units." Draw the arrow for the two correct descriptions and a third one not in the list, like this:



Point out that each arrow does, in fact, go up 2 and 4 to the right.

4.2 Reflecting Points on the Coordinate Plane

15 minutes (there is a digital version of this activity)

While the warm-up focuses on studying translations using a coordinate grid, the goal of this activity is for students to work through multiple examples of specific points reflected over the x -axis and then generalize to describe where a reflection takes *any* point (MP8). They also consider reflections over the y -axis with slightly less scaffolding. In the next activity, students will study 90 degree rotations on a coordinate grid, rounding out this preliminary investigation of how transformations work on the coordinate grid.

Watch for students who identify early the pattern for how reflections over the x -axis or y -axis influence the coordinates of a point. Make sure that they focus on explaining *why* the pattern holds as the goal here is to understand reflections better using the coordinate grid. The rule is less important than understanding how it is essential to see the coordinate grid and state the rule.

Building On

- 8.G.A.1

Addressing

- 8.G.A.3

Instructional Routines

- MLR7: Compare and Connect

Launch

Tell students that they will have 5 minutes of quiet think time to work on the activity, and tell them to pause after the second question.

Select 2–3 students to share their strategies for the first 2 questions. You may wish to start with students who are measuring distances of points from the x -axis or counting the number of squares a point is from the x -axis and then counting out the same amount to find the reflected point. These strategies work, but overlook the structure of the coordinate plane. To help point out the role of the coordinate plane, select a student who noticed the pattern of changing the sign of the y -coordinate when reflecting over the x -axis.

After this initial discussion, give 2–3 minutes of quiet work time for the remaining questions, which ask them to generalize how to reflect a point over the y -axis.

Classes using the digital version have an applet for graphing and labeling points.

Support for Students with Disabilities

Representation: Internalize Comprehension. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. For example, to get students started, provide a smaller bank of points and only the first two instructions. Once students have successfully completed the four steps for each, present the remaining questions, one at a time.

Supports accessibility for: Conceptual processing; Organization

Support for English Language Learners

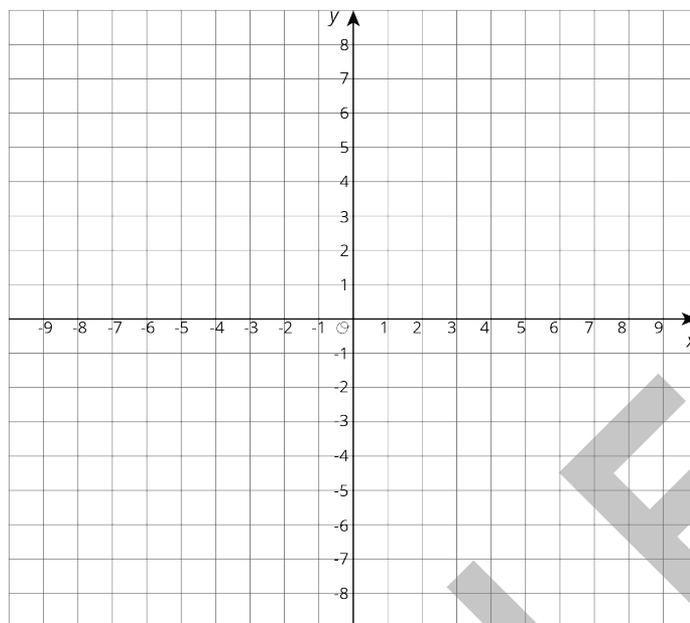
Speaking: MLR7 Compare and Connect. Use this routine when students present their strategies for reflecting points using the x -axis as the line of reflection before continuing on. Ask students to consider what is the same and what is different about the strategies. Draw students' attention to the different ways students reasoned to find the reflected coordinates. These exchanges strengthen students' mathematical language use and reasoning of reflections along the x -axis and y -axis.

Design Principle(s): Maximize meta-awareness

Anticipated Misconceptions

If any students struggle getting started because they are confused about where to plot the points, refer them back to the warm-up activity and practice plotting a few example points with them.

Student Task Statement

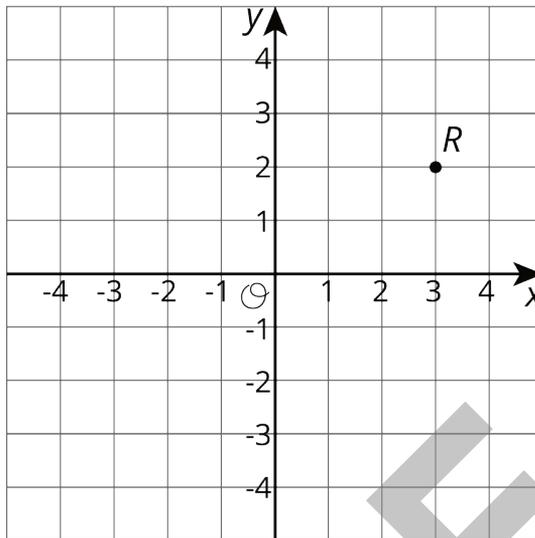


1. Here is a list of points

$$A = (0.5, 4) \quad B = (-4, 5) \quad C = (7, -2) \quad D = (6, 0) \quad E = (0, -3)$$

On the coordinate plane:

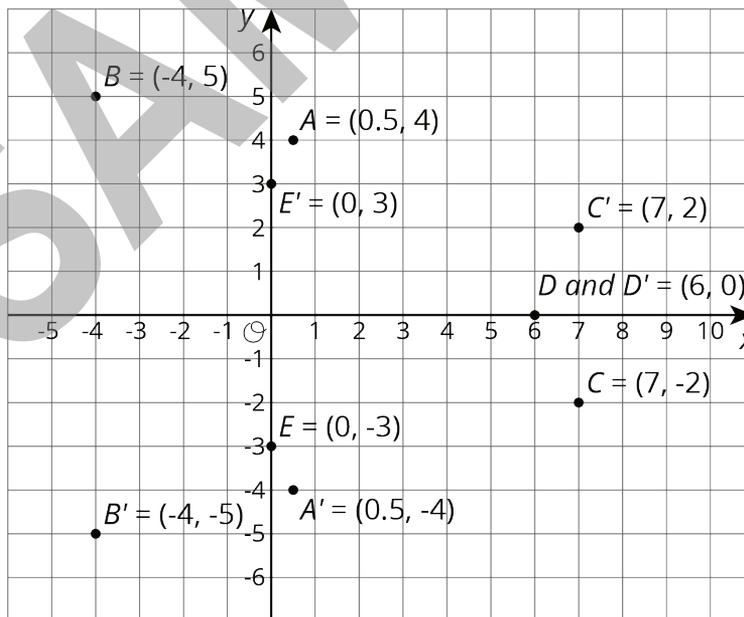
- Plot each point and label each with its coordinates.
 - Using the x -axis as the line of reflection, plot the image of each point.
 - Label the image of each point with its coordinates.
 - Include a label using a letter. For example, the image of point A should be labeled A' .
2. If the point $(13, 10)$ were reflected using the x -axis as the line of reflection, what would be the coordinates of the image? What about $(13, -20)$? $(13, 570)$? Explain how you know.
3. The point R has coordinates $(3, 2)$.
- Without graphing, predict the coordinates of the image of point R if point R were reflected using the y -axis as the line of reflection.
 - Check your answer by finding the image of R on the graph.



- c. Label the image of point R as R' .
 - d. What are the coordinates of R' ?
4. Suppose you reflect a point using the y -axis as line of reflection. How would you describe its image?

Student Response

1. The picture shows the points A, B, C, D, E and also their reflections over the x -axis:
 $A' = (0.5, -4), B' = (-4, -5), C' = (7, 2), D' = (6, 0), E' = (0, 3)$.



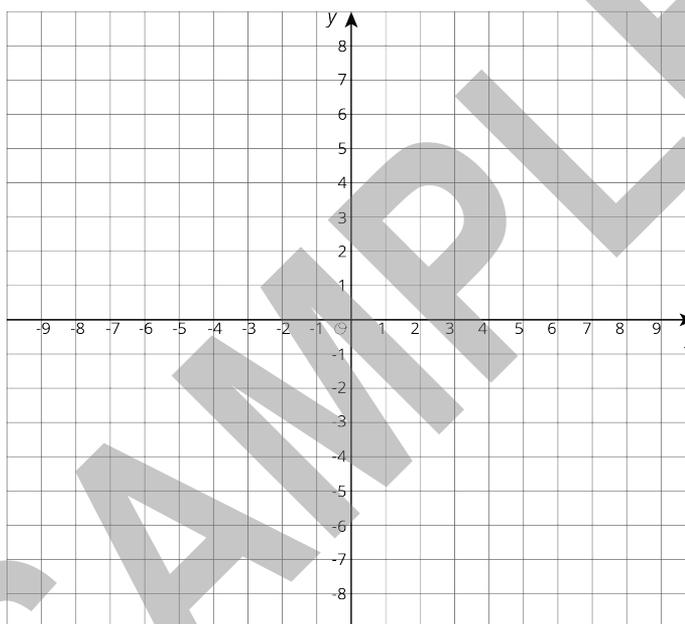
2. Using the x -axis as line of reflection, the reflection of $(13, 10)$ is $(13, -10)$, the reflection of $(13, -20)$ is $(13, 20)$ and the reflection of $(13, 570)$ is $(13, -570)$. Using the x -axis as line of reflection does not move points horizontally but it does move points which are not on the

x -axis vertically. In coordinates, the x -coordinate of the point stays the same while the y -coordinate changes sign.

- Using the y -axis as line of reflection does not move points vertically but it does move points that are not on the y -axis horizontally. In coordinates, the y -coordinate of the point stays the same while the x -coordinate changes sign. The point R has coordinates $(3, 2)$. When I reflect it over the y -axis it will go to $(-3, 2)$: the x -coordinate changes sign but the y -coordinate remains the same.
- The point will have the same y -coordinate but the x -coordinate will change signs. The distance from the y -axis does not change and the y -coordinate does not change.

Activity Synthesis

To facilitate discussion, display a blank coordinate grid.



Questions for discussion:

- "When you have a point and an axis of reflection, how do you find the reflection of the point?"
- "How can you use the coordinates of a point to help find the reflection?"
- "Are some points easier to reflect than others? Why?"
- "What patterns have you seen in these reflections of points on the coordinate grid?"

The goal of the activity is *not* to create a rule that students memorize. The goal is for students to notice the pattern of reflecting over an axis changing the sign of the coordinate (without having to graph). The coordinate grid can sometimes be a powerful tool for understanding and expressing structure and this is true for reflections over both the x -axis and y -axis.

4.3 Transformations of a Segment

15 minutes (there is a digital version of this activity)

This activity concludes looking at how the different basic transformations (translations, rotations, and reflections) behave when applied to points on a coordinate grid. In general, it is difficult to use coordinates to describe rotations. But when the center of the rotation is $(0, 0)$ and the rotation is 90 degrees (clockwise or counterclockwise), there is a straightforward description of rotations using coordinates.

Unlike translations and reflections over the x or y axis, it is more difficult to visualize where a 90 degree rotation takes a point. Tracing paper is a helpful tool, as is an index card.

Building On

- 8.G.A.1

Addressing

- 8.G.A.3

Instructional Routines

- MLR8: Discussion Supports

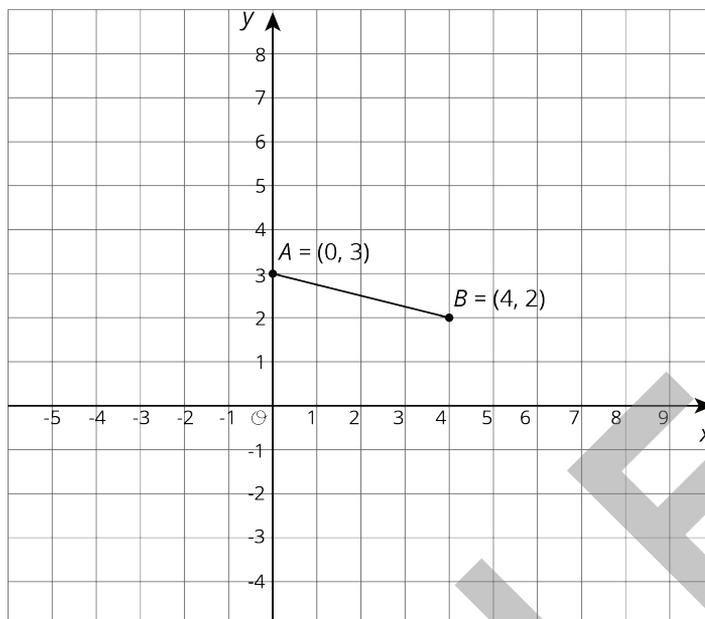
Launch

Demonstrate how to use tracing paper in order to perform a 90 degree rotation. It is helpful to put a small set of perpendicular axes (a + sign) on the piece of tracing paper and place their intersection point at the center of rotation. One of the small axes can be lined up with the segment being rotated and then the rotation is complete when the other small axis lines up with the segment.

An alternative method to perform rotations would be with the corner of an index card, which is part of the geometry toolkit.

Students using the digital version will see the segment being rotated by the computer as they manipulate the sliders.

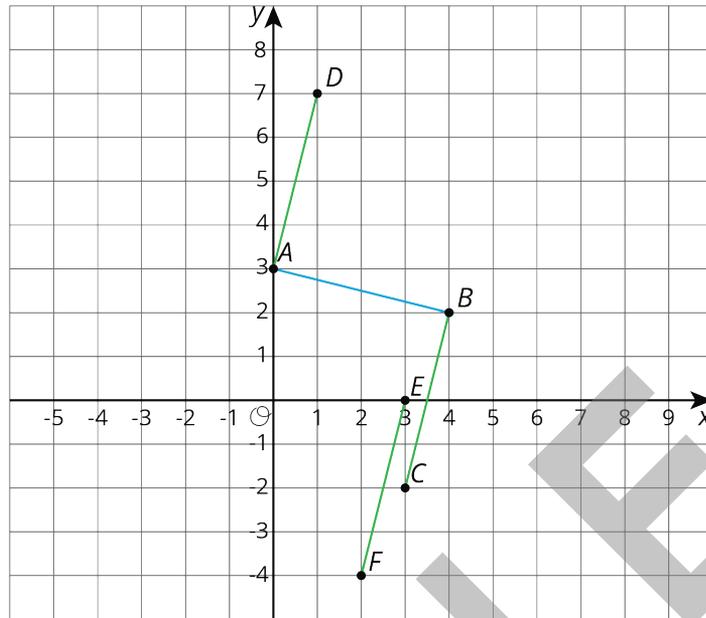
Student Task Statement



Apply each of the following transformations to segment AB .

1. Rotate segment AB 90 degrees counterclockwise around center B . Label the image of A as C . What are the coordinates of C ?
2. Rotate segment AB 90 degrees counterclockwise around center A . Label the image of B as D . What are the coordinates of D ?
3. Rotate segment AB 90 degrees clockwise around $(0, 0)$. Label the image of A as E and the image of B as F . What are the coordinates of E and F ?
4. Compare the two 90-degree counterclockwise rotations of segment AB . What is the same about the images of these rotations? What is different?

Student Response



1. $C = (3, -2)$
2. $D = (1, 7)$
3. $E = (3, 0), F = (2, -4)$
4. Answers vary. Sample response. The two counterclockwise rotations of AB are in different locations. The points A and B move different distances with the different rotations. One rotation can be mapped to the other by a translation.

Are You Ready for More?

Suppose EF and GH are line segments of the same length. Describe a sequence of transformations that moves EF to GH .

Student Response

Answers vary. For example, translate EF so that E lands on G , and then rotate EF with center G until (the image of) F lands on H .

Activity Synthesis

Ask students to describe or demonstrate how they found the rotations of segment AB . Make sure to highlight these strategies:

- Using tracing paper to enact a rotation through a 90 degree angle.
- Using an index card: Place the corner of the card at the center of rotation, align one side with the point to be rotated, and find the location of the rotated point along an adjacent side of the card. (Each point's distance from the corner needs to be equal.)

- Using the structure of the coordinate grid: All grid lines are perpendicular, so a 90 degree rotation with center at the intersection of two grid lines will take horizontal grid lines to vertical grid lines and vertical grid lines to horizontal grid lines.

The third strategy should only be highlighted if students notice or use this in order to execute the rotation, with or without tracing paper. This last method is the most accurate because it does not require any technology in order to execute, relying instead on the structure of the coordinate grid.

If some students notice that the three rotations of segment AB are all parallel, this should also be highlighted.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class.

Supports accessibility for: Language; Social-emotional skills; Attention

Support for English Language Learners

Speaking: MLR8 Discussion Supports. To support students in explaining the similarities and differences of the segment rotations for the last question, provide sentence frames for students to use when they are comparing segments, points, and rotations. For example, “___ is similar to ___ because ___” or “___ is different than ___ because ___.” Revoice student ideas using mathematical language use as needed.

Design Principle(s): Support sense-making; Optimize output for (comparison)

Lesson Synthesis

By this point, students should start to feel confident applying translations, reflections over either axis, and rotations of 90 degrees clockwise or counterclockwise to a point or shape in the coordinate plane.

To highlight working on the coordinate plane when doing transformations, ask:

- "What are some advantages to knowing the coordinates of points when you are doing transformations?"
- "What changes did we see when reflecting points over the x -axis? y -axis?"
- "How do you perform a 90 degree clockwise rotation of a point with center $(0, 0)$?"

Time permitting, ask students to apply a few transformations to a point. For example, where does $(1, 2)$ go when

- reflected over the x -axis? $(1, -2)$
- reflected over the y -axis? $(-1, 2)$
- rotated 90 degrees clockwise with center $(0, 0)$? $(2, -1)$

4.4 Rotation or Reflection

Cool Down: 5 minutes

Building On

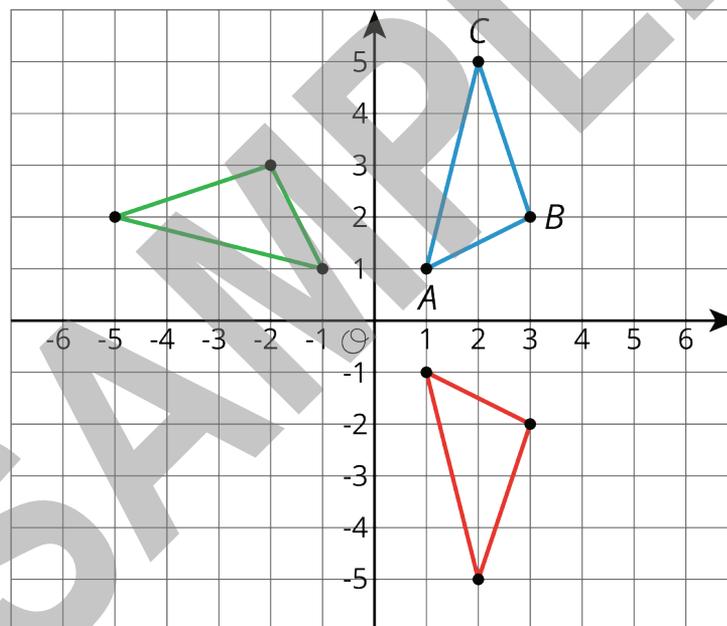
- 8.G.A.1

Addressing

- 8.G.A.3

Student Task Statement

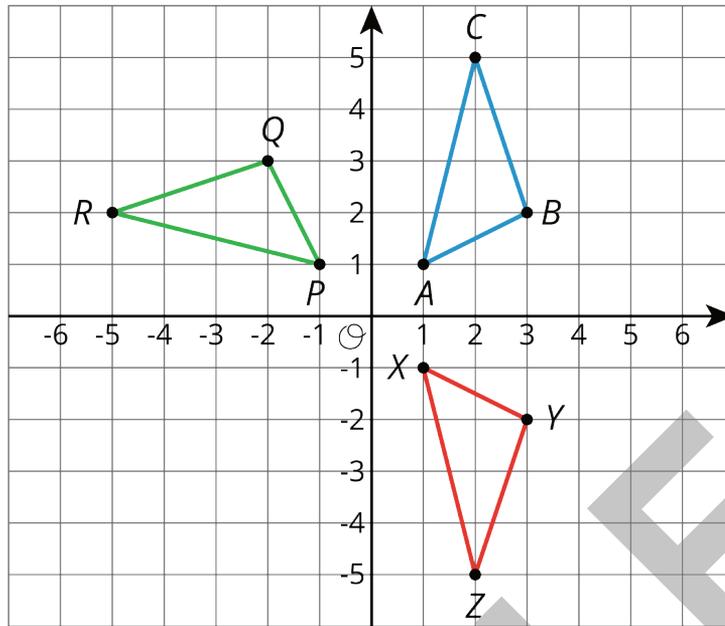
One of the triangles pictured is a rotation of triangle ABC and one of them is a reflection.



1. Identify the center of rotation, and label the rotated image PQR .
2. Identify the line of reflection, and label the reflected image XYZ .

Student Response

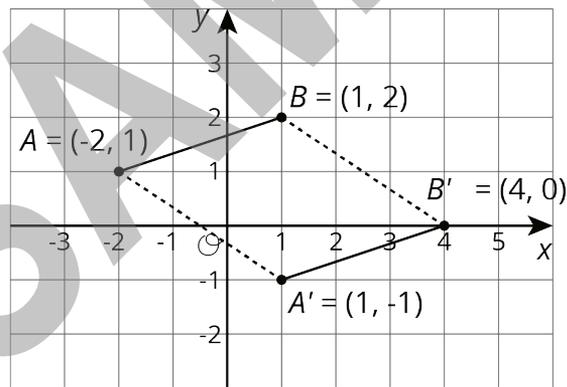
1. The center of the rotation taking $\triangle ABC$ to $\triangle PQR$ is $(0, 0)$, and the rotation is 90 degrees in a counterclockwise direction.
2. A reflection over the x -axis takes $\triangle ABC$ to $\triangle XYZ$.



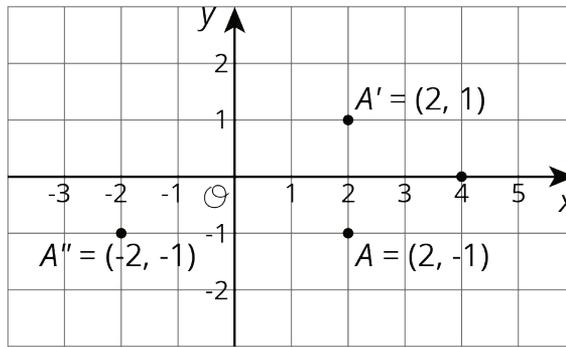
Student Lesson Summary

We can use coordinates to describe points and find patterns in the coordinates of transformed points.

We can describe a translation by expressing it as a sequence of horizontal and vertical translations. For example, segment AB is translated right 3 and down 2.

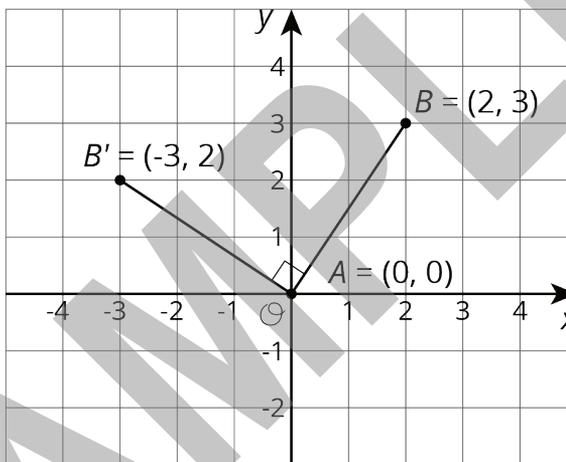


Reflecting a point across an axis changes the sign of one coordinate. For example, reflecting the point A whose coordinates are $(2, -1)$ across the x -axis changes the sign of the y -coordinate, making its image the point A' whose coordinates are $(2, 1)$. Reflecting the point A across the y -axis changes the sign of the x -coordinate, making the image the point A'' whose coordinates are $(-2, -1)$.



Reflections across other lines are more complex to describe.

We don't have the tools yet to describe rotations in terms of coordinates in general. Here is an example of a 90° rotation with center $(0, 0)$ in a counterclockwise direction.



Point A has coordinates $(0, 0)$. Segment AB was rotated 90° counterclockwise around A . Point B with coordinates $(2, 3)$ rotates to point B' whose coordinates are $(-3, 2)$.

Glossary

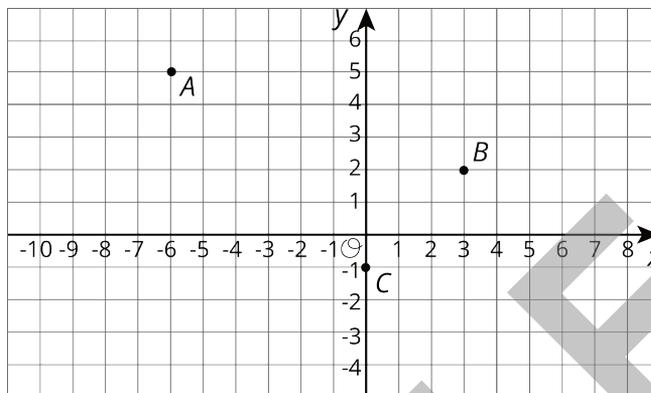
- coordinate plane

Lesson 4 Practice Problems

Problem 1

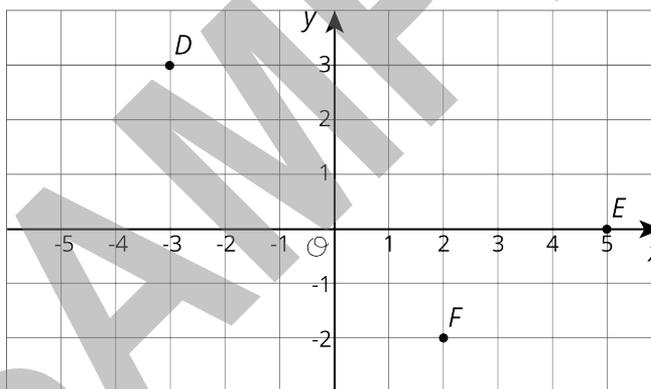
Statement

a. Here are some points.



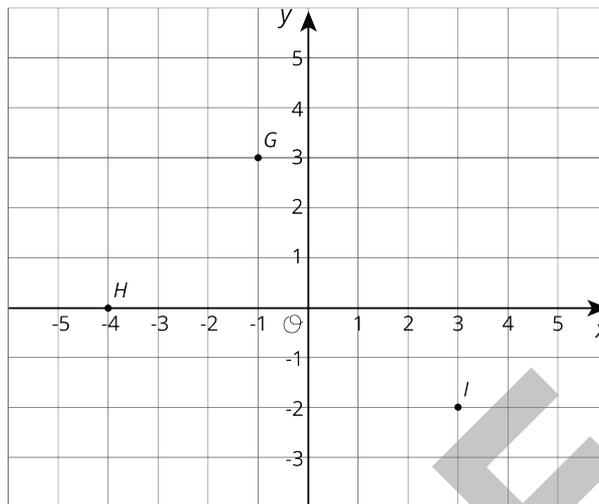
What are the coordinates of A , B , and C after a translation to the right by 4 units and up 1 unit? Plot these points on the grid, and label them A' , B' and C' .

b. Here are some points.



What are the coordinates of D , E , and F after a reflection over the y axis? Plot these points on the grid, and label them D' , E' and F' .

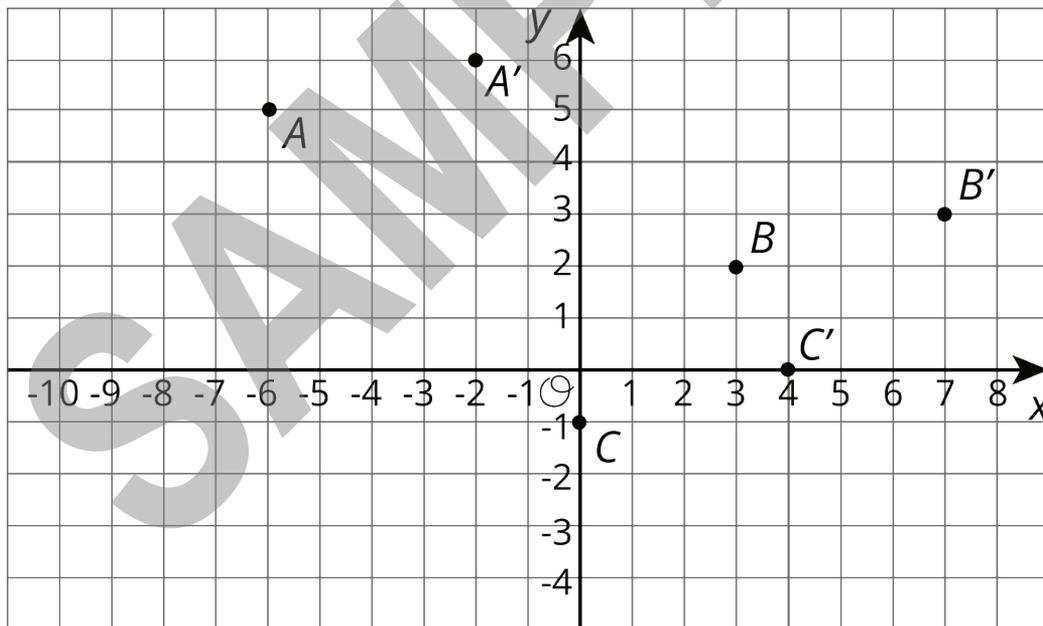
c. Here are some points.



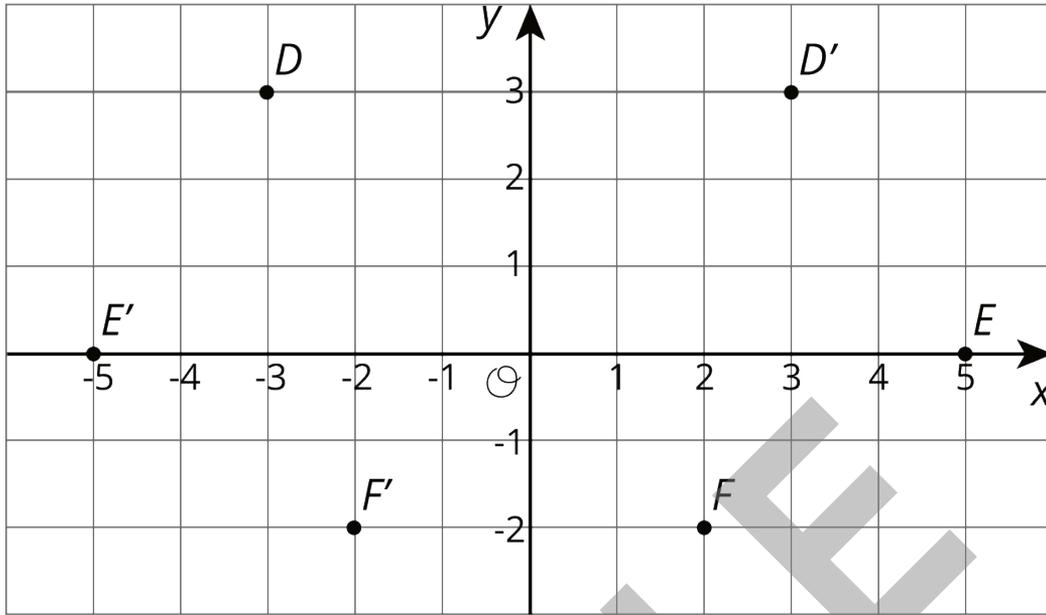
What are the coordinates of G , H , and I after a rotation about $(0, 0)$ by 90 degrees clockwise? Plot these points on the grid, and label them G' , H' and I' .

Solution

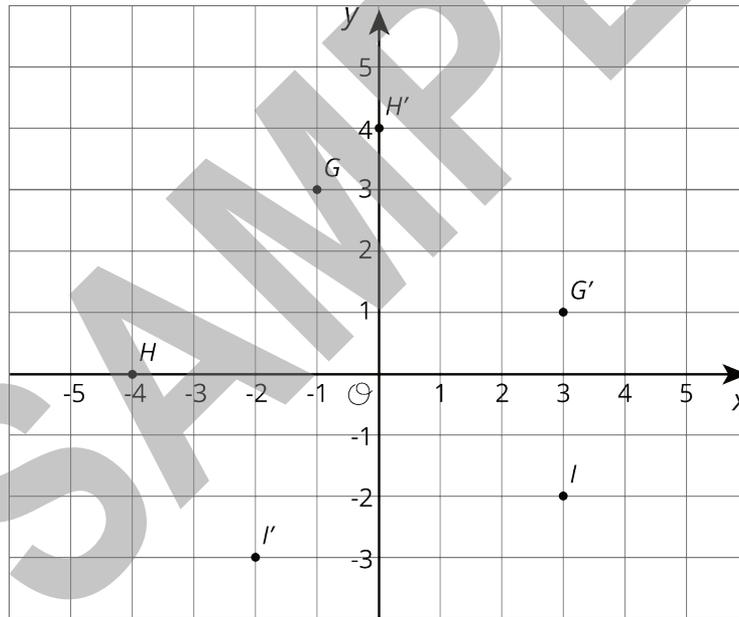
a. $A' = (-2, 6), B' = (7, 3), C' = (4, 0)$



b. $D' = (3, 3), E' = (-5, 0), F' = (-2, -2)$



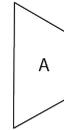
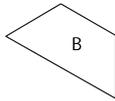
c. $G' = (3, 1)$, $H' = (0, 4)$, $I' = (-2, -3)$



Problem 2

Statement

Describe a sequence of transformations that takes trapezoid A to trapezoid B.



Solution

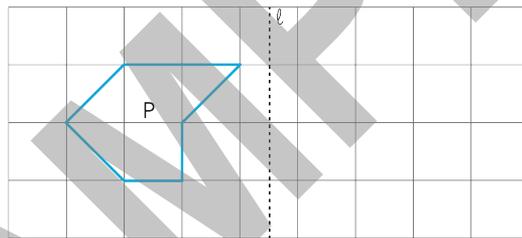
Answers vary. Sample response: Translate A up, then rotate it 60 degrees counter-clockwise (with center of rotation the bottom vertex), and then translate it left.

(From Unit 1, Lesson 3.)

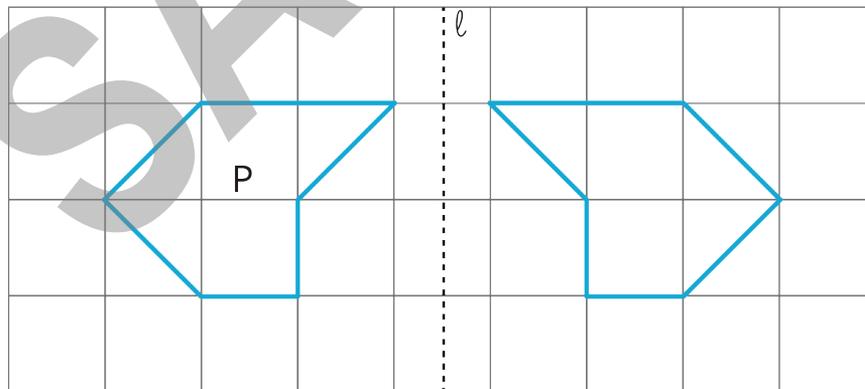
Problem 3

Statement

Reflect polygon P using line ℓ .



Solution



(From Unit 1, Lesson 3.)

Lesson 5: Describing Transformations

Goals

- Create a drawing on a coordinate grid of a transformed object using verbal descriptions.
- Identify what information is needed to transform a polygon. Ask questions to elicit that information.

Learning Targets

- I can apply transformations to a polygon on a grid if I know the coordinates of its vertices.

Lesson Narrative

Prior to this lesson, students have studied and classified different types of transformations (translations, rotations, reflections). They have practiced applying individual transformations and sequences of transformations to figures both on and off of a coordinate grid. In this lesson, they focus on communicating precisely the information needed to apply a sequence of transformations to a polygon on the coordinate grid. They must think carefully about what information they need (MP1) and request this information from their partner in a clear, precise way. They also explain *why* they need each piece of information (MP3). The coordinate grid plays a key role in this work, allowing students to communicate precisely about the locations of polygons and how they are transformed.

Alignments

Addressing

- 8.G.A.1: Verify experimentally the properties of rotations, reflections, and translations:
- 8.G.A.3: Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

Instructional Routines

- MLR4: Information Gap Cards
- Think Pair Share

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Pre-printed slips, cut from copies of the blackline master

Required Preparation

Print 1 copy of the blackline master for every 2 students. Cut them up ahead of time.

From the geometry toolkits, graph paper and tracing paper are especially helpful.

Student Learning Goals

Let's transform some polygons in the coordinate plane.

5.1 Finding a Center of Rotation

Warm Up: 5 minutes

Sometimes it is easy to forget to communicate all of the vital information about a transformation. In this case, the center of a rotation is left unspecified. Students do not need to develop a general method for finding the center of rotation, given a polygon and its rotated image. They identify the center in one situation and this can be done via geometric intuition and a little trial and error.

Addressing

- 8.G.A.1

Instructional Routines

- Think Pair Share

Launch

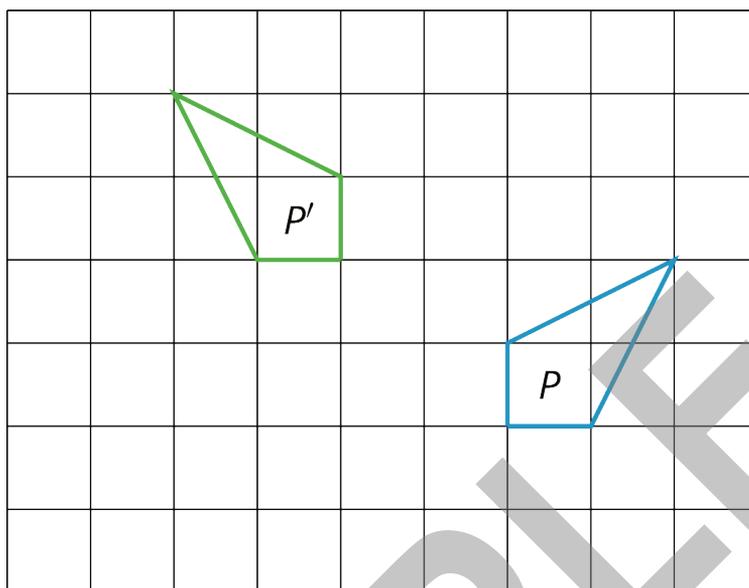
Arrange students in groups of 2 and provide access to geometry toolkits. Tell students that they have a diagram of a figure and its rotated image and that they need to identify the center of rotation. Give them 2 minutes of quiet work time and an opportunity to share with a partner, followed by a whole-class discussion.

Anticipated Misconceptions

Students may have trouble getting started. Suggest that they trace P on to tracing paper and try rotating it 90° . How must they rotate it to get it to land on P? Where is the center of the "spin"?

Student Task Statement

Andre performs a 90-degree counterclockwise rotation of Polygon P and gets Polygon P', but he does not say what the center of the rotation is. Can you find the center?



Student Response

The rotation takes the horizontal side of P to the vertical side of P', and the center of rotation is the intersection of the grid lines containing these two sides.

Activity Synthesis

Emphasize that it is important to communicate clearly. When we perform a transformation, we should provide the information necessary for others to understand what we have done. For a rotation, this means communicating:

- The center of the rotation.
- The direction of the rotation (clockwise or counterclockwise).
- The angle of rotation.

The grid provides extra structure that helps to identify these three parts of the rotation. Invite students to share how they identified the center of rotation. Methods may include:

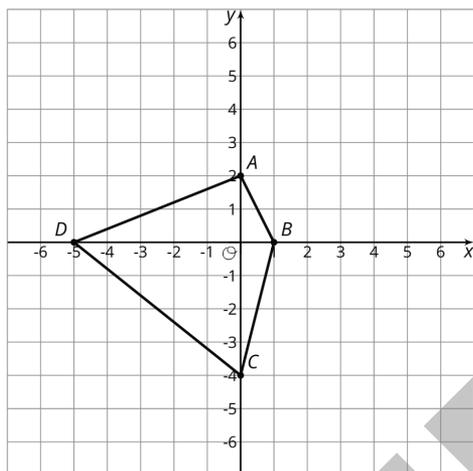
- Experimenting with tracing paper.
- Understanding that the rotation does not change the distance between the center of rotation and each vertex, so the center should be the same distance from each vertex and its image.

5.2 Info Gap: Transformation Information

30 minutes

This info gap activity gives students an opportunity to determine and request the information needed to perform a transformation in the coordinate plane. A sample pair of cards looks as follows:

Info Gap: Transformation Information
Problem Card 1



Polygon $A'B'C'D'$ is the image of $ABCD$ after some transformations.

Find $A'B'C'D'$.

Info Gap: Transformation Information
Data Card 1

Translation: 2 units up and 3 units to the right

Rotation: none

Reflection: over x -axis

Order of Transformations: Translation first and then reflection

Students likely need several rounds to determine the information they need.

- They need to know which transformations were applied (i.e., translation, rotation, or reflection)
- They need to determine the order in which the transformations were applied.
- They need to remember what information is needed to describe a translation, rotation, or reflection.

Monitor for students who successfully determine or remember each of these three important pieces of information as well as students who have partially but not completely solved the problem. Students may not realize that the order in which the transformations are applied is important, and this should be addressed in the Synthesis.

The info gap structure requires students to make sense of problems by determining what information is necessary, and then to ask for information they need to solve it. This may take several rounds of discussion if their first requests do not yield the information they need (MP1). It also allows them to refine the language they use and ask increasingly more precise questions until they get the information they need (MP6).

Addressing

- 8.G.A.1

- 8.G.A.3

Instructional Routines

- MLR4: Information Gap Cards

Launch

Tell students they will continue to describe transformations using coordinates. Explain the info gap structure, and consider demonstrating the protocol if students are unfamiliar with it.

Arrange students in groups of 2. Provide access to graph paper. In each group, distribute a problem card to one student and a data card to the other student. After you review their work on the first problem, give them the cards for a second problem and instruct them to switch roles.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Display or provide students with a physical copy of the written directions. Check for understanding by inviting students to rephrase directions in their own words. Keep the display of directions visible throughout the activity.

Supports accessibility for: Memory; Organization

Support for English Language Learners

Conversing: This activity uses MLR4 Information Gap to give students a purpose for discussing information necessary to transform a polygon. Display questions or question starters for students who need a starting point such as: “Can you tell me . . . (specific piece of information)”, and “Why do you need to know . . . (that piece of information)?”

Design Principle(s): Cultivate Conversation

Anticipated Misconceptions

Students may struggle to ask their partner for all of the information they need or may ask a question that is not sufficiently precise, such as, “What are the transformations?” Ask these students what kinds of transformations they have worked with. What information is needed to perform a translation? What about a rotation or reflection? Encourage them to find out *which* transformations they need to perform (Is there a translation? Is there a rotation?) and then find out the information they need for each transformation.

Student Task Statement

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.

Continue to ask questions until you have enough information to solve the problem.

4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

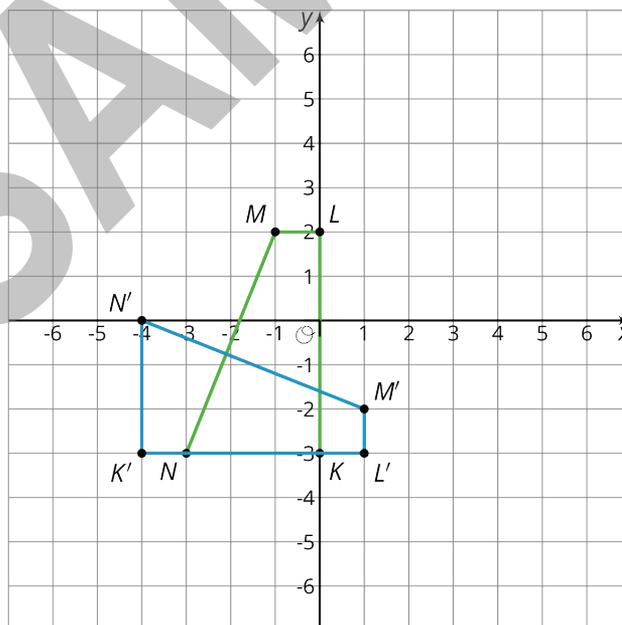
If your teacher gives you the *data card*:

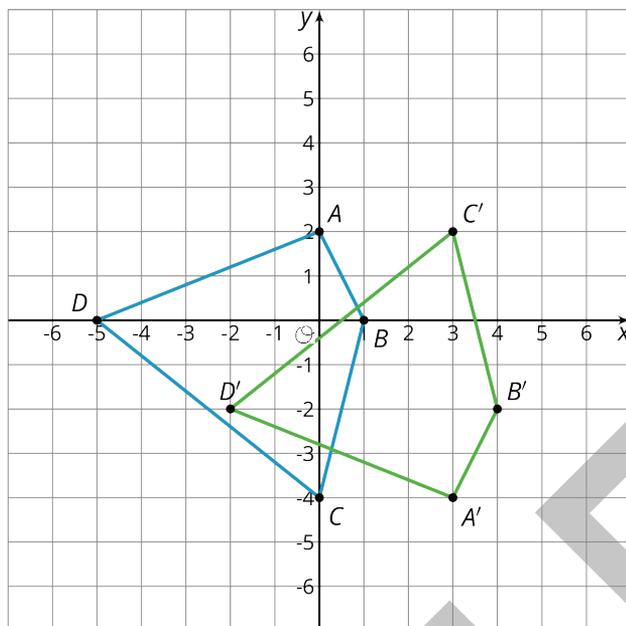
1. Silently read your card.
2. Ask your partner “*What specific information do you need?*” and wait for them to *ask* for information.

If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.

3. Before sharing the information, ask “*Why do you need that information?*” Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Student Response





Are You Ready for More?

Sometimes two transformations, one performed after the other, have a nice description as a single transformation. For example, instead of translating 2 units up followed by translating 3 units up, we could simply translate 5 units up. Instead of rotating 20 degrees counterclockwise around the origin followed by rotating 80 degrees clockwise around the origin, we could simply rotate 60 degrees clockwise around the origin.

Can you find a simple description of reflecting across the x -axis followed by reflecting across the y -axis?

Student Response

Reflecting across the x -axis followed by reflecting across the y -axis is the same as rotating 180 degrees (in either direction) around the origin.

Activity Synthesis

After students have completed their work, share the correct answers and ask students to discuss the process of solving the problems. Some guiding questions:

- "How did using coordinates help in talking about the problem?"
- "Was the order in which the transformations were applied important? Why?"
- "If this same problem were a picture on a grid without coordinates, how would you talk about the points?"

Highlight for students that one advantage of the coordinate plane is that it allows us to communicate information about transformations precisely. Here is what is needed for each type of transformation (consider showing one example of each while going through the different transformations):

- For a translation, the distance of vertical and horizontal components
- For a rotation, the center of rotation, the direction of rotation, and the angle of rotation
- For a reflection, the line of reflection

Lesson Synthesis

Ask students to choose which of the three transformations they have studied so far (translation, reflection, rotation) is their favorite and give 2–3 minutes for students to write a few sentences explaining why. Have students first share their explanations with a partner and then invite students to share their favorite with the class.

5.3 Describing a Sequence of Transformations

Cool Down: 5 minutes

Students describe what information is required to perform a translation and what information is required to perform a reflection. They also need to think about the order in which the two transformations are applied as they have just seen that switching the order can impact the outcome.

Addressing

- 8.G.A.1
- 8.G.A.3

Student Task Statement

Jada applies two transformations to a polygon in the coordinate plane. One of the transformations is a translation and the other is a reflection. What information does Jada need to provide to communicate the transformations she has used?

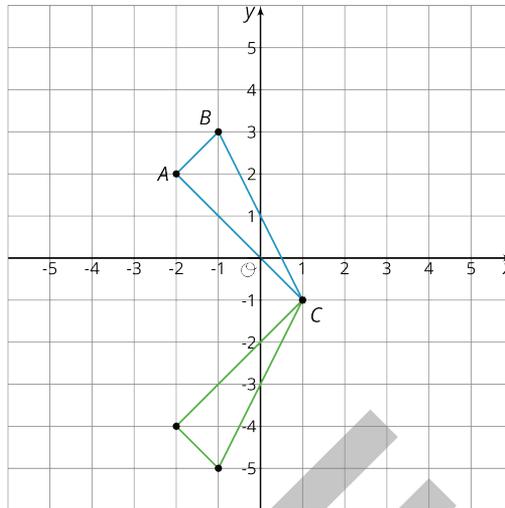
Student Response

For the translation, Jada needs to provide the distance and direction of the vertical displacement and the distance and direction of the horizontal displacement. For the reflection, Jada needs to give the line of reflection. It is also important for Jada to communicate the order in which the transformations are applied.

Student Lesson Summary

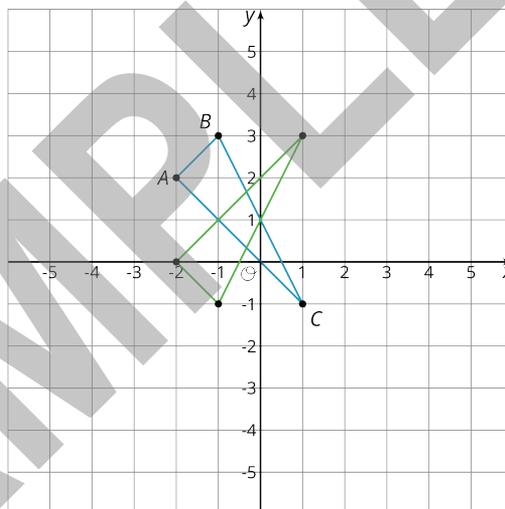
The center of a rotation for a figure doesn't have to be one of the points on the figure. To find a center of rotation, look for a point that is the same distance from two corresponding points. You will probably have to do this for a couple of different pairs of corresponding points to nail it down.

When we perform a sequence of transformations, the order of the transformations can be important. Here is triangle ABC translated up two units and then reflected over the x -axis.



Here is triangle ABC reflected over the x -axis and then translated up two units.

Triangle ABC ends up in different places when the transformations are applied in the opposite order!

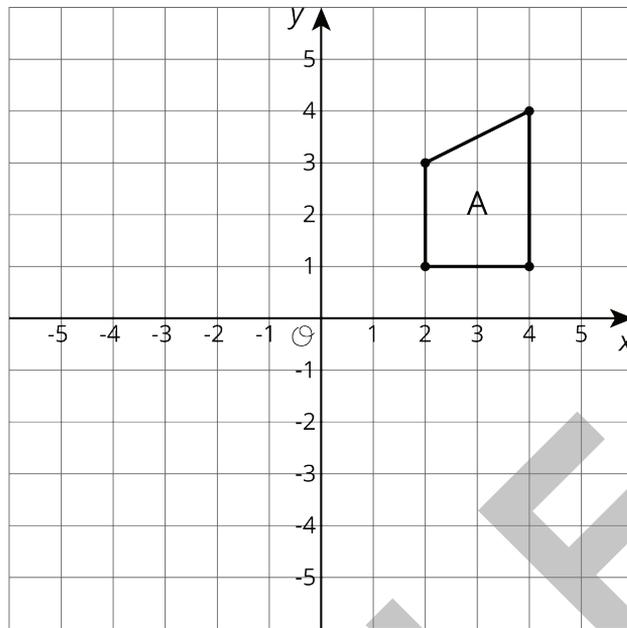


Lesson 5 Practice Problems

Problem 1

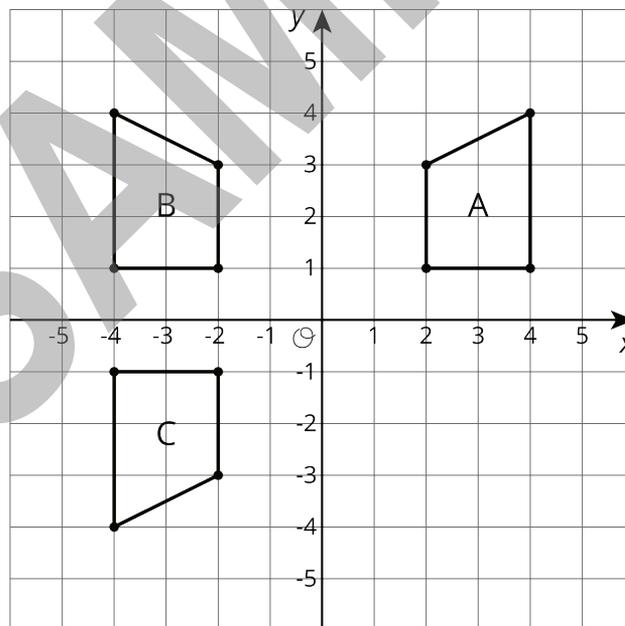
Statement

Here is Trapezoid A in the coordinate plane:



- Draw Polygon B, the image of A, using the y -axis as the line of reflection.
- Draw Polygon C, the image of B, using the x -axis as the line of reflection.
- Draw Polygon D, the image of C, using the x -axis as the line of reflection.

Solution



Polygon D is the same as B: reflecting a polygon twice over the x -axis returns it to its original position.

Problem 2

Statement

The point $(-4, 1)$ is rotated 180 degrees counterclockwise using center $(-3, 0)$. What are the coordinates of the image?

- A. $(-5, -2)$
- B. $(-4, -1)$
- C. $(-2, -1)$
- D. $(4, -1)$

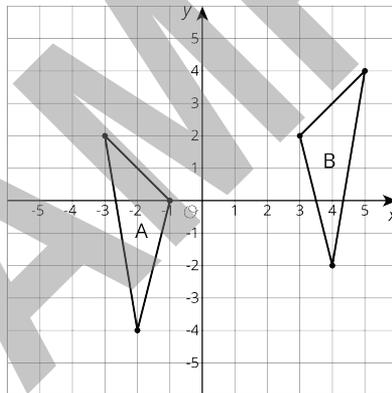
Solution

C

Problem 3

Statement

Describe a sequence of transformations for which Triangle B is the image of Triangle A.



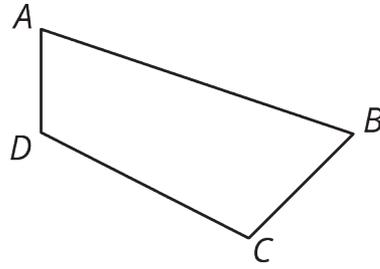
Solution

Answers vary. Sample response: B is the image of A under a reflection over the y-axis, then a translation 2 units to the right and 2 units up.

Problem 4

Statement

Here is quadrilateral $ABCD$.



Draw the image of quadrilateral $ABCD$ after each transformation.

- The translation that takes B to D .
- The reflection over segment BC .
- The rotation about point A by angle DAB , counterclockwise.

Solution

- Image of trapezoid moved to the left so that B lines up with D
- Image of trapezoid sharing segment BC with $ABCD$
- Image of trapezoid rotated so that the side corresponding to AD is now part of segment AB

(From Unit 1, Lesson 2.)

Section: Properties of Rigid Transformations

Lesson 6: No Bending or Stretching

Goals

- Comprehend that the phrase “rigid transformation” refers to a transformation where all pairs of “corresponding distances” and “corresponding angle” measures in the figure and its image are the same.
- Draw and label a diagram of the image of a polygon under a rigid transformation, including calculating side lengths and angle measures.
- Identify (orally and in writing) a sequence of rigid transformations using a drawing of a figure and its image.

Learning Targets

- I can describe the effects of a rigid transformation on the lengths and angles in a polygon.

Lesson Narrative

In this lesson, students begin to see that translations, rotations, and reflections preserve lengths and angle measures, and for the first time call them **rigid transformations**. In earlier lessons, students talked about corresponding points under a transformation. Now they will talk about **corresponding sides** and **corresponding angles** of a polygon and its image.

As students experiment with measuring corresponding sides and angles in a polygon and its image, they will need to use the structure of the grid (MP7) as well as appropriate technology, including protractors, rulers, and tracing paper.

Alignments

Building On

- 4.MD.A: Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.

Addressing

- 8.G.A.1.a: Lines are taken to lines, and line segments to line segments of the same length.
- 8.G.A.1.b: Angles are taken to angles of the same measure.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR2: Collect and Display
- MLR8: Discussion Supports

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's compare measurements before and after translations, rotations, and reflections.

6.1 Measuring Segments

Warm Up: 5 minutes

In this warm-up, students measure four line segments. They discuss the different aspects of making and recording accurate measurements. It is important to highlight the fractional markings and fraction and decimal equivalents used as students explain how they determined the length of the segment.

Building On

- 4.MD.A

Launch

Give students 2 minutes of quiet work time followed by whole-class discussion.

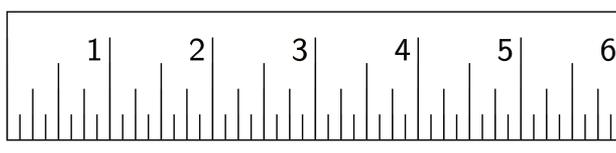
Anticipated Misconceptions

Students may struggle with the ruler that is not pre-partitioned into fractional units. Encourage these students to use what they know about eighths and tenths to partition the ruler and estimate their answer.

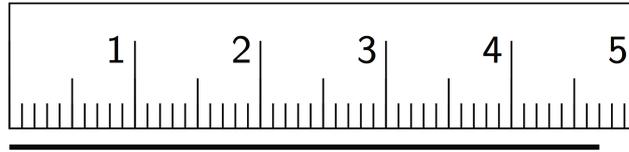
Student Task Statement

For each question, the unit is represented by the large tick marks with whole numbers.

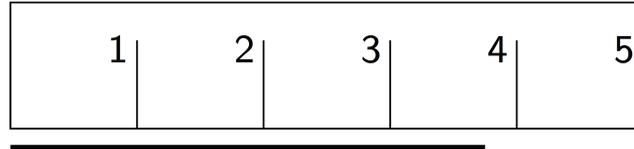
1. Find the length of this segment to the nearest $\frac{1}{8}$ of a unit.



2. Find the length of this segment to the nearest 0.1 of a unit.



3. Estimate the length of this segment to the nearest $\frac{1}{8}$ of a unit.



4. Estimate the length of the segment in the prior question to the nearest 0.1 of a unit.

Student Response

1. $4\frac{5}{8}$ units
2. 4.7 units
3. $3\frac{3}{4}$ (or $3\frac{6}{8}$ units)
4. 3.7 units (or 3.8 units)

Activity Synthesis

Invite students to share their responses and record them for all to see. Ask the class if they agree or disagree with each response. When there is a disagreement, have students discuss possible reasons for the different measurements.

Students are likely to have different answers for their measure of the third segment. The ruler shown is not as accurate as the question requires as it has not been pre-partitioned into fractional units. Ask 2–3 students with different answers to share their strategies for measuring the third segment. There will be opportunities for students to use measuring strategies later in this lesson.

6.2 Sides and Angles

15 minutes (there is a digital version of this activity)

The purpose of this activity is for students to see that translations, rotations, and reflections preserve lengths and angle measures. Students can use tracing paper to help them draw the figures and make observations about the preservation of side lengths and angle measures under transformations. While the grid helps measure lengths of horizontal and vertical segments, the students may need more guidance when asked to measure diagonal lengths. It is important in the launch to demonstrate for students how to either use the tracing paper or an index card to mark off unit lengths using the grid (MP5).

Since students are creating their own measuring tool, they can only give an estimate, and some flexibility should be allowed in the response. During the discussion, highlight different reasonable answers that students find for the lengths which are not whole numbers.

As students work individually, monitor and ask them to explain how they are performing their transformations and finding the side lengths and angle measures. During the discussion, select students who mention corresponding sides and angles, which they learned in grade 7 when making scaled copies, to share. Also select students who estimated the side lengths for Figure C correctly using either the tracing paper or index card.

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Instructional Routines

- MLR8: Discussion Supports

Launch

Tell students, “In this activity you will be performing transformations. You can use tracing paper to help you draw the images of the figures or to check your work.”

Point students to Figure C and tell them, “When you are asked to measure side lengths here, you will need to make a ruler on either tracing paper or on a blank edge of an index card.” This reinforces the strategies and estimates students made in the warm-up.

Give students 3 minutes of quiet think time. Be sure to save at least 5 minutes for the discussion.

For classrooms using the digital version of the activity, the applets contain tools for the three rigid transformations students need. They have to choose which tool to use in each problem. Caution students that a quick click is all that is needed to select a figure. If they move the cursor away, and the image does not seem “highlighted,” it is likely they selected and de-selected the figure.

Support for Students with Disabilities

Action and Expression: Internalize Executive Functions. To support development of organizational skills, check in with students within the first 2–3 minutes of work time. Look for students who make a ruler and use it appropriately in finding side lengths. Invite students to share how they found diagonal side lengths in the last problem.

Supports accessibility for: Memory; Organization

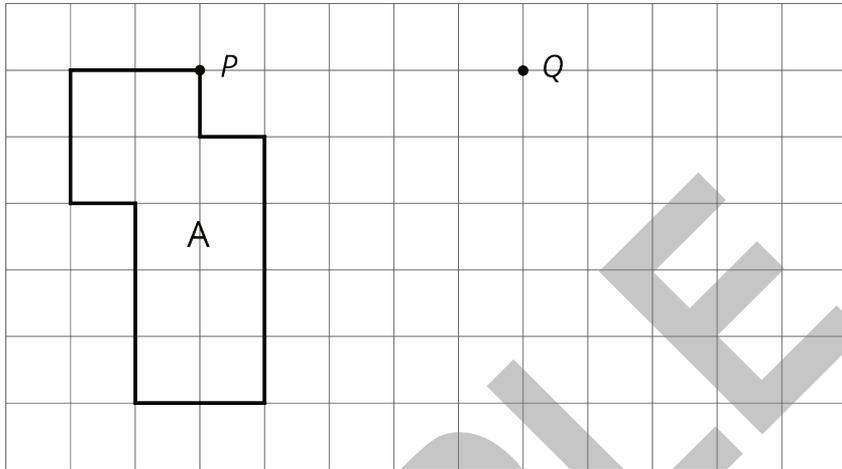
Anticipated Misconceptions

Students may try to count the grid squares on the diagonal side lengths. Remind students to measure these lengths with their tracing paper or index card. Students may also struggle estimating

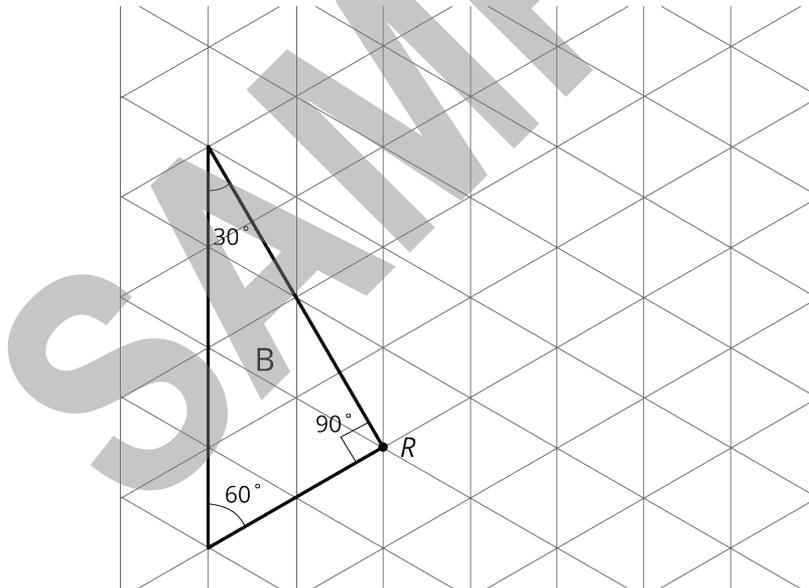
the diagonal side lengths on their self-marked index card or tracing paper. Remind students of how they estimated the lengths for the questions in the warm-up where the ruler was not marked.

Student Task Statement

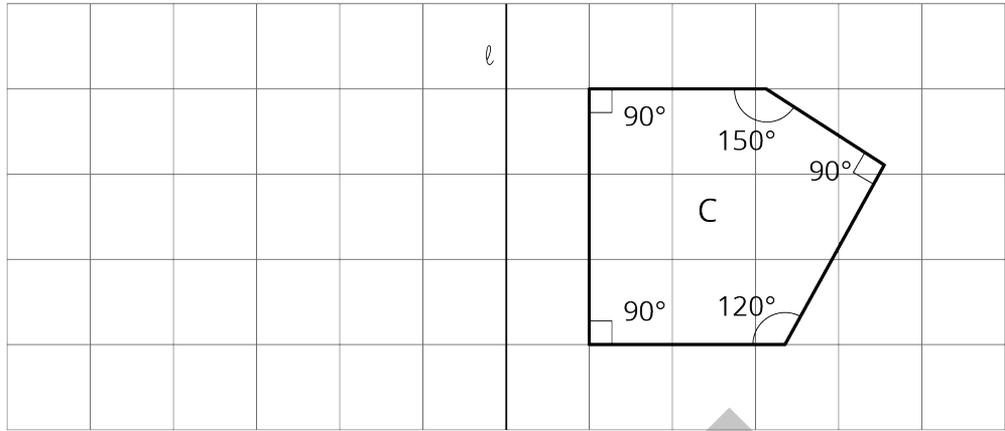
1. Translate Polygon A so point P goes to point Q . In the image, write the length of each side, in grid units, next to the side.



2. Rotate Triangle B 90 degrees clockwise using R as the center of rotation. In the image, write the measure of each angle in its interior.

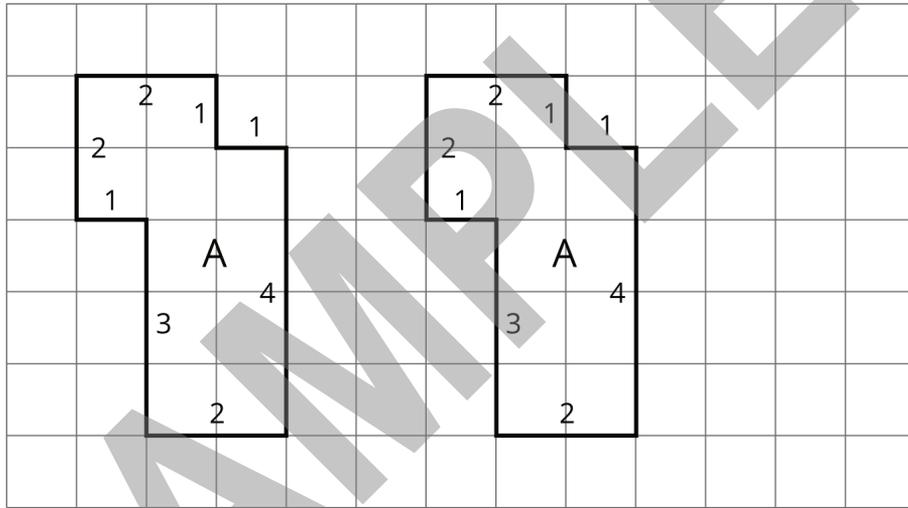


3. Reflect Pentagon C across line ℓ .
 - a. In the image, write the length of each side, in grid units, next to the side. You may need to make your own ruler with tracing paper or a blank index card.
 - b. In the image, write the measure of each angle in the interior.



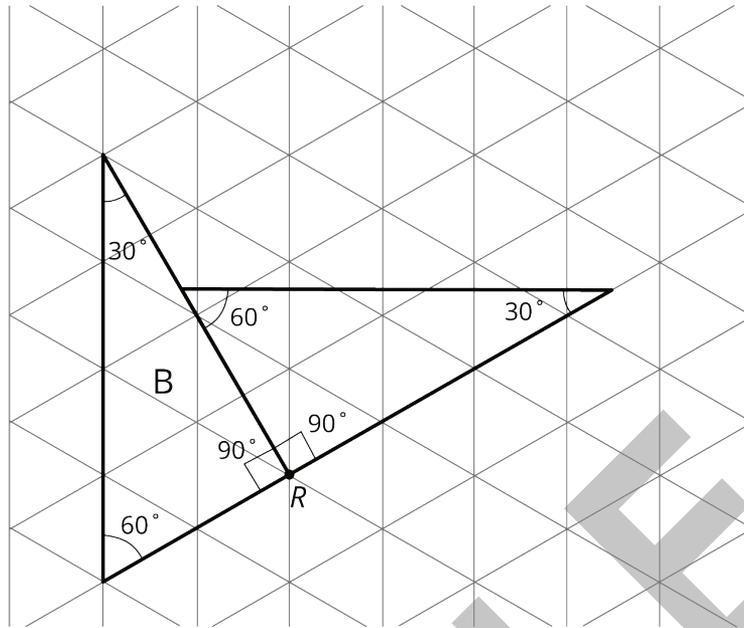
Student Response

1.

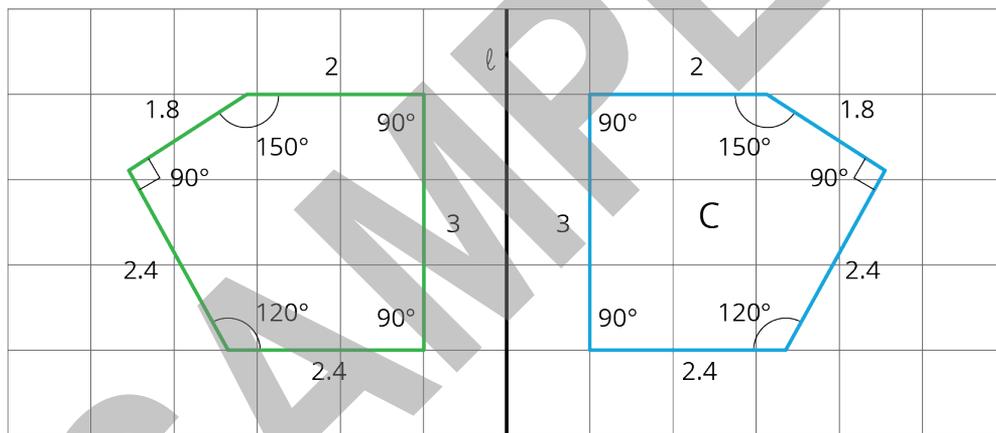


The side lengths are measured in units where one unit is the side length of the square in the grid.

2.



3.



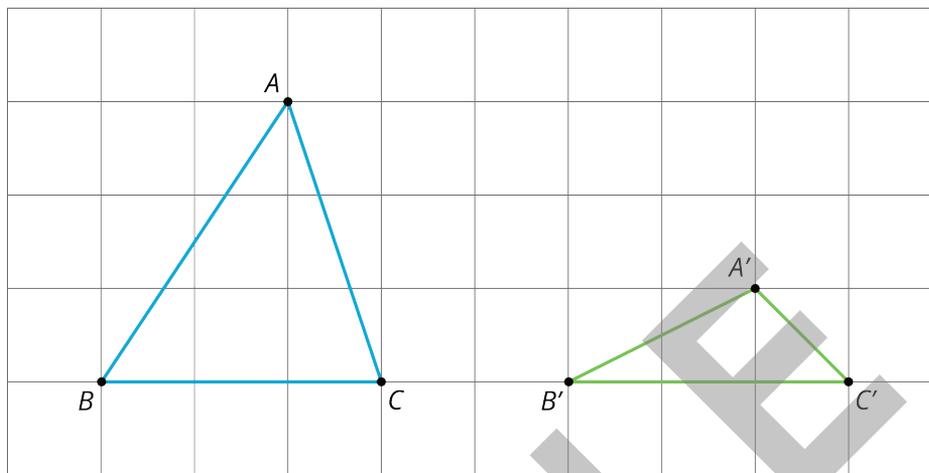
The lengths are measured in grid units. The sides that are not whole numbers have been rounded to the nearest tenth.

Activity Synthesis

Ask selected students to share how they performed the given transformation for each question. After each explanation, ask the class if they agree or disagree. Introduce students to the idea of **corresponding sides** and **corresponding angles**. Ask students to identify the corresponding angles in the first question and the corresponding side lengths in the second since they were not asked about these attributes the first time. The point here is not to find the actual values but to note that the corresponding measurements are equal. Since it is sometimes not possible to measure angles or side lengths exactly, student estimates for these values (both corresponding sides and corresponding angles) may be slightly different.

Point out that for each of the transformations in this activity, the lengths of the sides of the original figure equal the lengths of the corresponding sides in the image, and the measures of the angles in

the original figure equal the measures of the corresponding angles in the image. For this reason, we call these transformations **rigid transformations**: they behave as if we are moving the shapes around without stretching, bending, or breaking them. An example of a non-rigid transformation is one that compresses a figure vertically, like this:



Tell them that a rigid transformation is a transformation where all pairs of corresponding distances and angle measures in the figure and its image are equal. It turns out that translations, reflections, and rotations are the building blocks for *all* rigid transformations, and we will explore that next.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. As students describe their approaches, press for details in students' explanations by requesting that students challenge an idea, elaborate on an idea, or give an example of their process. Connect the terms corresponding sides and corresponding angles to students' explanations multi-modally by using different types of sensory inputs, such as demonstrating the transformation or inviting students to do so, using the images, and using gestures. This will help students to produce and make sense of the language needed to communicate their own ideas.

Design Principle(s): Optimize output (for explanation)

6.3 Which One?

10 minutes (there is a digital version of this activity)

The purpose of this activity is to decide if there is a sequence of translations, rotations, and reflections that take one figure to another and, if so, to produce one such sequence. Deciding whether or not such a sequence is possible uses the knowledge that translations, rotations, and reflections do not change side lengths or angle measures. The triangles ABC and CFG form part of a large pattern of images of triangle ABC that will be examined more closely in future lessons.

Monitor for students who use different transformations to take triangle ABC to triangle CFG and select them to share during the discussion. (There are two possible sequences in the Possible Responses section, but these are not the only two.)

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR2: Collect and Display

Launch

Provide access to geometry toolkits. Give students 4 minutes quiet work time, 2 minutes to discuss with partner, and then time for a whole-class discussion.

If using the digital activity, have a brief discussion of the previous activity to highlight the transformations that the students used. Then give students 4 minutes of individual work time, 2 minutes to discuss with a partner, and then time for a whole-class discussion.

Support for Students with Disabilities

Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, "Triangle ____ is a rigid transformation of Triangle ABC because..." "I agree/disagree because..." or "Another transformation is ____ because..."
Supports accessibility for: Language; Organization

Support for English Language Learners

Conversing, Speaking, Listening: Math Language Routine 2 Collect and Display. This is the first time Math Language Routine 2 is suggested as a support in this course. In this routine, the teacher circulates and listens to students talk while jotting down the words, phrases, or drawings students produce. The language collected is displayed visually for the whole class to use throughout the lesson and unit. Generally, the display contains different examples of students using features of the disciplinary language functions, such as interpreting, justifying, or comparing. The purpose of this routine is to capture a variety of students' words and phrases in a display that students can refer to, build on, or make connections with during future discussions, and to increase students' awareness of language used in mathematics conversations.

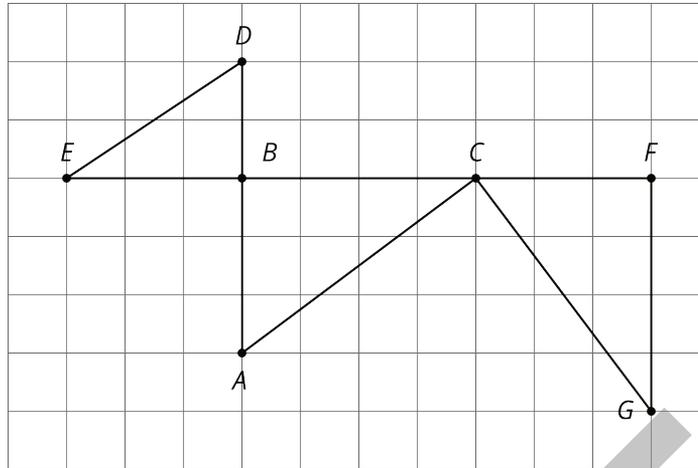
Design Principle(s): Support sense-making

How it Happens

1. As students discuss their work with a partner, listen for and collect the language students use to describe each transformation. Be sure to capture both mathematical and informal language students use, as well as phrases that capture a sequence of transformations.
2. To support the discussion, provide these sentence frames: "Triangle ____ is a rigid transformation of triangle ABC , because . . .", "Triangle ____ cannot be a rigid transformation of triangle ABC , because . . ."
3. As groups close their conversation, display the language collected for all to reference.
4. At the start of the synthesis, facilitate a whole-class discussion encouraging students to ask and respond to clarifying questions about the meaning of a word or phrase on the display. To prompt discussion, ask students, "From the language I collected, are there any words or phrases that are unclear?" or "which phrases include words that we can replace with mathematical terms?" At this point in the unit, students can be expected to use the terms, clockwise, counterclockwise, rotation, reflection, and translation. If necessary, revoice student ideas to demonstrate mathematical language, and invite students to chorally repeat phrases that include relevant vocabulary in context. Choral repetition facilitates the transfer of new vocabulary to students' long term memory.
5. Keep this display visible for the remainder of the lesson, and if possible, continue to add to it throughout the unit. Close this conversation by reminding students to use the display as a reference.

Student Task Statement

Here is a grid showing triangle ABC and two other triangles.



You can use a **rigid transformation** to take triangle ABC to *one* of the other triangles.

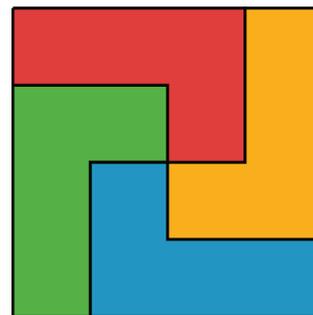
1. Which one? Explain how you know.
2. Describe a rigid transformation that takes ABC to the triangle you selected.

Student Response

1. It's triangle CFG . Triangle DBE is smaller than $\triangle ABC$, so no sequence of rigid transformations can take $\triangle ABC$ to triangle DBE .
2. Answers vary. Here are two possible sequences:
 - Translate triangle ABC 7 units right so that B matches up with F . Then rotate 90 degrees clockwise around F .
 - Rotate triangle ABC 90 degrees counterclockwise around point C , and then rotate 180 degrees around the midpoint of M of segment CG .

Are You Ready for More?

A square is made up of an L-shaped region and three transformations of the region. If the perimeter of the square is 40 units, what is the perimeter of each L-shaped region?



Student Response

25 units.

Activity Synthesis

Ask a student to explain why triangle ABC *cannot* be taken to triangle DBE . (We are only using rigid transformations and therefore the corresponding lengths have to be equal and they are not.) If a student brings up that they think triangle DBE is a scale drawing of ABC , bring the discussion back to translations, rotations, and reflections, rather than talking about how or why triangle DBE isn't actually a scale drawing of ABC .

Offer as many methods for transforming triangle ABC as possible as time permits, selecting previously identified students to share their methods. Include at least two different sequences of transformations. Make sure students attend carefully to specifying each transformation with the necessary level of precision. For example, for a rotation, that they specify the center of rotation, the direction, and the angle of rotation.

If time allows, consider asking the following questions:

- "Can triangle ABC be taken to triangle CFG with only a translation?" (No, since CFG is rotated.)
- "What about with only a reflection?" (No, because they have the same orientation.)
- "What about with a single rotation?" (The answer is yes, but this question does not need to be answered now as students will have an opportunity to investigate this further in a future lesson.)

Lesson Synthesis

Remind students that a **rigid transformation** is a transformation for which all pairs of **corresponding** lengths and angle measures in the original figure and its image are equal. Translations, rotations, and reflections have this property, so they are rigid transformations. Sequences of these are as well—for example, if you translate a figure then reflect the image, the side lengths and angle measures stay the same.

Ask students to think of ways they could look at two shapes and tell that one is not the image of the other under a rigid transformation. Give a moment of quiet think time, and then invite students to share their ideas (If two shapes have different side lengths or angle measures then there is no rigid transformation taking one shape to the other).

When there is a rigid transformation taking one figure to another, there are many ways to do this. Ask students:

- "What are some good ways to tell whether one shape can be taken to another with a sequence of rigid transformations?" (Measure all of the side lengths and angle measure and ensure that corresponding measurements are equal. Use tracing paper to see if one shape matches up exactly with the other.)
- "What are the three basic types of rigid transformations?" (rotations, translations, and reflections)

6.4 Translated Trapezoid

Cool Down: 5 minutes

Students use key defining properties of rigid motions, namely that they preserve side lengths and angle measures, in order to calculate side lengths and angle measures in a polygon and its image under a rigid transformation.

Addressing

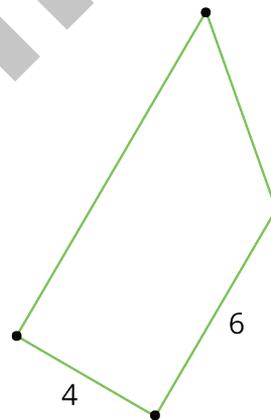
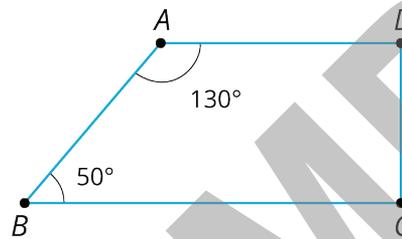
- 8.G.A.1.a
- 8.G.A.1.b

Launch

Provide access to a geometry toolkit.

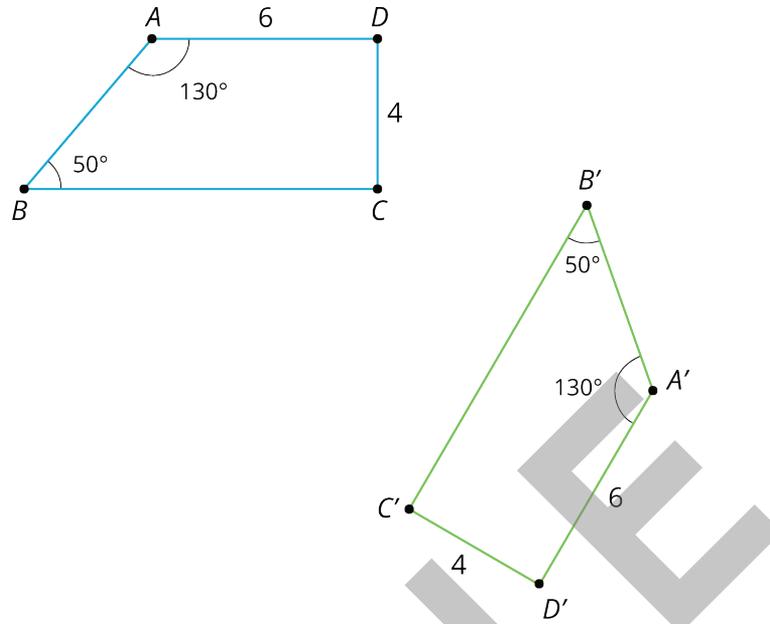
Student Task Statement

Trapezoid $A'B'C'D'$ is the image of trapezoid $ABCD$ under a rigid transformation.



1. Label all vertices on trapezoid $A'B'C'D'$.
2. On both figures, label all known side lengths and angle measures.

Student Response

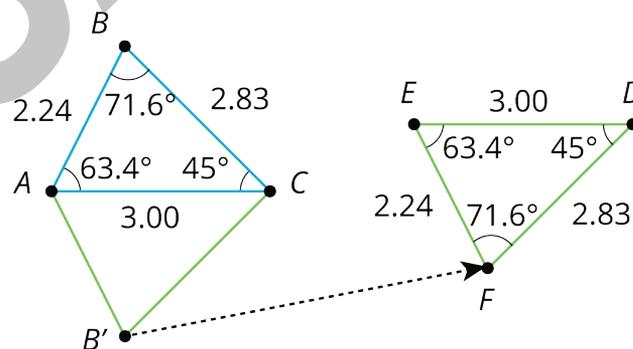


Student Lesson Summary

The transformations we've learned about so far, translations, rotations, reflections, and sequences of these motions, are all examples of **rigid transformations**. A rigid transformation is a move that doesn't change measurements on any figure.

Earlier, we learned that a figure and its image have corresponding points. With a rigid transformation, figures like polygons also have **corresponding sides** and corresponding angles. These corresponding parts have the same measurements.

For example, triangle EFD was made by reflecting triangle ABC across a horizontal line, then translating. Corresponding sides have the same lengths, and corresponding angles have the same measures.



measurements in triangle ABC	corresponding measurements in image EFD
$AB = 2.24$	$EF = 2.24$
$BC = 2.83$	$FD = 2.83$
$CA = 3.00$	$DE = 3.00$
$m\angle ABC = 71.6^\circ$	$m\angle EFD = 71.6^\circ$
$m\angle BCA = 45.0^\circ$	$m\angle FDE = 45.0^\circ$
$m\angle CAB = 63.4^\circ$	$m\angle DEF = 63.4^\circ$

Glossary

- corresponding
- rigid transformation

Lesson 6 Practice Problems

Problem 1

Statement

Is there a rigid transformation taking Rhombus P to Rhombus Q? Explain how you know.



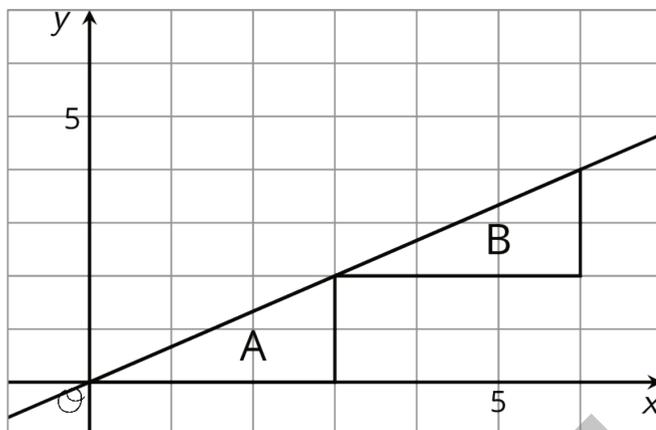
Solution

No, because the angle measures of the two polygons are different, and a rigid transformation must preserve all lengths and angle measures.

Problem 2

Statement

Describe a rigid transformation that takes Triangle A to Triangle B.



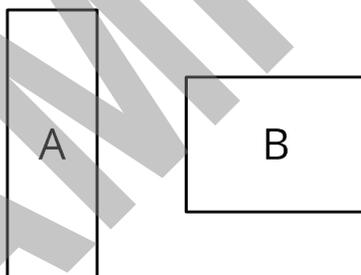
Solution

Translate three units right and two units up.

Problem 3

Statement

Is there a rigid transformation taking Rectangle A to Rectangle B? Explain how you know.



Solution

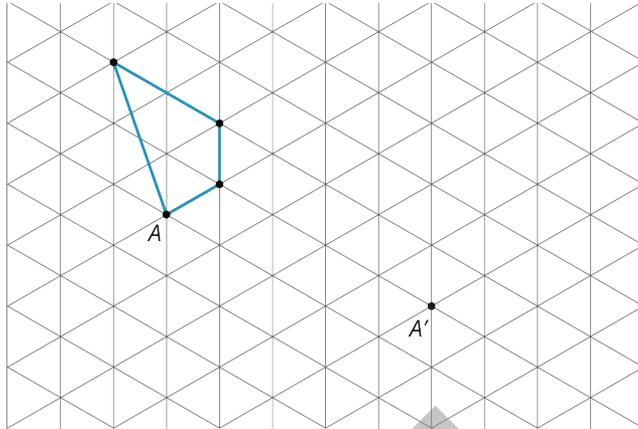
No, because the side lengths of the two rectangles are different, and a rigid transformation must preserve all lengths and angle measures.

Problem 4

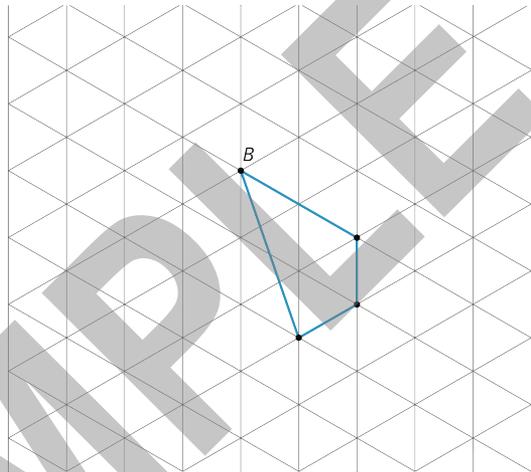
Statement

For each shape, draw its image after performing the transformation. If you get stuck, consider using tracing paper.

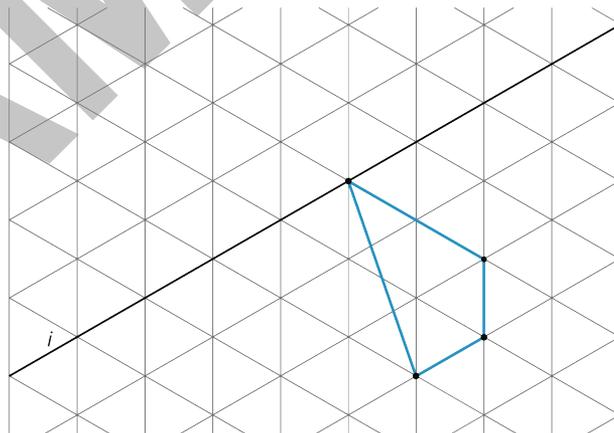
- a. Translate the shape so that A goes to A' .



- a. Rotate the shape 180 degrees counterclockwise around B .

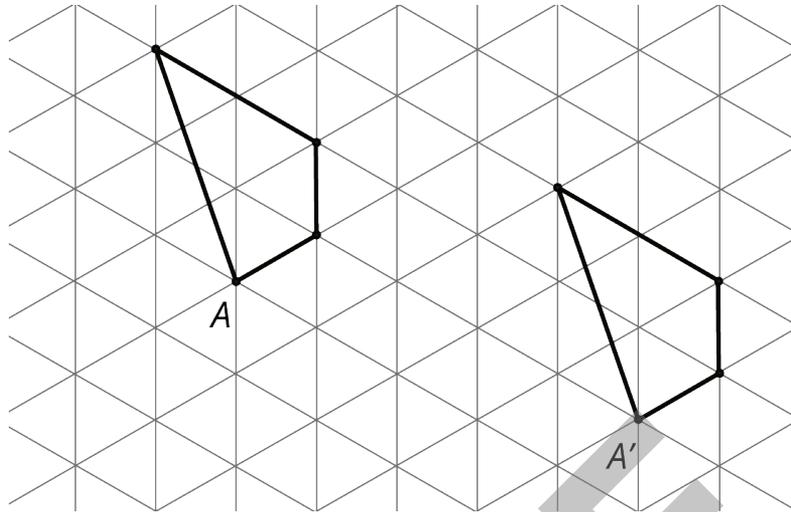


- a. Reflect the shape over the line shown.

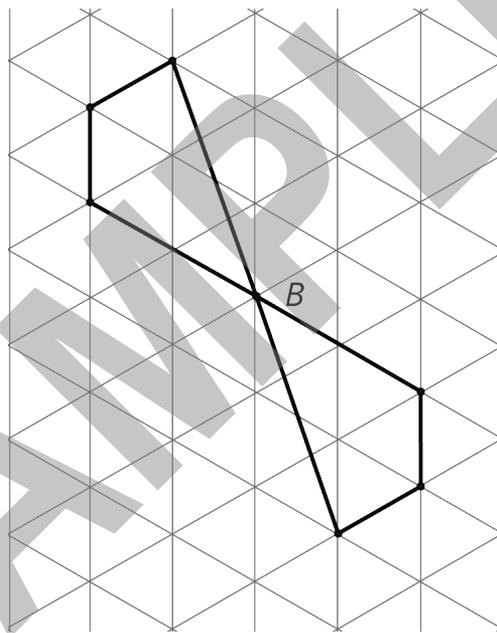


Solution

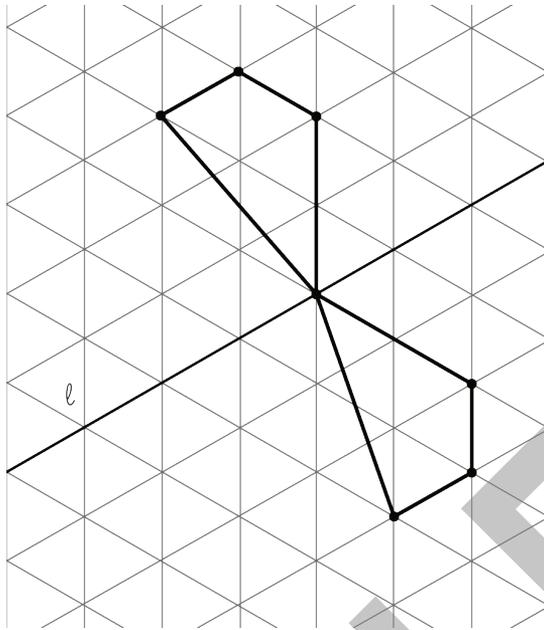
- a.



b.



c.



(From Unit 1, Lesson 3.)

SAMPLE

Lesson 7: Rotation Patterns

Goals

- Draw and label rotations of 180 degrees of a line segment from centers of the midpoint, a point on the segment, and a point not on the segment.
- Generalize (orally and in writing) the outcome when rotating a line segment 180 degrees.
- Identify (orally and in writing) the rigid transformations that can build a diagram from one starting figure.

Learning Targets

- I can describe how to move one part of a figure to another using a rigid transformation.

Lesson Narrative

In this lesson, rigid transformations are applied to line segments and triangles. For line segments, students examine the impact of a 180 degree rotation. This is important preparatory work for studying parallel lines and rigid transformations, the topic of the next lesson. For triangles students look at a variety of transformations where rotations of 90 degrees and 180 degrees are again a focus. This work and the patterns that students build will be important later when they study the Pythagorean Theorem.

Throughout the lesson, students use the properties of rigid transformations (they do not change distances or angles) in order to make conclusions about the objects they are transforming (MP7).

Alignments

Addressing

- 8.G.A.1.a: Lines are taken to lines, and line segments to line segments of the same length.
- 8.G.A.1.b: Angles are taken to angles of the same measure.

Building Towards

- 8.G.A.1.c: Parallel lines are taken to parallel lines.

Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's rotate figures in a plane.

7.1 Building a Quadrilateral

Warm Up: 5 minutes

Students rotate a copy of a right isosceles triangle four times to build a quadrilateral. It turns out that the quadrilateral is a square. Students are not asked or expected to justify this but it can be addressed in the discussion. The fourth question about rotational symmetry of the quadrilateral will help students conclude that it is a square.

There are many more opportunities to build figures using rigid transformations in other lessons.

Addressing

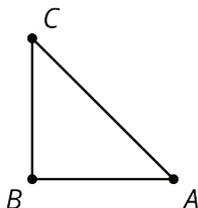
- 8.G.A.1.a
- 8.G.A.1.b

Launch

Provide access to geometry toolkits, particularly tracing paper.

Student Task Statement

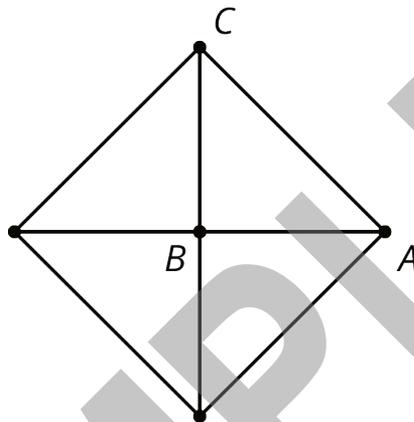
Here is a right isosceles triangle:



1. Rotate triangle ABC 90 degrees clockwise around B .
2. Rotate triangle ABC 180 degrees clockwise round B .
3. Rotate triangle ABC 270 degrees clockwise around B .
4. What would it look like when you rotate the four triangles 90 degrees clockwise around B ? 180 degrees? 270 degrees clockwise?

Student Response

1–3.



4. The overall figure would look the same. These rotations just interchange the 4 triangles.

Activity Synthesis

Ask students what they notice and wonder about the quadrilateral that they have built. Likely responses include:

- It looks like a square.
- Rotating it 90 degrees clockwise or counterclockwise interchanges the 4 copies of triangle ABC .
- Continuing the pattern of rotations, the next one will put ABC back in its original position.

Ask the students how they know the four triangles fit together without gaps or overlaps to make a quadrilateral. Here the key point is that the triangle is isosceles, so the rotations match up these sides perfectly. The four right angles make a complete 360 degrees, so the shape really is a quadrilateral. The fact that the quadrilateral is a square can be deduced from the fact that it is mapped to itself by a 90 degree rotation, but this does not need to be stressed or addressed.

7.2 Rotating a Segment

15 minutes (there is a digital version of this activity)

The purpose of this activity is to allow students to explore special cases of rotating a line segment 180° . In general, rotating a segment 180° produces a parallel segment the same length as the original. This activity also treats two special cases:

- When the center of rotation is the midpoint, the rotated segment is the same segment as the original, except the vertices are switched.
- When the center of rotation is an endpoint, the segment together with its image form a segment twice as long as the original.

As students look to make general statements about what happens when a line segment is rotated 180° they engage in MP8. They are experimenting with a particular line segment but the conclusions that they make, especially in the last problem, are for *any* line segment.

Watch for how students explain that the 180° rotation of segment CD in the second part of the question is parallel to CD . Some students may say that they “look parallel” while others might try to reason using the structure of the grid. Tell them that they will investigate this further in the next lesson.

Addressing

- 8.G.A.1.a

Building Towards

- 8.G.A.1.c

Instructional Routines

- MLR8: Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to geometry toolkits. Give 3 minutes of quiet work time, followed by sharing with a partner and a whole-class discussion.

Support for Students with Disabilities

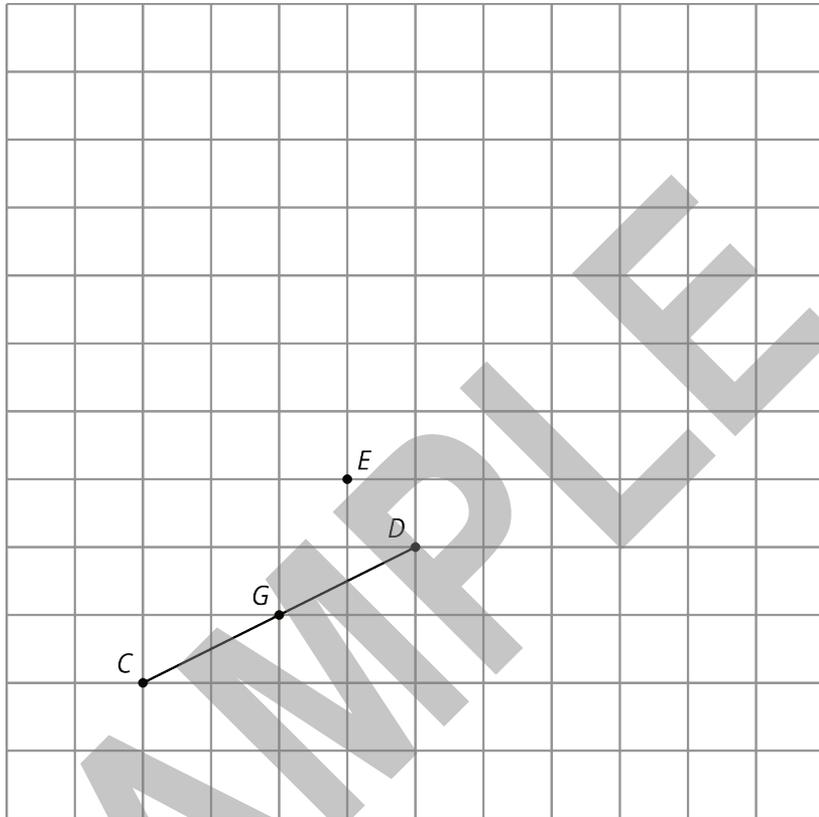
Representation: Internalize Comprehension. Begin the activity with concrete or familiar contexts. As in previous lessons, use tracing paper or digital software to rotate a segment 180 degrees around a point. Lead the class in a think aloud considering the point of rotation to be the midpoint or endpoint of the segment as an entry point for this activity.

Supports accessibility for: Conceptual processing; Memory

Anticipated Misconceptions

Students may be confused when rotating around the midpoint because they think the image cannot be the same segment as the original. Assure students this can occur and highlight that point in the discussion.

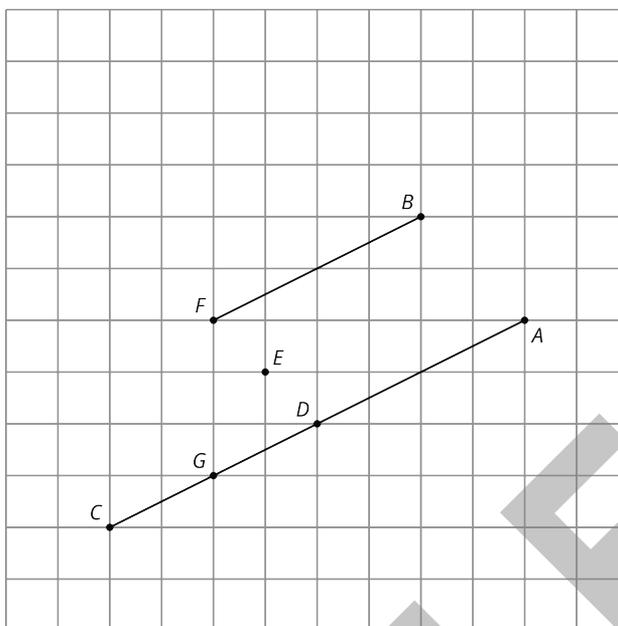
Student Task Statement



1. Rotate segment CD 180 degrees around point D . Draw its image and label the image of C as A .
2. Rotate segment CD 180 degrees around point E . Draw its image and label the image of C as B and the image of D as F .
3. Rotate segment CD 180 degrees around its midpoint, G . What is the image of C ?
4. What happens when you rotate a segment 180 degrees around a point?

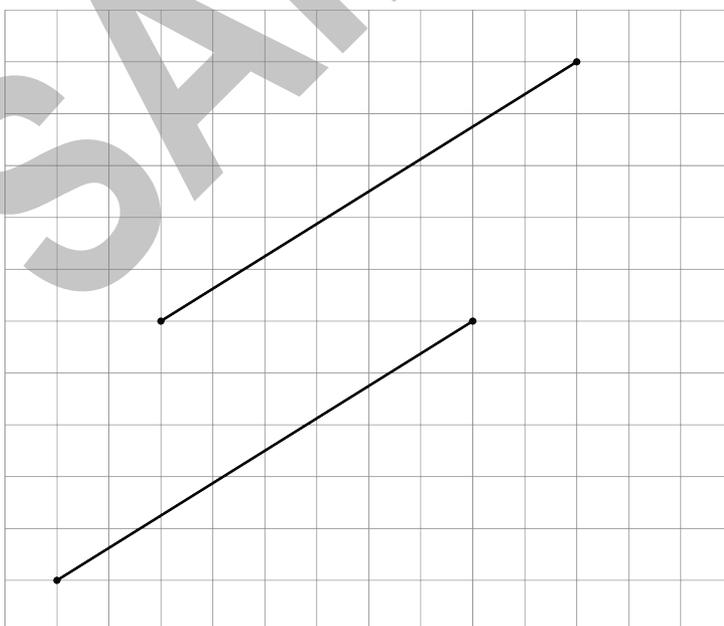
Student Response

1.



- The image of the segment lines up with itself, but the endpoints are switched. D is now where C was and C is where D was.
- The new segment may change its location, but it remains the same length. The new segment is parallel to the original segment. When the point of rotation is the midpoint of the segment, then the rotated segment is the same as the original (the endpoints trade places) and when the point of rotation is an end point of the segment, the image connects to the original to form a segment twice as long.

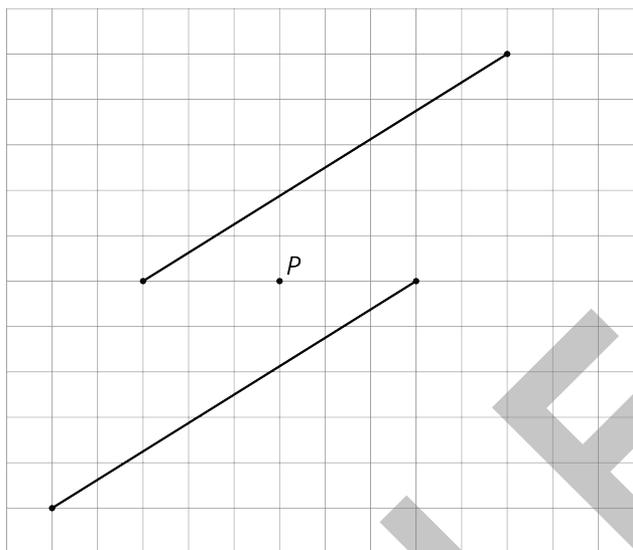
Are You Ready for More?



Here are two line segments. Is it possible to rotate one line segment to the other? If so, find the center of such a rotation. If not, explain why not.

Student Response

Yes



Activity Synthesis

Ask students why it is not necessary to specify the direction of a 180 degree rotation (because a 180 degree clockwise rotation around point P has the same effect as a 180 degree counterclockwise rotation around P). Invite groups to share their responses. Ask the class if they agree or disagree with each response. When there is a disagreement, have students discuss possible reasons for the differences.

Three important ideas that emerge in the discussion are:

- Rotating a segment 180° around a point that is not on the original line segment produces a parallel segment the same length as the original.
- When the center of rotation is the midpoint, the rotated segment is the same segment as the original, except the vertices are switched.
- When the center of rotation is an endpoint, the segment together with its image form a segment twice as long.

If any of the ideas above are not brought up by the students during the class discussion, be sure to make them known.

All of these ideas can be emphasized dynamically by carrying out a specified rotation in the applet and then moving the center of rotation or an endpoint of the original line segment. Even if students are not using the digital version of the activity, you may want to display and demonstrate with the applet.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to support whole-class discussion when students discuss whether it is necessary to specify the direction of a 180 degree rotation. After a student speaks, call on students to restate and/or revoice what was shared using mathematical language (e.g., rotation, line segment, midpoint, etc.). This will provide more students with an opportunity to produce language as they explore special cases of rotating a line segment 180°.

Design Principle(s): Support sense-making; Maximize meta-awareness

7.3 A Pattern of Four Triangles

10 minutes (there is a digital version of this activity)

In this activity, students use rotations to build a pattern of triangles. In the previous lesson, students examined a right triangle and a rigid transformation of the triangle. In this activity, several rigid transformations of the triangle form an interesting pattern.

Triangle ABC can be mapped to each of the three other triangles in the pattern with a single rotation. As students work on the first three questions, watch for any students who see that a single rotation can take triangle ABC to CDE . The center for the rotation is not drawn in the diagram: it is the intersection of segment AE and segment CG . For students who finish early, guide them to look for a *single* transformation taking ABC to each of the other triangles.

This pattern will play an important role later when students use this shape to understand a proof of the Pythagorean Theorem.

Identify students who notice that they have already solved the first question in an earlier activity. Watch for students who think that $CAGE$ is a square and tell them that this will be addressed in a future lesson. However, encourage them to think about what they conclude about $CAGE$ now. Also watch for students who repeat the same steps to show that ABC can be mapped to each of the other three triangles.

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students in groups of 2–4. Provide access to geometry toolkits.

If using the digital activity, give students individual work time before allowing them to converse with a partner.

Support for Students with Disabilities

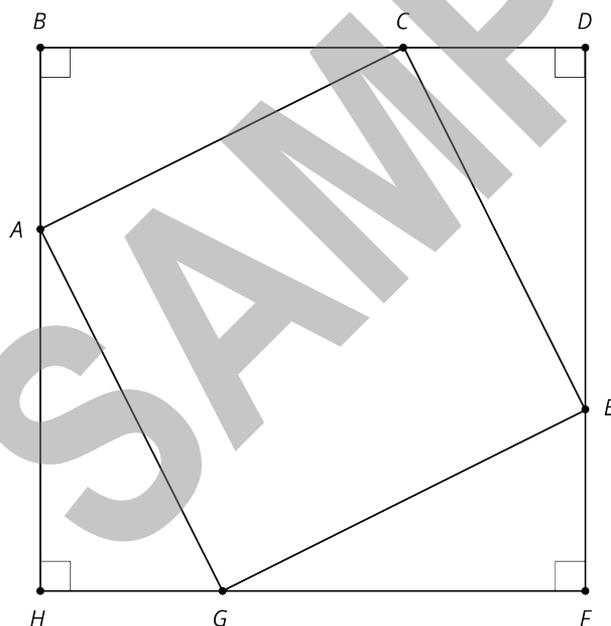
Representation: Internalize Comprehension. Demonstrate and encourage students to use color coding and annotations to draw students' attention to the triangles in the diagram and the corresponding question. For example, highlight and isolate triangles ABC and CDE to help make connections to work in the previous activity.

Supports accessibility for: Visual-spatial processing

Anticipated Misconceptions

Some students might not recognize how this work is similar to the previous activity. For these students, ask them to step back and consider only triangles ABC and CDE , perhaps covering the bottom half of the diagram.

Student Task Statement



You can use rigid transformations of a figure to make patterns. Here is a diagram built with three different transformations of triangle ABC .

1. Describe a rigid transformation that takes triangle ABC to triangle CDE .
2. Describe a rigid transformation that takes triangle ABC to triangle EFG .
3. Describe a rigid transformation that takes triangle ABC to triangle GHA .
4. Do segments AC , CE , EG , and GA all have the same length? Explain your reasoning.

Student Response

1. Answers vary. Sample responses:

- Translate point B to point D , then rotate 90 degrees clockwise using D as center.
- Rotate counterclockwise using C as center until segment CA matches up perfectly with segment CE , then rotate 180 degrees using the midpoint of segment CE as center.

1. Answers vary. Sample responses:

- Translate B to F and then rotate 180 degrees with center F .
- Translate so segment AC matches up with segment GE and then rotate 180 degrees with the midpoint of segment GE as center of rotation.

1. Answers vary. Sample responses:

- Translate B to H and then rotate 90 degrees counterclockwise with center H .
- Rotate with center A so that segment AC matches up with segment AG and then rotate 180 degrees with the midpoint of segment AG as center.

1. Yes, because the size and shape of triangle ABC did not change under the rigid transformation. Segment AC can be matched up exactly with segments CE , EG , and GA so the lengths of these segments are all the same.

Activity Synthesis

Select a student previously identified who noticed how the first question relates to a previous activity to share their observation. Discuss here how previous work can be helpful in new work, since students may not be actively looking for these connections. The next questions are like the first, but the triangles have a different orientation and different transformations are needed.

Discuss rigid transformations. Focus especially on the question about lengths. A key concept in this section is the idea that lengths and angle measures are preserved under rigid transformations.

Some students may claim $CAGE$ is a square. If this comes up, leave it as an open question for now. This question will be revisited at the end of this unit, once the angle sum in a triangle is known. The last question establishes that $CAGE$ is a rhombus.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Give students additional time to make sure that everyone in their group can explain whether the segments AC , CE , EG , and GA all have the same lengths.

Then, vary who is called on to represent the ideas of each group. This routine will prepare students for the role of group representative and to support each other to take on that role.

Design Principle(s): Optimize output (for explanation)

Lesson Synthesis

Ask students to describe the possible outcomes when a line segment AB is rotated 180 degrees.

- AB is mapped to itself, when the center of rotation is the midpoint of the segment
- AB is mapped to another segment collinear with the first, when the center of rotation is A or B (or any other point on segment AB)
- AB is mapped to a parallel segment, when the center of rotation is not on line AB .

7.4 Is it a rotation?

Cool Down: 5 minutes

Having studied rotations in detail throughout this lesson, students look at a triangle and its image after a rigid transformation. They decide whether or not one is a rotation of the other. It turns out to be a reflection rather than a rotation. Students can use tracing paper to verify their conjectures, but at this point they should start to have an intuition for the effects of a rotation versus a reflection.

Addressing

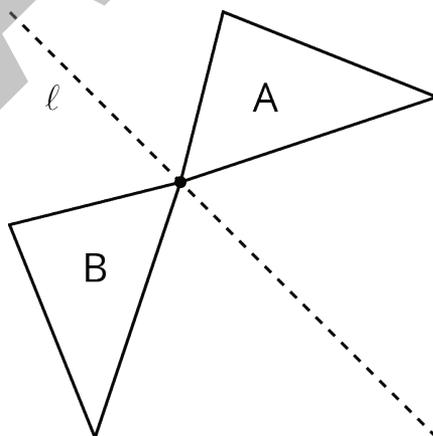
- 8.G.A.1.a
- 8.G.A.1.b

Launch

Make tracing paper available.

Student Task Statement

Here are two triangles.



Is Triangle B a rotation of Triangle A? Explain your reasoning.

Student Response

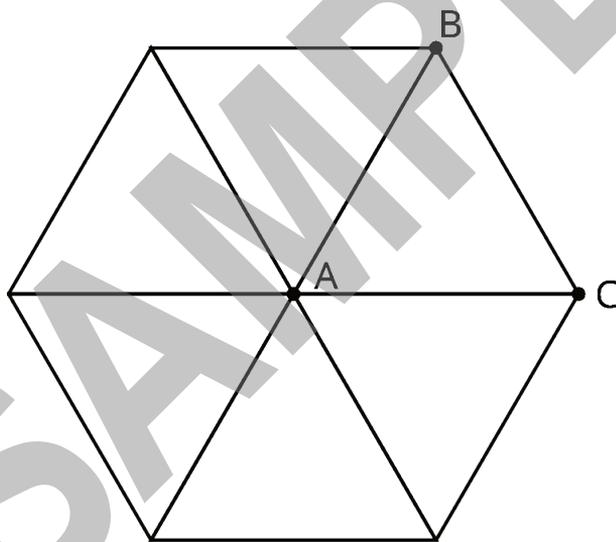
No, Triangle B is a reflection of Triangle A over line ℓ . A rotation can be used to match two sides of the triangles but will not match one up perfectly with the other.

Student Lesson Summary

When we apply a 180-degree rotation to a line segment, there are several possible outcomes:

- The segment maps to itself (if the center of rotation is the midpoint of the segment).
- The image of the segment overlaps with the segment and lies on the same line (if the center of rotation is a point on the segment).
- The image of the segment does not overlap with the segment (if the center of rotation is *not* on the segment).

We can also build patterns by rotating a shape. For example, triangle ABC shown here has $m(\angle A) = 60$. If we rotate triangle ABC 60 degrees, 120 degrees, 180 degrees, 240 degrees, and 300 degrees clockwise, we can build a hexagon.



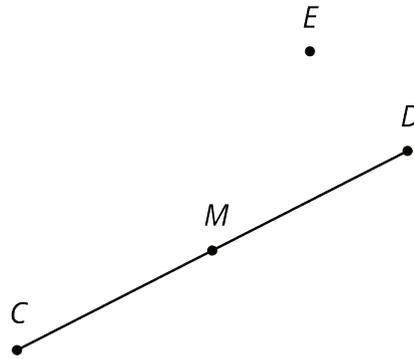
Lesson 7 Practice Problems

Problem 1

Statement

For the figure shown here,

- Rotate segment CD 180° around point D .
- Rotate segment CD 180° around point E .
- Rotate segment CD 180° around point M .



Solution

- The segment is attached at point D and is an extension of segment CD .
- The segment is above point E and is parallel to segment CD .
- The segment is identical to segment CD .

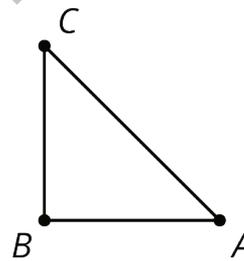
Problem 2

Statement

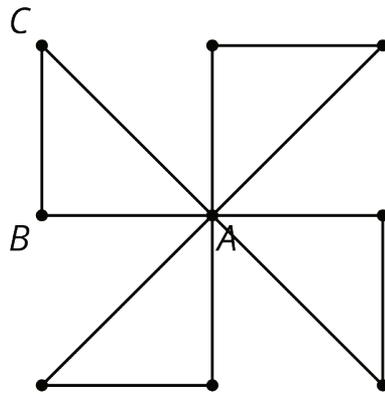
Here is an isosceles right triangle:

Draw these three rotations of triangle ABC together.

- Rotate triangle ABC 90° degrees clockwise around A .
- Rotate triangle ABC 180° degrees around A .
- Rotate triangle ABC 270° degrees clockwise around A .



Solution

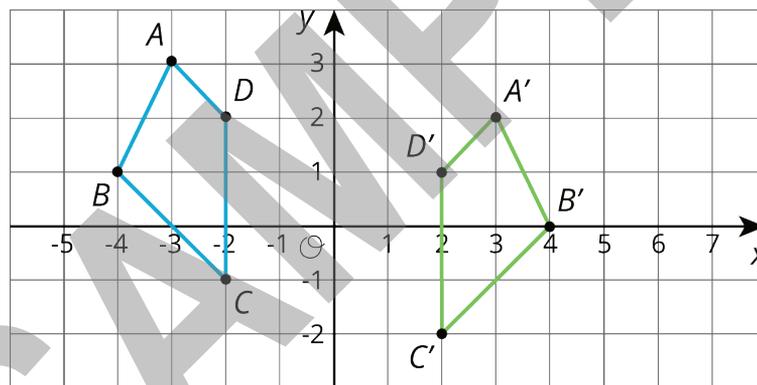


Problem 3

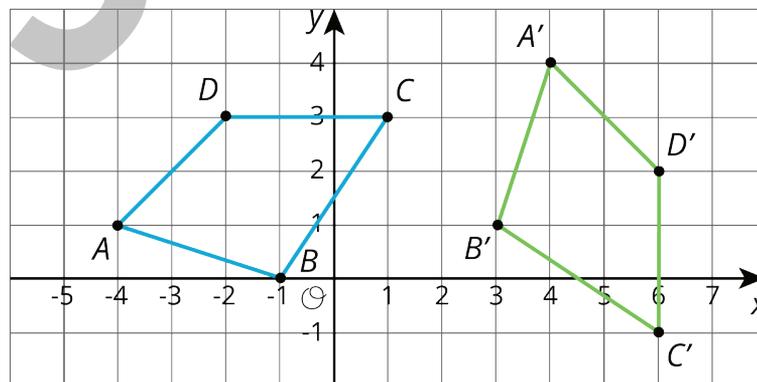
Statement

Each graph shows two polygons $ABCD$ and $A'B'C'D'$. In each case, describe a sequence of transformations that takes $ABCD$ to $A'B'C'D'$.

a.



b.



Solution

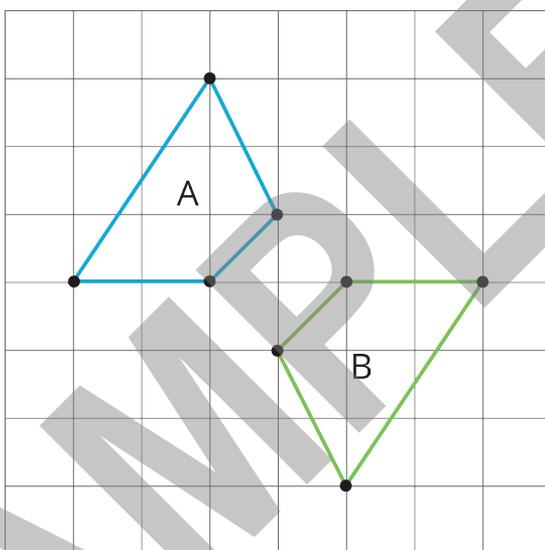
- Reflect $ABCD$ over the y -axis, and then translate down 1.
- Rotate $ABCD$ 90 degrees clockwise with center $B = (-1, 0)$, and then translate $(-1, 0)$ to $(3, 1)$.

(From Unit 1, Lesson 4.)

Problem 4

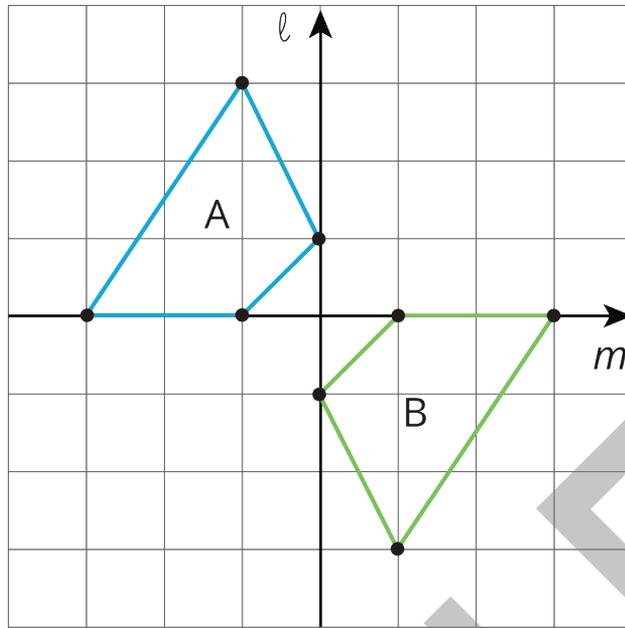
Statement

Lin says that she can map Polygon A to Polygon B using *only* reflections. Do you agree with Lin? Explain your reasoning.



Solution

I agree with Lin. If Polygon A is reflected first over the vertical line ℓ and then over the horizontal line m , this takes Polygon A to Polygon B.



(From Unit 1, Lesson 3.)

SAMPLE

Lesson 8: Moves in Parallel

Goals

- Comprehend that a rotation by 180 degrees about a point of two intersecting lines moves each angle to the angle that is vertical to it.
- Describe (orally and in writing) observations of lines and parallel lines under rigid transformations, including lines that are taken to lines and parallel lines that are taken to parallel lines.
- Draw and label rigid transformations of a line and explain the relationship between a line and its image under the transformation.
- Generalize (orally) that “vertical angles” are congruent using informal arguments about 180 degree rotations of lines.

Learning Targets

- I can describe the effects of a rigid transformation on a pair of parallel lines.
- If I have a pair of vertical angles and know the angle measure of one of them, I can find the angle measure of the other.

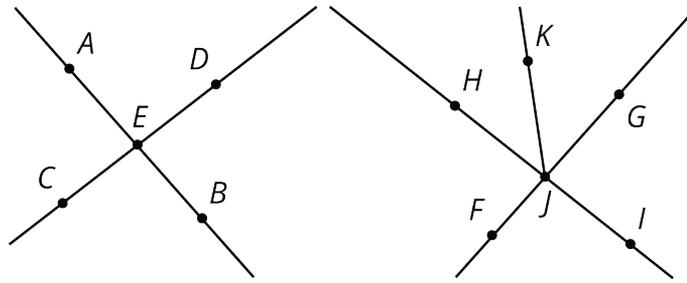
Lesson Narrative

The previous lesson examines the impact of rotations on line segments and polygons. This lesson focuses on the effects of rigid transformations on lines. In particular, students see that parallel lines are taken to parallel lines and that a 180° rotation about a point on the line takes the line to itself. In grade 7, students found that **vertical angles** have the same measure, and they justify that here using a 180° rotation.

As they investigate how 180° rotations influence parallel lines and intersecting lines, students are looking at specific examples but their conclusions hold for all pairs of parallel or intersecting lines. No special properties of the two intersecting lines are used so the 180° rotation will show that vertical angles have the same measure for *any* pair of vertical angles.

Teacher Notes for IM 6–8 Math Accelerated

This lesson is the first time students see the term **vertical angles** in IM 6–8 Math Accelerated. They learn that vertical angles are two angles with the same measure formed by two intersecting lines and use transformations to understand why these pairs of angles have the same measure. If students need additional practice identifying vertical angles, use the lesson synthesis to display this image and ask students to identify four pairs of vertical angles. In particular, students may have trouble seeing that angles FJI and HJG are vertical angles.



Alignments

Building On

- 7.G.B.5: Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

Addressing

- 8.G.A.1.a: Lines are taken to lines, and line segments to line segments of the same length.
- 8.G.A.1.b: Angles are taken to angles of the same measure.
- 8.G.A.1.c: Parallel lines are taken to parallel lines.

Instructional Routines

- MLR7: Compare and Connect

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's transform some lines.

8.1 Line Moves

Warm Up: 10 minutes

In this warm-up, students continue their work with transformations by shifting from applying rigid transformations to shapes to applying them specifically to lines. Each image in this activity has the same starting line and students are asked to name the translation, rotation, or reflection that takes this line to the second marked line. Because of their infinite and symmetric nature, different

transformations of lines look the same unless specific points are marked, so 1–2 points on each line are marked.

While students have experience transforming a variety of figures, this activity provides the opportunity to use precise language when describing transformations of lines while exploring how sometimes different transformations can result in the same final figures. During the activity, encourage students to look for more than one way to transform the original line.

Addressing

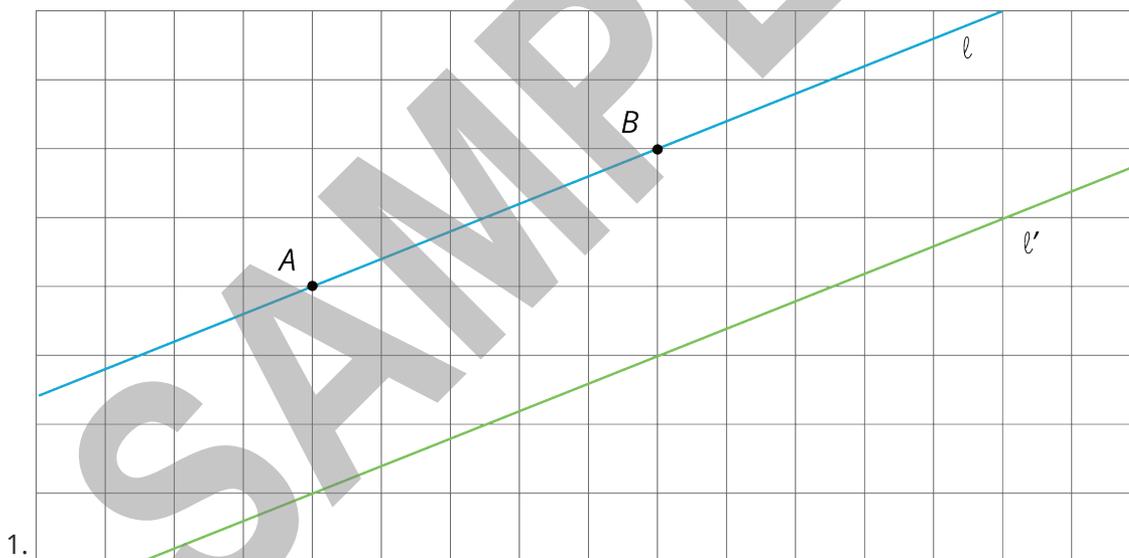
- 8.G.A.1.a

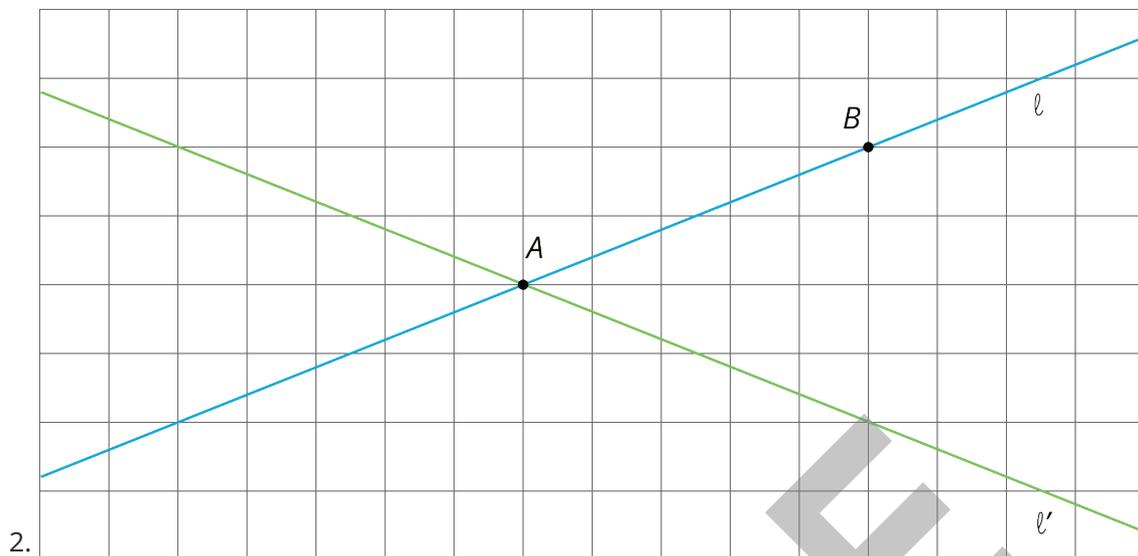
Launch

Provide access to tracing paper. Give students 2 minutes of quiet work time followed by whole-class discussion.

Student Task Statement

For each diagram, describe a translation, rotation, or reflection that takes line ℓ to line ℓ' . Then plot and label A' and B' , the images of A and B .





Student Response

1. Answers vary. Possible responses:
 - Translation in many possible directions, for example, down 3 units
 - Reflection over a line parallel to l halfway between l and l'
 - Rotation using a point halfway between l and l' as the center of rotation and an angle of 180°
2. Answers vary. Possible responses:
 - Reflection across the vertical line through point A
 - Reflection across the horizontal line through point A
 - Counterclockwise rotation about point A by the obtuse angle whose vertex is at A
 - Clockwise rotation about point A by the acute angle whose vertex is at A

Activity Synthesis

Invite students to share the transformations they choose for each problem. Each diagram has more than one possible transformation that would result in the final figure. If the class only found one, pause for 2–3 minutes and encourage students to see if they can find another. For the first diagram, look for a single translation, single rotation, and single reflection that work. For the second diagram, look for a single rotation and a single reflection.

- "Will a translation work for the second diagram? Explain your reasoning." (A translation will not work. Since translations do not incorporate a turn, translations of a line are parallel to the original line or are the same line.)

8.2 Parallel Lines

15 minutes

In this activity, students will investigate the question, “What happens to parallel lines under rigid transformations?” by performing three different transformations on a set of parallel lines. After applying each transformation, they will jot down what they notice by answering the questions for each listed transformation.

As students work through these problems they may remember essential features of parallel lines (they do not meet, they remain the same distance apart). Rigid transformations do not change either of these features which means that the image of a set of parallel lines after a rigid transformation is another set of parallel lines (MP7).

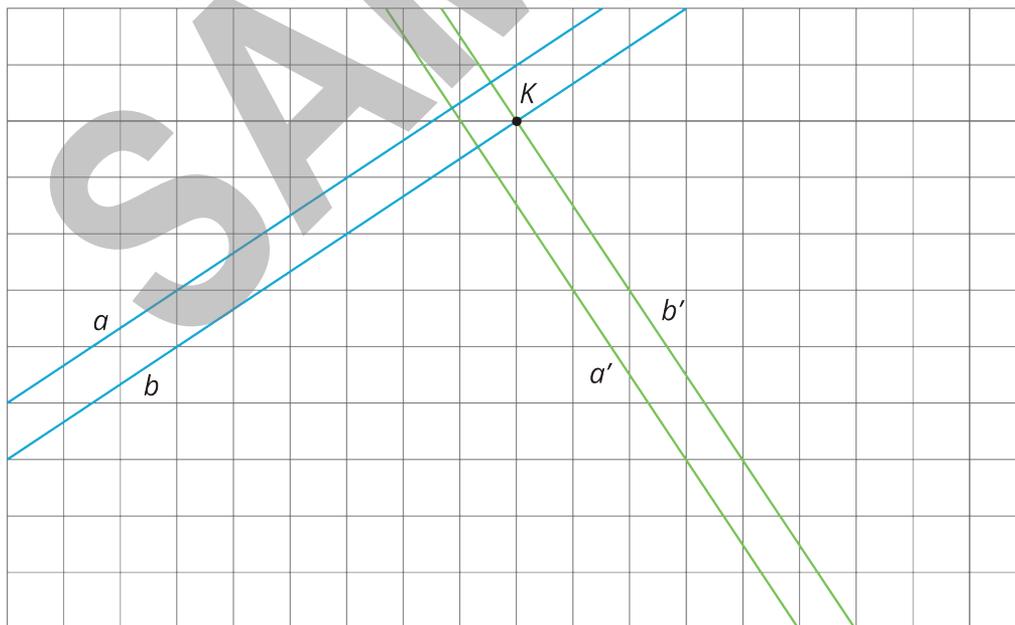
Identify the students who saw that the orientation of the lines changes but the lines remain parallel to each other regardless and select them to share during the discussion.

Addressing

- 8.G.A.1.c

Launch

Before beginning, review with students what happens when we perform a rigid transformation. Demonstrate by moving the tracing paper on top of the image to replicate an example transformation (for example, rotation of the lines clockwise 90° around the center K). Tell students that the purpose of this activity is to investigate, “What happens to parallel lines when we perform rigid transformations on them?”



Arrange students in groups of 3. Provide access to tracing paper. Each student in the group does *one* of the problems and then the group discusses their findings.

Support for Students with Disabilities

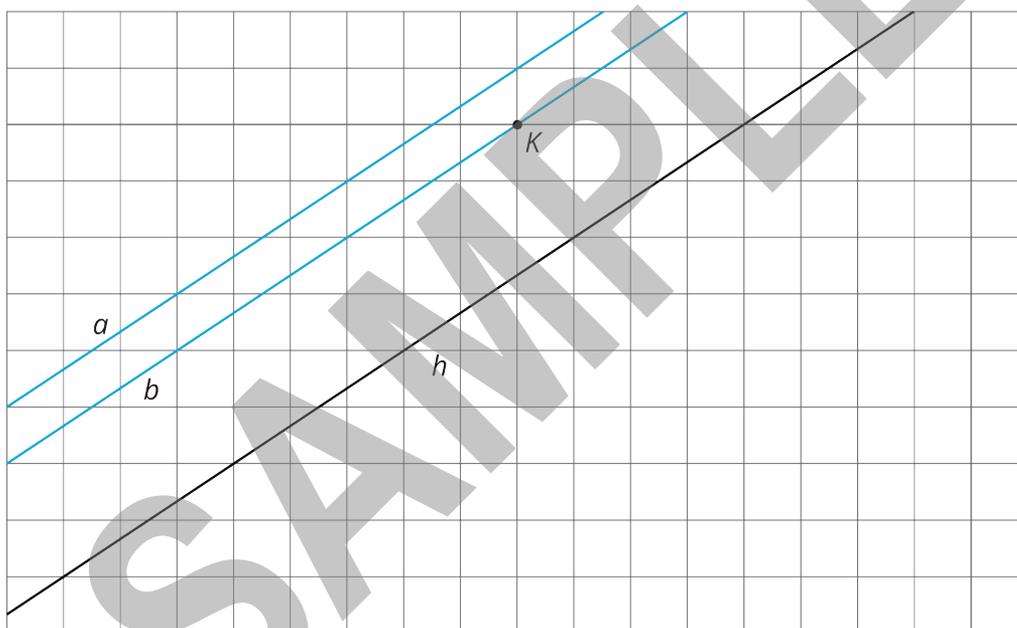
Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organizational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.

Supports accessibility for: Organization; Attention

Anticipated Misconceptions

Students may not perform the transformations on top of the original image. Ask these students to place the traced lines over the original and perform each transformation from there.

Student Task Statement



Use a piece of tracing paper to trace lines a and b and point K . Then use that tracing paper to draw the images of the lines under the three different transformations listed.

As you perform each transformation, think about the question:

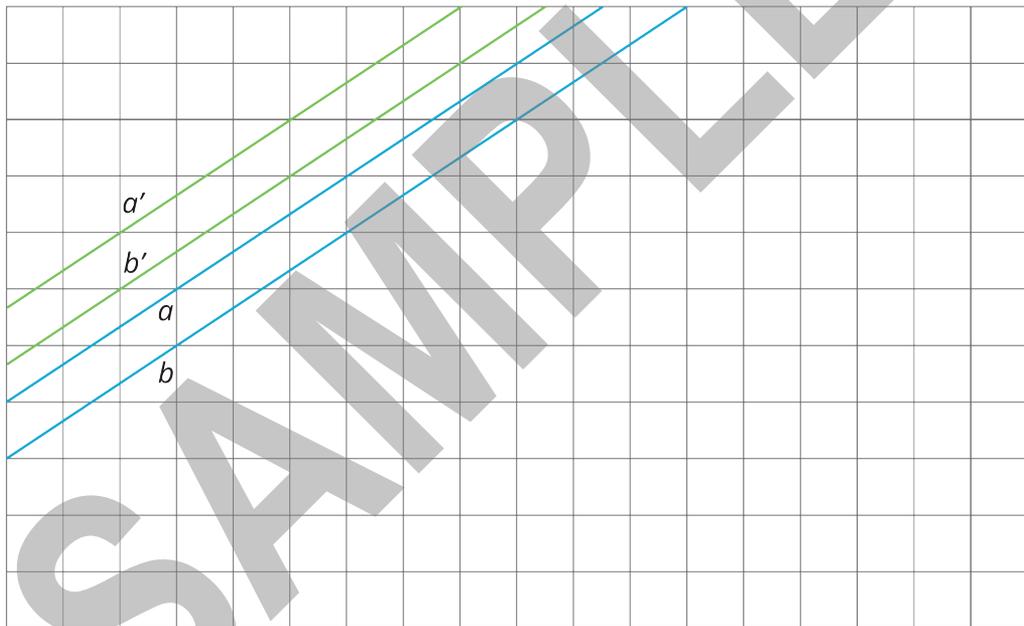
What is the image of two parallel lines under a rigid transformation?

1. Translate lines a and b 3 units up and 2 units to the right.
 - a. What do you notice about the changes that occur to lines a and b after the translation?
 - b. What is the same in the original and the image?

2. Rotate lines a and b counterclockwise 180 degrees using K as the center of rotation.
 - a. What do you notice about the changes that occur to lines a and b after the rotation?
 - b. What is the same in the original and the image?
3. Reflect lines a and b across line h .
 - a. What do you notice about the changes that occur to lines a and b after the reflection?
 - b. What is the same in the original and the image?

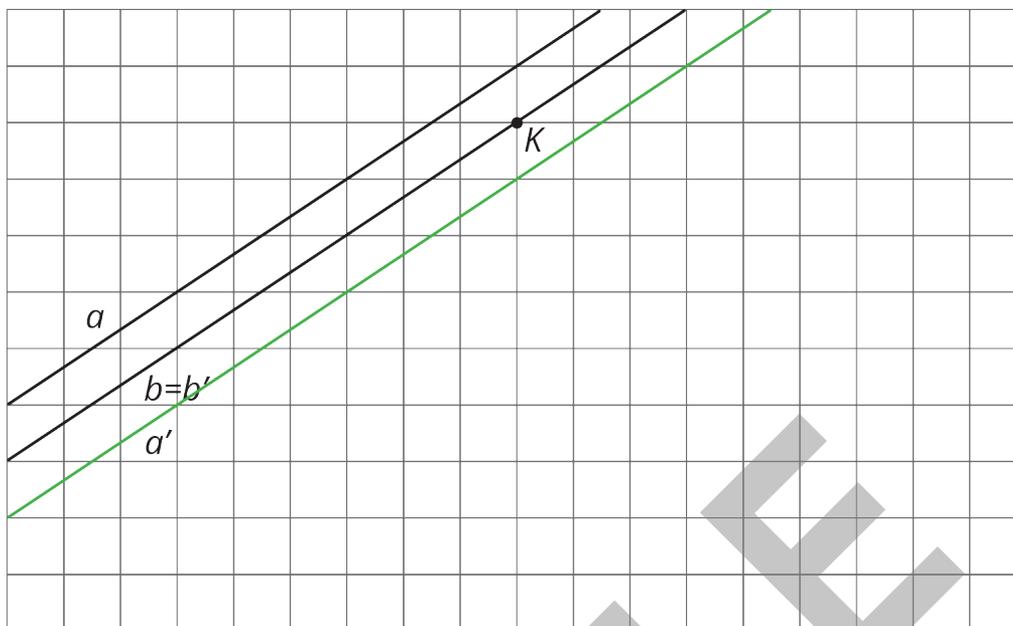
Student Response

1. Translation: 3 grid square units up and 2 grid square units to the right



Answers vary. Sample Responses:

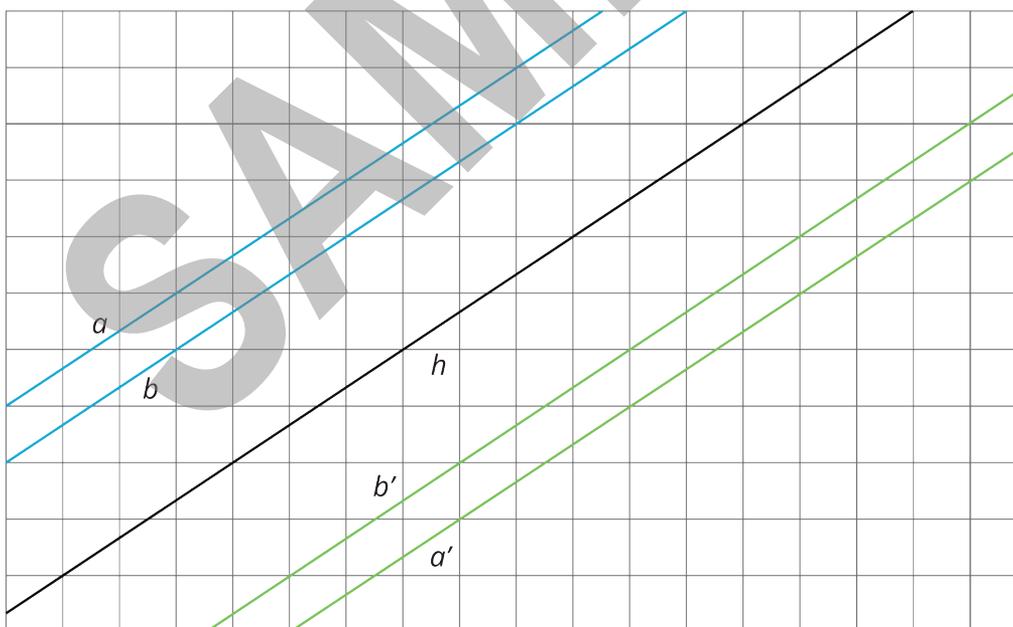
- a. All 4 lines, a , b , a' , and b' are parallel. The lines a' and b' look like a and b but shifted upward.
 - b. The pair of lines remain parallel. The distance between the lines did not change.
2. Rotation around K



Answers vary. Sample responses:

- a. The new pair of lines a' and b' are parallel to the original lines a and b .
- b. The lines a' and b' are still parallel and they are the same distance apart as a and b .

3. Reflection over line h .



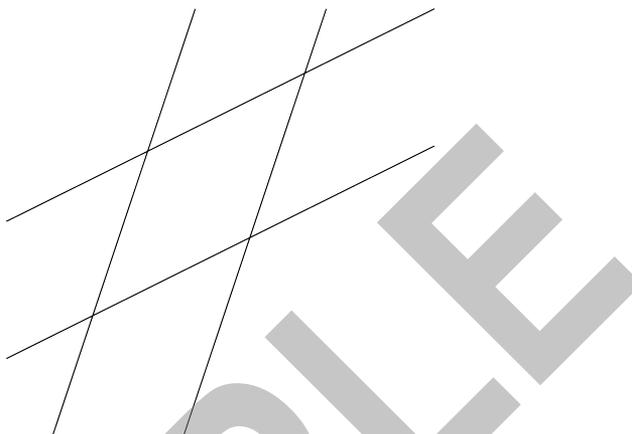
Answers vary. Sample responses:

- a. Line a is above line b whereas line b' is above line a' .

- b. Lines a' and b' are still parallel and are the same distance apart as lines a and b . All four lines are parallel to one another.

Are You Ready for More?

When you rotate two parallel lines, sometimes the two original lines intersect their images and form a quadrilateral. What is the most specific thing you can say about this quadrilateral? Can it be a square? A rhombus? A rectangle that isn't a square? Explain your reasoning.



Student Response

The quadrilateral is always a rhombus. It can be a square if the two pair of parallel lines are perpendicular. It can not be a rectangle that is not a square because the distance between the two sets of parallel lines is the same.

Activity Synthesis

Ask previously selected students who saw that the images of the parallel lines were parallel to the original in all three cases to share how they would answer the main question "What is the image of two parallel lines under a rigid transformation?" Make sure students understand that in general if ℓ and m are parallel lines and ℓ' and m' are their images under a rigid transformation then:

- ℓ' and m' are parallel.
- ℓ and m are *not* necessarily parallel to ℓ' and m' (refer to the 90 degree rotation shown during the launch).

In addition to the fact that the parallel lines remain parallel to each other when rigid transformations are performed, the distance between the lines stay the same. What can change is the position of the lines in the plane, in relative terms (i.e., which line is 'on top') or in absolute terms (i.e., does a line contain a particular point in the plane).

Give students 1–2 minutes of quiet time to write a response to the main question, "What is the image of two parallel lines under a rigid transformation?"

8.3 Let's Do Some 180's

15 minutes

In this activity, students apply their understanding of the properties of rigid transformations to 180° rotations of a line about a point on the line in order to establish the vertical angle theorem.

Students have likely already used this theorem in grade 7, but this lesson informally demonstrates why the theorem is true. The demonstration of the vertical angle theorem exploits the structure of parallel lines and properties of both 180 degree rotations (studied in the previous lesson) and rigid transformations. This lesson is a good example of MP7, investigating the structure of different mathematical objects.

Students begin the activity by rotating a line with marked points 180° about a point on the line. Unlike the whole-class example discussed in the launch, this line contains marked points other than the center of rotation. Then students rotate an angle 180° about a point on the line to draw conclusions about lengths and angles. Finally, students are asked to consider the intersection of two lines, the angles formed, and how the measurements of those angles can be deduced using a 180° rotation about the intersection of the lines, which is the vertical angle theorem.

While students are working, encourage the use of tracing paper to show the transformations directly over the original image in order to help students keep track of what is happening with lines in each 180° rotation.

Teacher Notes for IM 6–8 Math Accelerated

This activity is a student's first introduction to the term **vertical angles**. Instead of asking students, “What does this transformation informally prove?”, display the image from the last question of the activity and use it to define the term **vertical angles** as a pair of angles, formed by two intersecting lines, that are opposite each other. Students will learn the terms “complementary” and “supplementary” for pairs of angles in a future lesson, so disregard discussing those connections at this time if not brought up by students who may already know those terms.

Building On

- 7.G.B.5

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Instructional Routines

- MLR7: Compare and Connect

Launch

Rotations require the students to think about rotating an entire figure. It would be good to remind students about this before the start of this activity. This might help students see what is happening in the first question better.

Before students read the activity, draw a line ℓ with a marked point D for all to see. Ask students to picture what the figure rotated 180° around point D looks like. After a minute of quiet think time, invite students share what they think the transformed figure would look like. Make sure all students agree that ℓ' looks “the same” as the original. If no students bring it up in their explanations, ask for suggestions of features that would make it possible to quickly tell the difference between the ℓ' and ℓ , such as another point or if the line were different colors on each side of point D .

Provide access to tracing paper.

Support for Students with Disabilities

Engagement: Internalize Self Regulation. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. For example, to get students started, display only the first problem and diagram. Once students have successfully completed the three parts, introduce the remaining problems, one at a time. Invite students to think aloud about the relationships between the angles after the 180° rotation.

Supports accessibility for: Organization; Attention

Anticipated Misconceptions

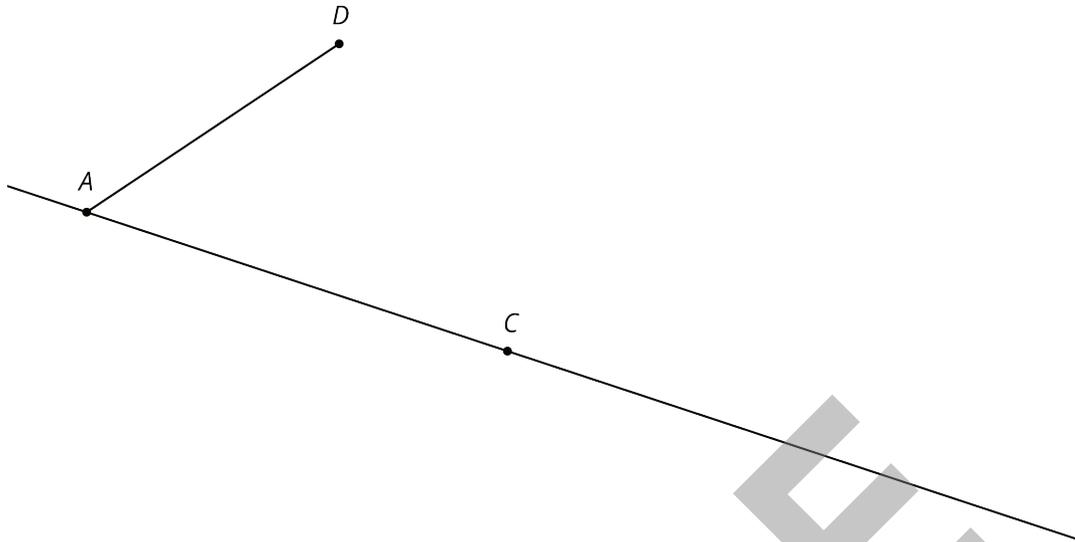
In the second question, students may not understand that rotating the figure includes both segment CA and segment AD since they have been working with rotating one segment at a time. Explain to these students that the figure refers to both of the segments. Encourage them to use tracing paper to help them visualize the rotation.

Student Task Statement

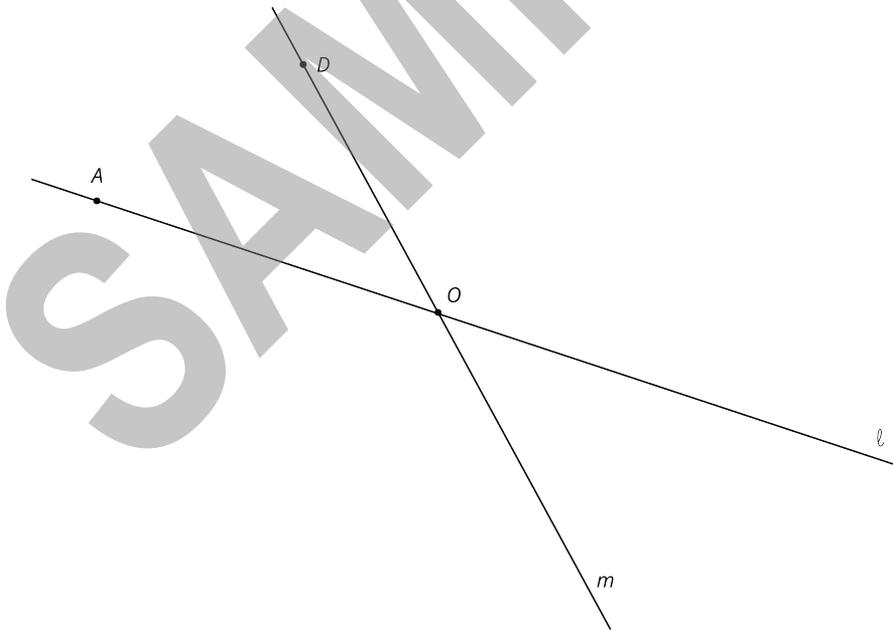
1. The diagram shows a line with points labeled A , C , D , and B .
 - a. On the diagram, draw the image of the line and points A , C , and B after the line has been rotated 180° around point D .
 - b. Label the images of the points A' , B' , and C' .
 - c. What is the order of all seven points? Explain or show your reasoning.



2. The diagram shows a line with points A and C on the line and a segment AD where D is not on the line.
 - a. Rotate the figure 180° about point C . Label the image of A as A' and the image of D as D' .
 - b. What do you know about the relationship between angle CAD and angle $CA'D'$? Explain or show your reasoning.



3. The diagram shows two lines ℓ and m that intersect at a point O with point A on ℓ and point D on m .
- Rotate the figure 180 degrees around O . Label the image of A as A' and the image of D as D' .
 - What do you know about the relationship between the angles in the figure? Explain or show your reasoning.

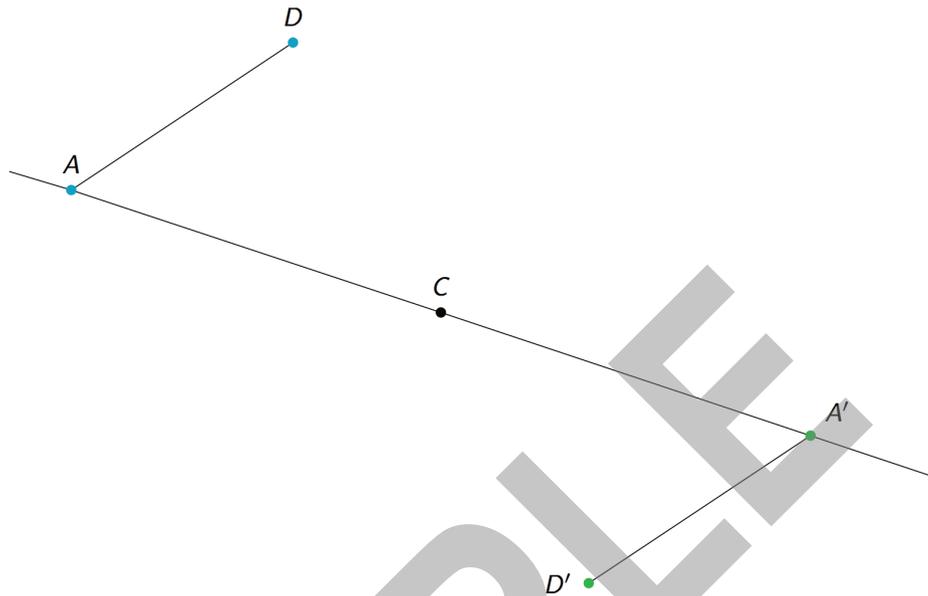


Student Response

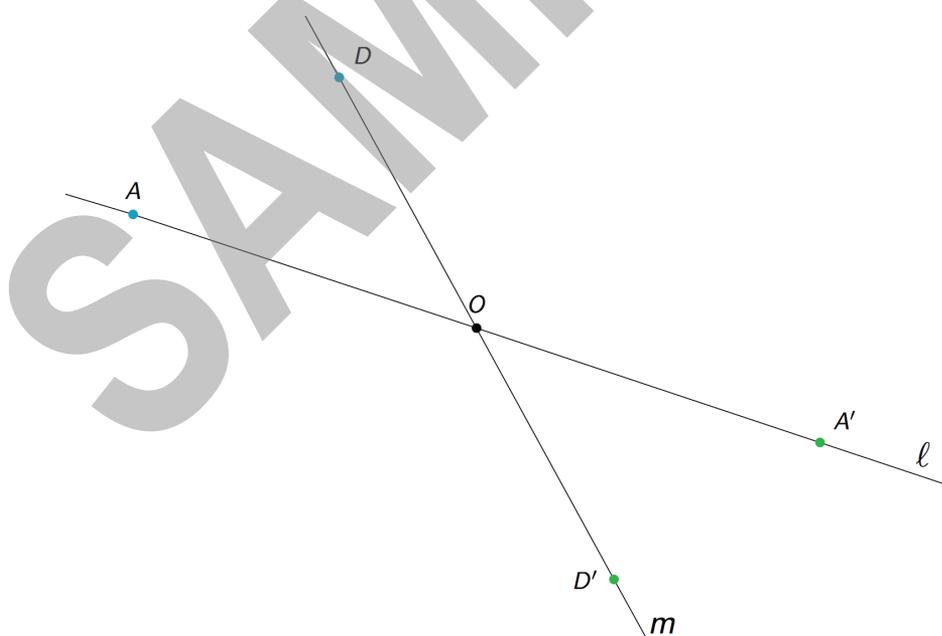
1. A, C, B', D, B, C', A'



2. Lengths of segment CA and segment CA' are the same, lengths of segment AD and segment $A'D'$ are the same, and angles CAD and $CA'D'$ have the same measure because both distances and angle measures are preserved under rigid transformations.



3. Possible responses: Angles AOD and $A'OD'$ have the same measure. Angles DOA' and $D'OA$ have the same measure.



Activity Synthesis

The focus of the discussion should start with the relationships students find between the lengths of segments and angle measures and then move to the final problem, which establishes the vertical

angle theorem as understood through rigid transformations. Questions to connect the discussion include:

- "What relationships between lengths did we find after performing transformations?" (They are the same.)
- "What relationships between angle measures did we find after performing transformations?" (They are the same.)
- "What does this transformation informally prove?" (Vertical angles are congruent.)

If time permits, consider discussing how the vertical angle theorem was approached in grade 7, namely by looking for pairs of supplementary angles. Pairs of vertical angles have the same measure because they are both supplementary to the *same* angle. The argument using 180 degree rotations is different because no reference needs to be made to the supplementary angle. The 180 degree rotation shows that both pairs of vertical angles have the same measure directly by mapping them to each other!

Support for English Language Learners

Representing, Speaking: MLR7 Compare and Connect. Use this routine when students share what they noticed about the relationships between the angle measures. Ask students to consider what changes and what stays the same when rigid transformations are applied to lines and segments. Draw students' attention to the associations between the rigid transformation, lengths of segments, and angle measures. These exchanges strengthen students' mathematical language use and reasoning based on rigid transformations of lines and will lead to the informal argument of the vertical angle theorem.

Design Principle(s): Maximize meta-awareness

Lesson Synthesis

In this lesson, students apply different rigid transformations to lines with a focus on parallel lines. They should be able to articulate what happens to parallel lines when a rigid transformation is performed on them. In addition, students gain a better understanding of why the **vertical angle** theorem they learned in grade 7 is true.

To highlight how transformations affect parallel lines, ask students:

- "When we perform rigid transformations on parallel lines, what do we know about their image?"
- "Does the distance between the lines change?"

To help students make a connection to how rotations affect lines in the second activity, ask:

- "When we rotate a line 180° around a point on the line where does the line land?"

- "How does the rotation affect the angle measurements for a pair of intersecting lines?"
- "How does this help us prove the vertical angle theorem?"

Students should see that a rotation of two intersecting lines about the point of intersection by 180° moves each angle to the angle that is vertical to it. Since rotation is a rigid transformation, the vertical angles must have the same measure.

In general, rigid transformations help us see that when we transform lines it might change the orientation but the lines retain their original properties.

8.4 Finding Missing Measurements

Cool Down: 5 minutes

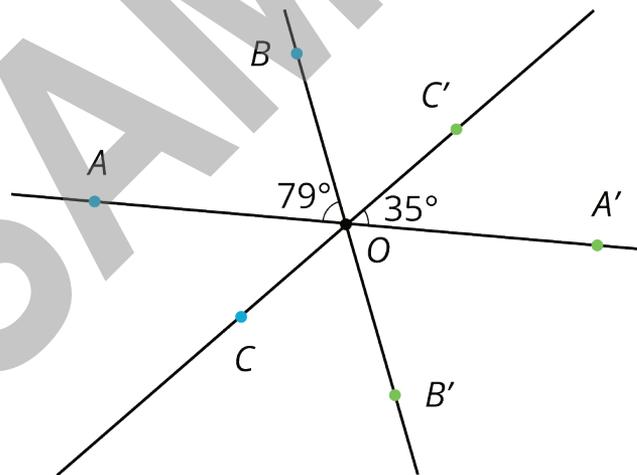
Building directly from the previous activity, students fill in missing measurements using their understanding of both rigid transformations and the vertical angle theorem.

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Student Task Statement

Points A' , B' , and C' are the images of 180° -degree rotations of A , B , and C , respectively, around point O .



Answer each question and explain your reasoning *without* measuring segments or angles.

1. Name a segment whose length is the same as segment AO .
2. What is the measure of angle $A'OB'$?

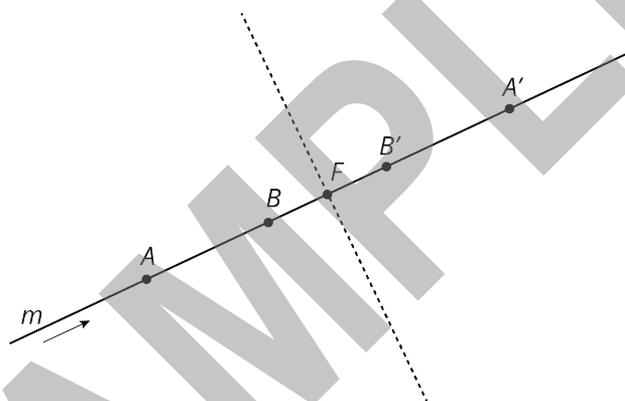
Student Response

1. Segment $A'O$, because A' is the image of A after a 180 degree rotation with center at O . This rotation preserves distances and takes segment AO to segment $A'O$.
2. 79 degrees, the same measure as $\angle AOB$, because the 180 degree rotation with center at O takes $\angle AOB$ to $\angle A'OB'$. The rotation preserves angle measures.

Student Lesson Summary

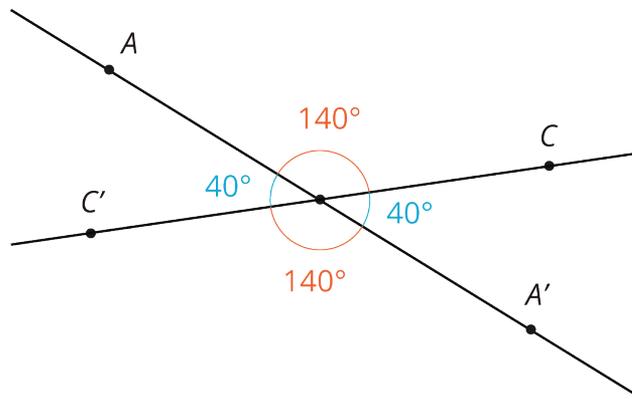
Rigid transformations have the following properties:

- A rigid transformation of a line is a line.
- A rigid transformation of two parallel lines results in two parallel lines that are the same distance apart as the original two lines.
- Sometimes, a rigid transformation takes a line to itself. For example:



- A translation parallel to the line. The arrow shows a translation of line m that will take m to itself.
- A rotation by 180° around any point on the line. A 180° rotation of line m around point F will take m to itself.
- A reflection across any line perpendicular to the line. A reflection of line m across the dashed line will take m to itself.

These facts let us make an important conclusion. If two lines intersect at a point, which we'll call O , then a 180° rotation of the lines with center O shows that **vertical angles** are congruent. Here is an example:



Rotating both lines by 180° around O sends angle AOC to angle $A'OC'$, proving that they have the same measure. The rotation also sends angle AOC' to angle $A'OC$.

Glossary

- vertical angles

Lesson 8 Practice Problems

Problem 1

Statement

- Draw parallel lines AB and CD .
- Pick any point E . Rotate AB 90 degrees clockwise around E .
- Rotate line CD 90 degrees clockwise around E .
- What do you notice?

Solution

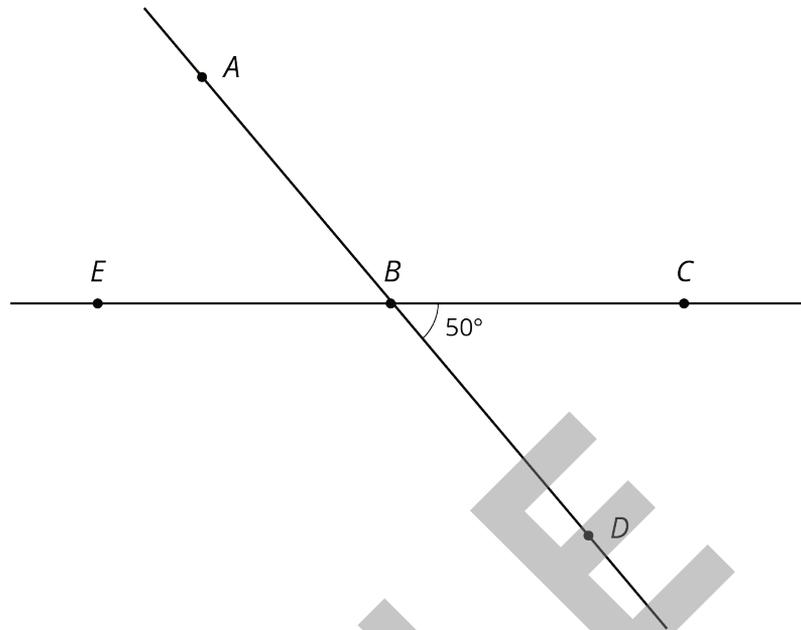
- Answers vary.
- Answers vary. The new line should be perpendicular to AB .
- Answers vary. The new line should be perpendicular to CD and parallel to $A'B'$.
- Answers vary. Sample response: the two new rotated lines are parallel.

Problem 2

Statement

Use the diagram to find the measures of each angle. Explain your reasoning.

- a. $m\angle ABC$
- b. $m\angle EBD$
- c. $m\angle ABE$



Solution

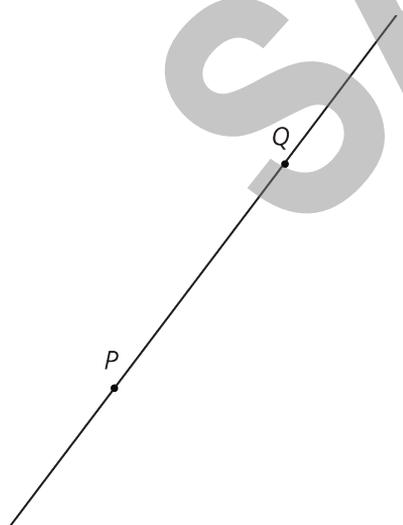
- a. 130 degrees. $\angle ABC$ and $\angle CBD$ make a line, so they add up to 180 degrees.
- b. 130 degrees. $\angle EBD$ and $\angle CBD$ make a line, so they add up to 180 degrees.
- c. 50 degrees. $\angle ABE$ and $\angle ABC$ make a line, so they add up to 180 degrees.

Problem 3

Statement

Points P and Q are plotted on a line.

- a. Find a point R so that a 180-degree rotation with center R sends P to Q and Q to P .
- b. Is there more than one point R that works for part a?



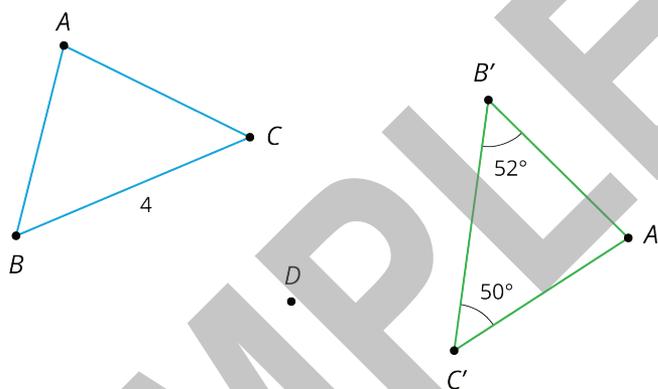
Solution

- If R is the midpoint of segment PQ , then a rotation of 180 degrees with center R sends P to Q and Q to P .
- No (The midpoint of PQ is the only point that works. 180 rotations with any other center do not send P to Q or Q to P .)

Problem 4

Statement

In the picture triangle $A'B'C'$ is an image of triangle ABC after a rotation. The center of rotation is D .



- What is the length of side $B'C'$? Explain how you know.
- What is the measure of angle B ? Explain how you know.
- What is the measure of angle C ? Explain how you know.

Solution

- 4 units. Rotations preserve side lengths, and side $B'C'$ corresponds to side BC under this rotation.
- 52 degrees. Rotations preserve angle measures, and angles B and B' correspond to each other under this rotation.
- 50 degrees. Rotations preserve angle measures, and angles C and C' correspond to each other under this rotation.

(From Unit 1, Lesson 6.)

Problem 5

Statement

The point $(-4, 1)$ is rotated 180 degrees counterclockwise using center $(0, 0)$. What are the coordinates of the image?

- A. $(-1, -4)$
- B. $(-1, 4)$
- C. $(4, 1)$
- D. $(4, -1)$

Solution

D

(From Unit 1, Lesson 5.)

SAMPLE

Lesson 9: Composing Figures

Goals

- Draw and label images of triangles under rigid transformations and then describe (orally and in writing) properties of the composite figure created by the images.
- Generalize that lengths and angle measures are preserved under any rigid transformation.
- Identify side lengths and angles that have equivalent measurements in composite shapes and explain (orally and in writing) why they are equivalent.

Learning Targets

- I can find missing side lengths or angle measures using properties of rigid transformations.

Lesson Narrative

In this lesson, students create composite shapes using translations, rotations, and reflections of polygons and continue to observe that the side lengths and angle measures do not change. They use this understanding to draw conclusions about the composite shapes. Later, they will use these skills to construct informal arguments, for example about the sum of the angles in a triangle.

When students rotate around a vertex or reflect across the side of a figure, it is easy to lose track of the center of rotation or line of reflection since they are already part of the figure. It can also be challenging to name corresponding points, segments, and angles when a figure and its transformation share a side. Students attend to these details carefully in this lesson (MP6).

Consider using the optional activity if you need to reinforce students' belief that rigid transformations preserve distances and angle measures after main activities.

Alignments

Addressing

- 8.G.A.1.a: Lines are taken to lines, and line segments to line segments of the same length.
- 8.G.A.1.b: Angles are taken to angles of the same measure.

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR3: Clarify, Critique, Correct
- MLR7: Compare and Connect
- Think Pair Share

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's use reasoning about rigid transformations to find measurements without measuring.

9.1 Angles of an Isosceles Triangle

Warm Up: 10 minutes

Isosceles triangles are triangles with (at least) one pair of congruent sides. Isosceles triangles also have (at least) one pair of congruent angles. In this warm-up, students show why this is the case using rigid motions by exploiting the fact that rigid motions of the plane do not change angle measures (MP7).

Addressing

- 8.G.A.1.b

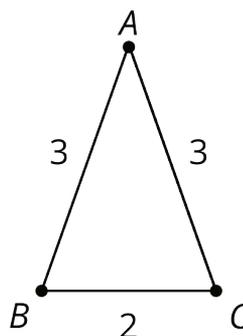
Launch

Give students 3 minutes of quiet work time followed by whole-class discussion.

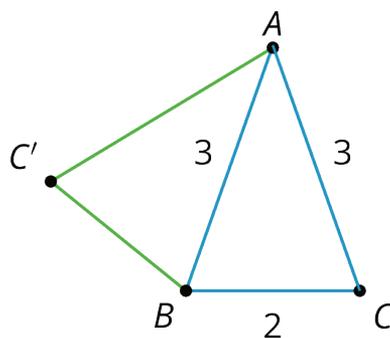
Student Task Statement

Here is a triangle.

1. Reflect triangle ABC over line AB . Label the image of C as C' .
2. Rotate triangle ABC' around A so that C' matches up with B .
3. What can you say about the measures of angles B and C ?



Student Response



- 1.
2. Rotating ABC' as described takes ABC' back to the original triangle.
3. The measures of angles B and C are the same. Neither the rotation nor the reflection change the angle measures, and so since these transformations take the angle at C to the angle at B , they must have the same measure.

Activity Synthesis

Select students to share their images and conclusions about the measures of angles B and C .

Time permitting, mention that it is also true that when a triangle has two angles with the same measure then the sides opposite those angles have the same length (i.e., the triangle is isosceles). This can also be shown with rigid transformations. Reflect the triangle first and then line up the sides containing the pairs of angles with the same measure.

9.2 Triangle Plus One

10 minutes

The purpose of this task is to use rigid transformations to describe an important picture that students have seen in grade 6 when they developed the formula for the area of a triangle. They first found the area of a parallelogram to be $\text{base} \cdot \text{height}$ and then, to find $\frac{1}{2} \text{base} \cdot \text{height}$ for the area of a triangle, they “composed” two copies of a triangle to make a parallelogram. The language “compose” is a grade 6 appropriate way of talking about a 180° rotation. The focus of this activity is on developing this precise language to describe a familiar geometric situation.

Students need to remember and use an important property of 180° rotations, namely that the image of a line after a 180° rotation is parallel to that line. This is what allows them to conclude that the shape they have built is a parallelogram (MP7).

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to geometry toolkits. A few minutes of quiet work time, followed by sharing with a partner and a whole-class discussion.

Support for Students with Disabilities

Action and Expression: Internalize Executive Functions. To support development of organizational skills, check in with students within the first 2–3 minutes of work time. Look for students who struggle with visualizing the 180 degree rotation using center M . Consider pausing for a brief discussion to invite 1–2 pairs of students to demonstrate and explain how to do the rotation.
Supports accessibility for: Memory; Organization

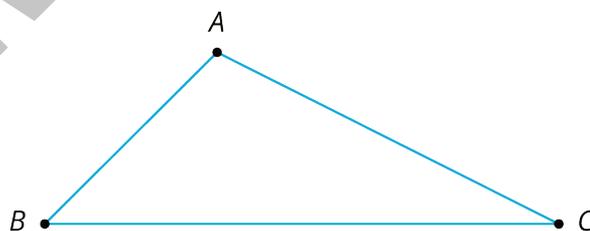
Anticipated Misconceptions

Students may struggle to see the 180° rotation using center M . This may be because they do not understand that M is the center of rotation or because they struggle with visualizing a 180° rotation. Offer these students patty paper, a transparency, or the rotation overlay from earlier in this unit to help them see the rotated triangle.

Student Task Statement

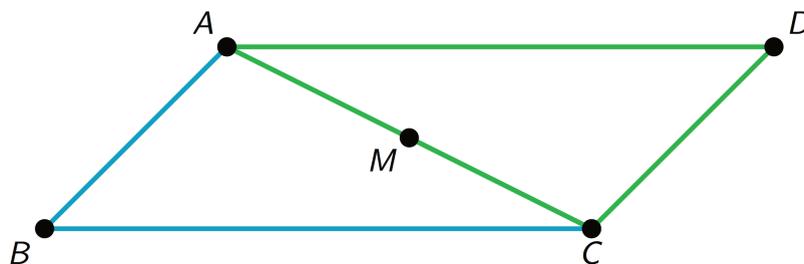
Here is triangle ABC .

1. Draw midpoint M of side AC .
2. Rotate triangle ABC 180 degrees using center M to form triangle CDA . Draw and label this triangle.
3. What kind of quadrilateral is $ABCD$? Explain how you know.



Student Response

- 1.



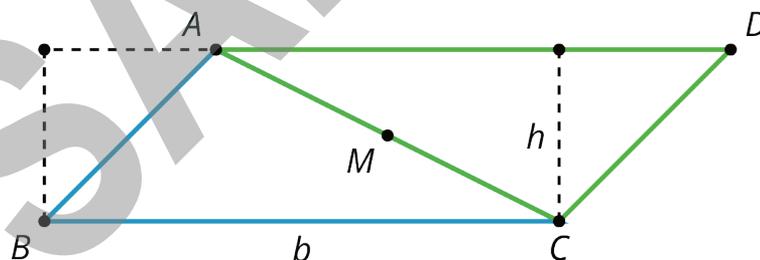
2. A parallelogram. The 180 degree rotation around M takes line AB to line CD and so these are parallel. It also takes line BC to line AD so these lines are also parallel. That means that $ABCD$ is a parallelogram.

Are You Ready for More?

In the activity, we made a parallelogram by taking a triangle and its image under a 180-degree rotation around the midpoint of a side. This picture helps you justify a well-known formula for the area of a triangle. What is the formula and how does the figure help justify it?

Student Response

A formula for the area of a triangle is $A = \frac{1}{2}bh$, where b is a base of the triangle and h the corresponding height. In the image we have added b and h to the parallelogram, and marked off two triangles of interest with dashed lines. Since the left and right triangles have the same area, the area of the parallelogram displayed is the same as the area of a rectangle with height h and base b , namely, an area of bh . Since the area of each triangle is half the area of the parallelogram, each triangle has area $A = \frac{1}{2}bh$.



Activity Synthesis

Begin the discussion by asking, "What happens to points A and C under the rotation?" (They end up at C and A , respectively.) This type of rotation and analysis will happen several times in upcoming lessons.

Next ask, "How do you know that the lines containing opposite sides of $ABCD$ are parallel?" (They are taken to one another by a 180 degree rotation.) As seen in the previous lesson, the image of a 180° rotation of a line ℓ is parallel to ℓ . Students also saw that when 180° rotations were applied to a pair of parallel lines it resulting in a (sometimes) new pair of parallel lines which are also parallel

to the original lines. The logic here is the same, except that only one line is being rotated 180° rather than a pair of lines. This does not need to be mentioned unless it is brought up by students.

Finally, ask students "How is the area of parallelogram $ABCD$ related to the area of triangle ABC ?" (The area of the parallelogram $ABCD$ is twice the area of triangle ABC because it is made up of ABC and CDA which has the same area as ABC .) Later in this unit, area of shapes and their images under rigid transformations will be studied further.

SAMPLE

Support for English Language Learners

Writing, Speaking: Math Language Routine 1 Stronger and Clearer Each Time. This is the first time Math Language Routine 1 is suggested as a support in this course. In this routine, students are given a thought-provoking question or prompt and are asked to create a first draft response. Students meet with 2–3 partners to share and refine their response through conversation. While meeting, listeners ask questions such as, “What did you mean by . . .?” and “Can you say that another way?” Finally, students write a second draft of their response reflecting ideas from partners and improvements on their initial ideas. The purpose of this routine is to provide a structured and interactive opportunity for students to revise and refine their ideas through verbal and written means.

Design Principle(s): Optimize output (for justification)

How It Happens:

1. Use this routine to provide students a structured opportunity to refine their justification for the question asking “What kind of quadrilateral is $ABCD$? Explain how you know.” Give students 2–3 minutes to individually create first draft responses in writing.
2. Invite students to meet with 2–3 other partners for feedback.

Instruct the speaker to begin by sharing their ideas without looking at their written draft, if possible. Listeners should press for details and clarity.

Provide students with these prompts for feedback that will help individuals strengthen their ideas and clarify their language: “What do you mean when you say...?”, “Can you describe that another way?”, “How do you know the lines are parallel?”, and “What happens to lines under rotations?” Be sure to have the partners switch roles. Allow 1–2 minutes to discuss.

3. Signal for students to move on to their next partner and repeat this structured meeting.
4. Close the partner conversations and invite students to revise and refine their writing in a second draft. Students can borrow ideas and language from each partner to strengthen the final product.

Provide these sentence frames to help students organize their thoughts in a clear, precise way: “Quadrilateral $ABCD$ is a ___ because....” and “Another way to verify this is....”.

Here is an example of a second draft:

“ $ABCD$ is a parallelogram, I know this because a 180-degree rotation creates new lines that are parallel to the original lines. In this figure, the 180-degree rotation takes line AB to line CD and line BC to line AD . I checked this by copying the triangle onto patty paper

and rotating it 180 degrees. This means the new shape will have two pairs of parallel sides. Quadrilaterals that have two pairs of parallel sides are called parallelograms.”

5. If time allows, have students compare their first and second drafts. If not, have the students move on by discussing other aspects of the activity.
-

9.3 Triangle Plus Two

15 minutes

This activity continues the previous one, building a more complex shape this time by adding an additional copy of the original triangle. The three triangle picture in the task statement will be important later in this unit when students show that the sum of the three angles in a triangle is 180° . To this end, encourage students to notice that the points E , A , and D all lie on a line.

As with many of the lessons applying transformations to build shapes, students are constantly using their structural properties (MP7) to make conclusions about their shapes. Specifically, that rigid transformations preserve angles and side lengths.

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Instructional Routines

- MLR3: Clarify, Critique, Correct

Launch

Keep students in the same groups. Provide access to geometry toolkits. Allow for a few minutes of quiet work time, followed by sharing with a partner and a whole-class discussion.

Support for Students with Disabilities

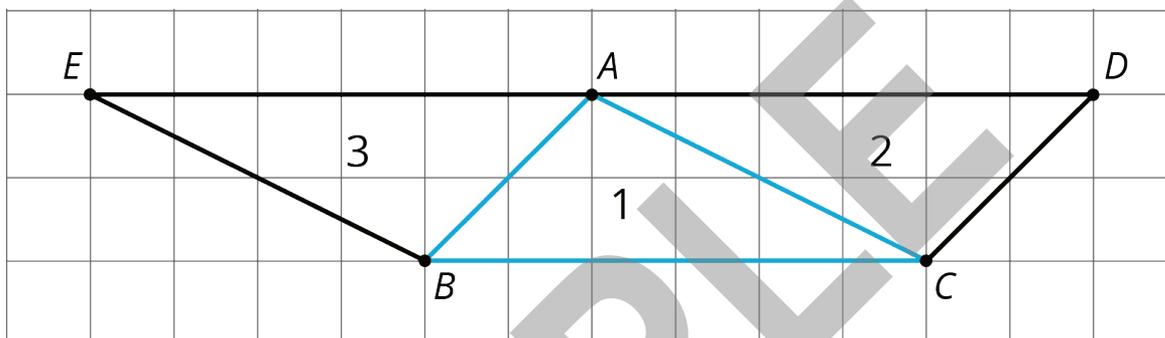
Representation: Internalize Comprehension. Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem. For example, use the same color to highlight corresponding points in different triangles and consider redrawing the triangles in the same orientation to emphasize corresponding parts.
Supports accessibility for: Visual-spatial processing

Anticipated Misconceptions

Students may have trouble understanding which pairs of points correspond in the first two questions, particularly the fact that point A in one triangle may not correspond to point A in another. Use tracing paper to create a transparency of triangle ABC , with its points labeled, and let students perform their rigid transformation. They should see A , B , and C on top of points in the new triangle.

Student Task Statement

The picture shows 3 triangles. Triangle 2 and Triangle 3 are images of Triangle 1 under rigid transformations.



1. Describe a rigid transformation that takes Triangle 1 to Triangle 2. What points in Triangle 2 correspond to points A , B , and C in the original triangle?
2. Describe a rigid transformation that takes Triangle 1 to Triangle 3. What points in Triangle 3 correspond to points A , B , and C in the original triangle?
3. Find two pairs of line segments in the diagram that are the same length, and explain how you know they are the same length.
4. Find two pairs of angles in the diagram that have the same measure, and explain how you know they have the same measure.

Student Response

1. Answers vary. Sample response: a 180-degree rotation using the midpoint of side AC as center. In triangle 2, point C corresponds to A in the original, D corresponds to B in the original, and A corresponds to C in the original.
2. Answers vary. Sample responses: a 180-degree rotation using the midpoint of side AB as center or a 180 degree rotation using the midpoint of segment AC as center followed by a translation taking A to E . In triangle 3, point A corresponds to B in the original, B corresponds to A in the original, and E corresponds to C in the original.
3. Answers vary. Sample response: segment AE and segment BC are the same length and segments AB and CD are also the same length. This is true because a rigid transformation doesn't change a figure's side lengths.

4. Answers vary. Sample response: $\angle D$ and $\angle ABC$ have the same measure and so do $\angle E$ and $\angle ACB$. This is true because a rigid transformation doesn't change a figure's angle measures.

Activity Synthesis

Ask students to list as many different pairs of matching line segments as they can find. Then, do the same for angles. Record these for all to see. Students may wonder why there are fewer pairs of line segments: this is because of shared sides AB and AC . If they don't ask, there's no reason to bring it up.

If you create a visual display of these pairs, hang on to the information about angles that have the same measure. The same diagram appears later in this unit and is used for a proof about the sum of the angle measures in a triangle.

After this activity, ask students to summarize their understanding about lengths and angle measures under rigid transformations. If students don't say it outright, you should: "Under any rigid transformation, lengths and angle measures are preserved."

SAMPLE

Support for English Language Learners

Writing: Math Language Routine 3 Clarify, Critique, Correct. This is the first time Math Language Routine 3 is suggested as a support in this course. In this routine, students are given an incorrect or incomplete piece of mathematical work. This may be in the form of a written statement, drawing, problem-solving steps, or another mathematical representation. Pairs of students analyze, reflect on, and improve the written work by correcting errors and clarifying meaning. Typical prompts are: “Is anything unclear?” or “Are there any reasoning errors?” The purpose of this routine is to engage students in analyzing mathematical thinking that is not their own and to solidify their knowledge through communicating about conceptual errors and ambiguities in language.

Design Principle(s): Maximize meta-awareness

How It Happens:

1. In the class discussion for this activity, present this incomplete description of a rigid transformation that takes Triangle 1 to Triangle 2:

“ CB and AD are the same because you turn ABC .”

Prompt students to identify the ambiguity of this response. Ask students, “What do you think this person is trying to say? What is unclear?”

2. Give students 1 minute of individual time to respond to the questions in writing, and then 3 minutes to discuss with a partner.

As pairs discuss, provide these sentence frames for scaffolding: “I think that what the author meant by ‘turn ABC ’ was...”, “The part that is the most unclear to me is ... because...”, and “I think this person is trying to say...”. Encourage the listener to press for detail by asking follow-up questions to clarify the intended meaning of the statement. Allow each partner to take a turn as the speaker and listener.

Listen for students using appropriate geometry terms such as “transformation” and “rotation” in explaining why the two sides are equivalent.

3. Then, ask students to write a more precise version and explain their reasoning in writing with their partner. Improved responses should include for each step an explanation, order/time transition words (first, next, then, etc.), and/or reasons for decisions made during steps.

Here are two sample improved responses:

“First, I used tracing paper to create a copy of triangle ABC because I wanted to transform it onto the new triangle. Next, I labeled the vertices on my tracing paper. Then,

using the midpoint of side AC as my center, I rotated the tracing paper 180 degrees, because then it matched up on top of Triangle 2. So, sides CB and AD are equivalent.”

or

“First, I accessed my geometry technology tool to create the drawing of only only Triangle 1 and 2. Next, I tried different transformations of Triangle 1 to make the points A , B , and C fall on top of points in Triangle 2. The one that worked was rotating Triangle 1 180 degrees using the midpoint of side AC . Finally, I know that sides CB and AD are equivalent because Triangle 1 is exactly on top of Triangle 2.”

4. Ask each pair of students to contribute their improved response to a poster, the whiteboard, or digital projection. Call on 2–3 pairs of students to present their response to the whole class, and invite the class to make comparisons among the responses shared and their own responses.

Listen for responses that identify the correct pair of equivalent sides and explain how they know. In this conversation, also allow students the opportunity to name other equivalent sides and angles.

5. Close the conversation with the generalization that lengths and angle measures are preserved under any rigid transformation, and then move on to the next lesson activity.

9.4 Triangle ONE Plus

Optional: 10 minutes

This activity builds upon the ideas of the previous one. This time a pattern is built from a single triangle via reflections and the goal is to study the angles and side lengths in this pattern as it grows. If they build the pattern carefully, students may notice after putting together 6 triangles, like in the previous activity, that two of the sides of the triangles (one side of the original and one side of the 6th) lie on the same line. The 6 triangle pattern can be reflected over this line to make it “complete” with 12 copies of the original triangle. Alternatively, students may notice the right angle made by 3 triangles and reason that they can complete a circle with 4 right angles. Both of these arguments are good examples of MP8. Rather than repeating the reflecting procedure 12 times, it is possible to use the structure of what they have learned along the way to accurately predict how many copies make a circle.

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Instructional Routines

- MLR7: Compare and Connect

Launch

Provide access to geometry toolkits. Allow for 8 minutes of work time, then a brief whole-class discussion.

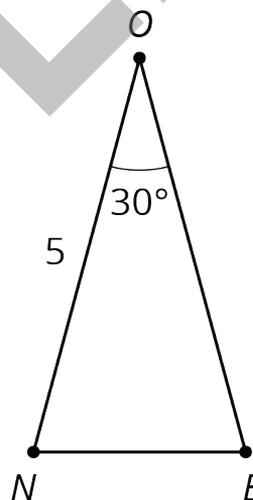
Anticipated Misconceptions

If students are stuck with the first reflection, suggest that they use tracing paper. If you need to, show them the first reflected triangle, then have them continue to answer the problems and do the next reflection on their own.

Some students may have difficulty with the length of OT , since it uses the initial information that triangle ONE is isosceles (otherwise, OT is unlabeled). Ask students what other information is given and if they can use it to figure out the missing length.

Student Task Statement

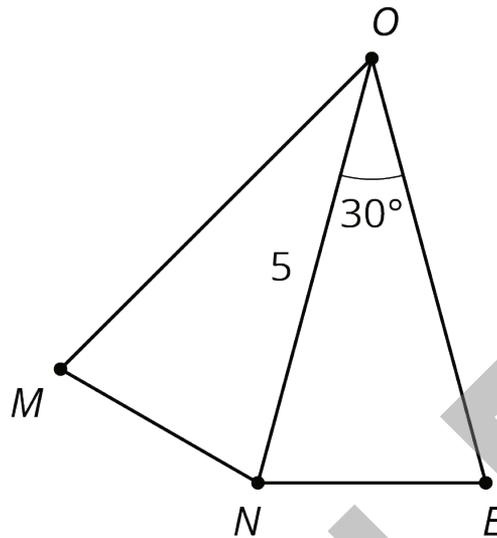
Here is isosceles triangle ONE . Its sides ON and OE have equal lengths. Angle O is 30 degrees. The length of ON is 5 units.



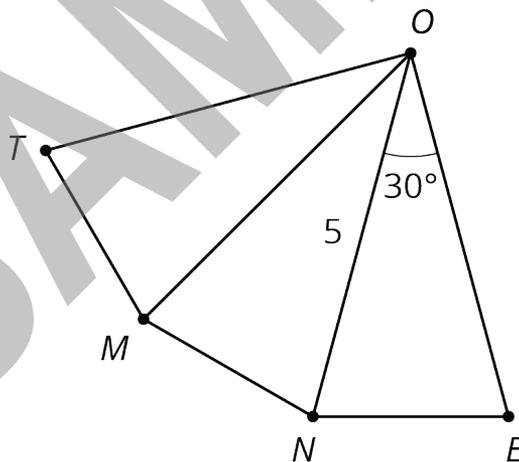
1. Reflect triangle ONE across segment ON . Label the new vertex M .
2. What is the measure of angle MON ?
3. What is the measure of angle MOE ?
4. Reflect triangle MON across segment OM . Label the point that corresponds to N as T .
5. How long is \overline{OT} ? How do you know?
6. What is the measure of angle TOE ?
7. If you continue to reflect each new triangle this way to make a pattern, what will the pattern look like?

Student Response

1.



2. 30° because angle measures are the same after applying a rigid transformation and angle NOE becomes angle NOM after reflection.
3. 60° because $\angle MOE$ is gotten by putting together $\angle MON$ and $\angle NOE$ and these are both 30 degree angles.
- 4.



5. Segment OT measures 5 units because it is the image of segment ON after a reflection so it has the same length as segment ON .
6. 90° , a right angle because $\angle MOE$ measures 60° and $\angle TOM$ measures 30° . Since $\angle TOE$ is gotten by putting together $\angle TOM$ and $\angle MOE$ so it is a 90 degree angle.
7. Answers vary. One possible description: Eventually point O will be completely surrounded by triangles, with the 12th triangle touching \overline{OE} again. There are 4 right angles in a full circle and each right angle has 3 copies of the original triangle.

Activity Synthesis

The main goal of this activity is to apply and reinforce students' belief that rigid transformations preserve distances and angle measures. Watch and make sure students are doing well with this. If they are not, reinforce the concept. It is critical that this is understood by students before moving forward.

Identify a few students, each with a different response, to share their description of the pattern they saw in the last question. The way in which each student visualizes and explains this shape may give insight into the different strategies used to create the final pattern.

Note: It is *not* important nor required that students know or understand how to find the base angle measures of the isosceles triangles, or even that the base angles have the same measure (though students do study this in the warm-up). Later in this unit, students will learn and prove that the sum of the angle measures in a triangle is 180 degrees.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class.

Supports accessibility for: Attention; Social-emotional skills

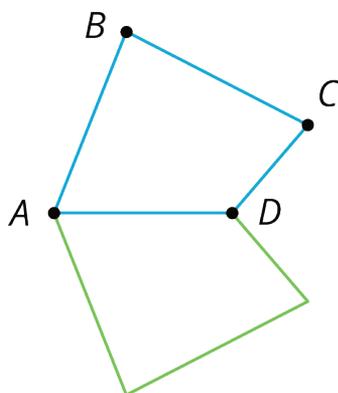
Support for English Language Learners

Speaking: MLR7 Compare and Connect. Use this routine when students describe the pattern they saw when reflecting the triangle multiple times. Ask students to consider what is the same and what is different about each description. Draw students' attention to the different ways the pattern is explained. Some students may benefit from the use of gestures to support their understanding of the descriptions. These exchanges can strengthen students' mathematical language use and reasoning to make sense of strategies used to create composite shapes.

Design Principle(s): Maximize meta-awareness

Lesson Synthesis

Briefly review the properties of rigid transformations (they preserve lengths and angle measures). Go over some examples of corresponding sides and angles in figures that share points. For example, talk about which sides and angles correspond if this image is found by reflecting $ABCD$ across line AD .



Point out to students that sides on a reflection line do not move, so they are their own image when we reflect across a side. Also, the center of rotation does not move, so it is its own image when we rotate around it. All points move with a translation.

9.5 Identifying Side Lengths and Angle Measures

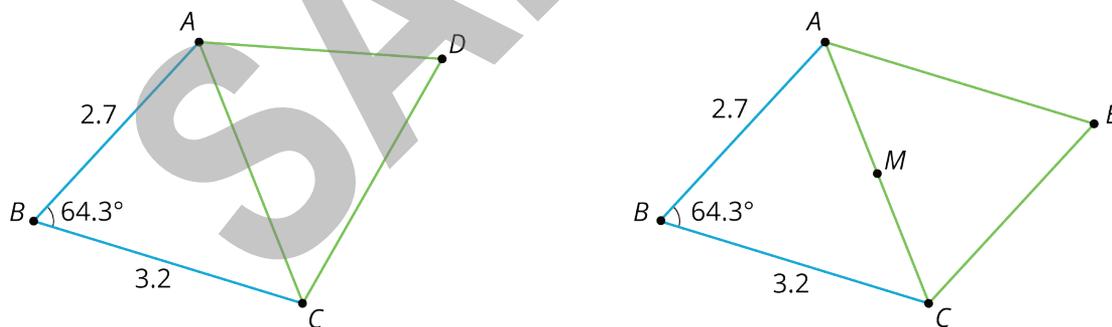
Cool Down: 5 minutes

Students apply the fact that rigid motions preserve side lengths and angles. They are presented with a figure and two transformed images, and they use what they know to find the side lengths and angles of the transformed figures.

Addressing

- 8.G.A.1.a
- 8.G.A.1.b

Student Task Statement

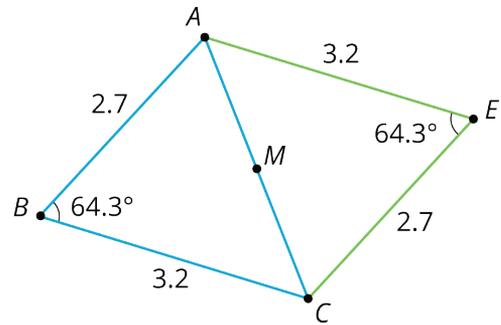
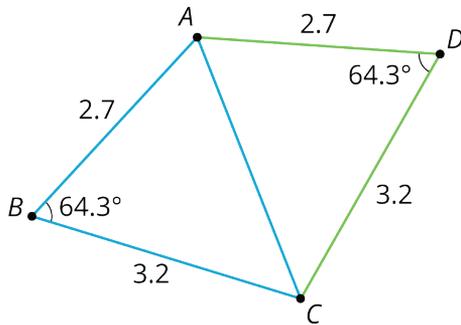


Here is a diagram showing triangle ABC and some transformations of triangle ABC .

On the left side of the diagram, triangle ABC has been *reflected* across line AC to form quadrilateral $ABCD$. On the right side of the diagram, triangle ABC has been *rotated* 180 degrees using midpoint M as a center to form quadrilateral $ABCE$.

Using what you know about rigid transformations, side lengths and angle measures, label as many side lengths and angle measures as you can in quadrilaterals $ABCD$ and $ABCE$.

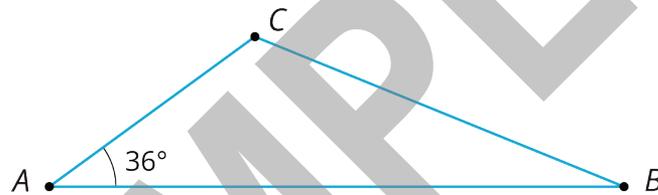
Student Response



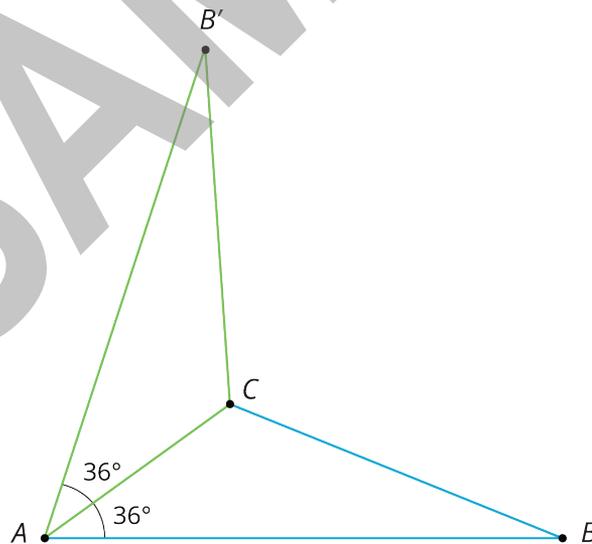
Student Lesson Summary

Earlier, we learned that if we apply a sequence of rigid transformations to a figure, then corresponding sides have equal length and corresponding angles have equal measure. These facts let us figure out things without having to measure them!

For example, here is triangle ABC .



We can reflect triangle ABC across side AC to form a new triangle:



Because points A and C are on the line of reflection, they do not move. So the image of triangle ABC is $AB'C$. We also know that:

- Angle $B'AC$ measures 36° because it is the image of angle BAC .
- Segment AB' has the same length as segment AB .

When we construct figures using copies of a figure made with rigid transformations, we know that the measures of the images of segments and angles will be equal to the measures of the original segments and angles.

Lesson 9 Practice Problems

Problem 1

Statement

Here is the design for the flag of Trinidad and Tobago.



Describe a sequence of translations, rotations, and reflections that take the lower left triangle to the upper right triangle.

Solution

Answers vary. Sample response: 180 degree rotation around the center point of the flag. Another sample response: The lower left triangle is first translated to the right so that it shares an edge with the upper right triangle. Then it's rotated 180 degrees around the midpoint of the common side.

Problem 2

Statement

Here is a picture of an older version of the flag of Great Britain. There is a rigid transformation that takes Triangle 1 to Triangle 2, another that takes Triangle 1 to Triangle 3, and another that takes Triangle 1 to Triangle 4.



- Measure the lengths of the sides in Triangles 1 and 2. What do you notice?
- What are the side lengths of Triangle 3? Explain how you know.
- Do all eight triangles in the flag have the same area? Explain how you know.

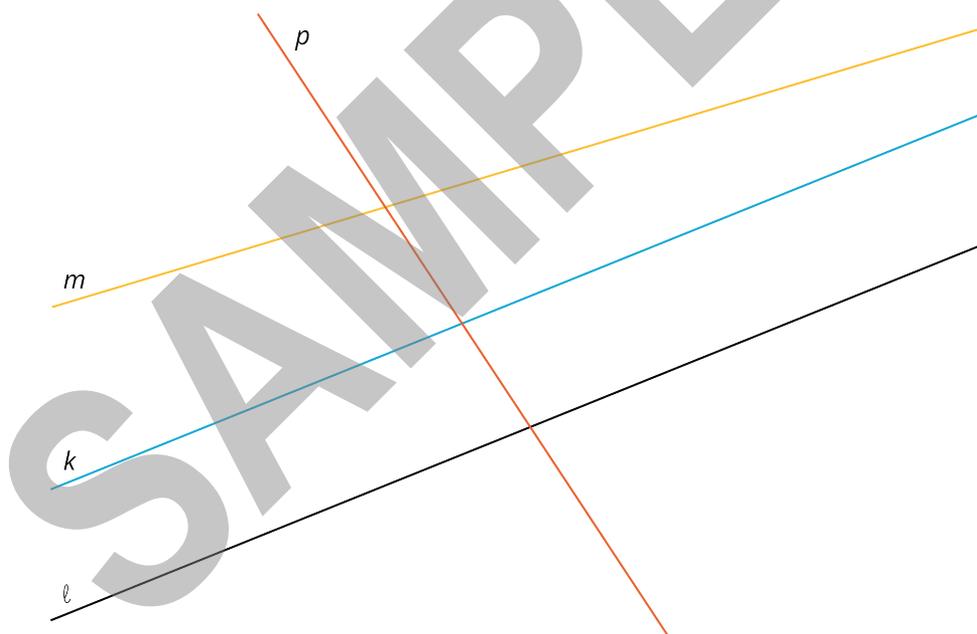
Solution

- Answers vary. The side lengths of the two triangles are the same.
- The side lengths will be the same as Triangle 1, because there is a rigid transformation taking Triangle 1 to Triangle 3.
- No. The four triangles without number labels are larger, so they will not have the same area as the smaller labeled triangles.

Problem 3

Statement

- Which of the lines in the picture is parallel to line ℓ ? Explain how you know.

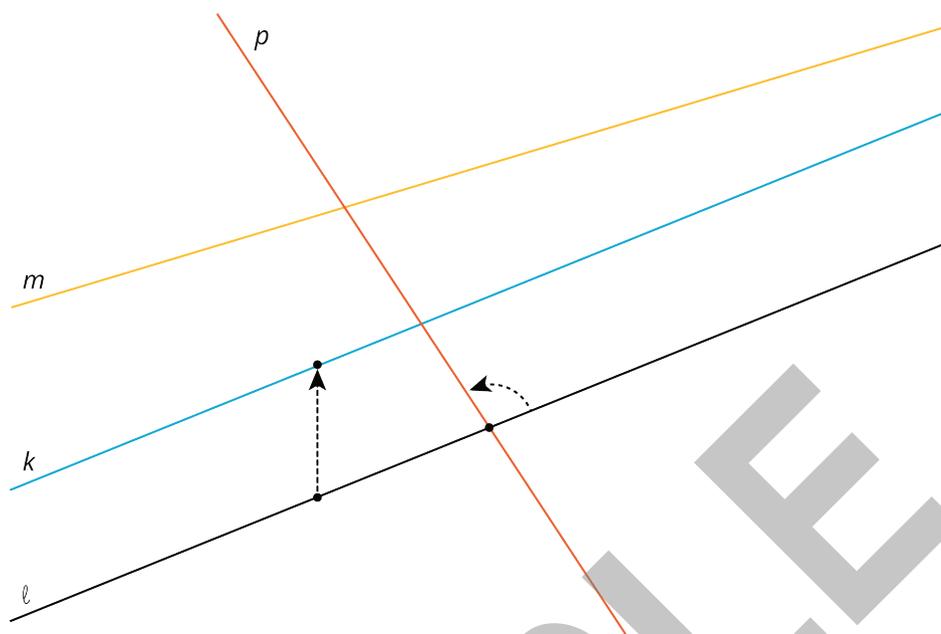


- Explain how to translate, rotate or reflect line ℓ to obtain line k .
- Explain how to translate, rotate or reflect line ℓ to obtain line p .

Solution

- k . These two lines do not intersect no matter how far out they extend.
- Line k can be obtained by translating line ℓ .
- Line p can be obtained by rotating line ℓ .

The picture shows how to translate ℓ to get k and how to rotate ℓ to get p .



(From Unit 1, Lesson 8.)

Problem 4

Statement

Point A has coordinates $(3, 4)$. After a translation 4 units left, a reflection across the x -axis, and a translation 2 units down, what are the coordinates of the image?

Solution

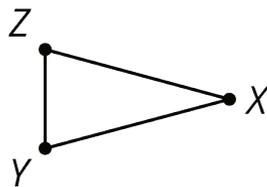
$(-1, -6)$

(From Unit 1, Lesson 5.)

Problem 5

Statement

Here is triangle XYZ :



Draw these three rotations of triangle XYZ together.

- a. Rotate triangle XYZ 90 degrees clockwise around Z .
- b. Rotate triangle XYZ 180 degrees around Z .
- c. Rotate triangle XYZ 270 degrees clockwise around Z .

Solution

Each rotation shares vertex Z with triangle XYZ . The four triangles together look like a pinwheel.

(From Unit 1, Lesson 7.)

SAMPLE

Section: Congruence

Lesson 10: What Is the Same?

Goals

- Compare and contrast (orally and in writing) side lengths, angle measures, and areas using rigid transformations to explain why a shape is or is not congruent to another.
- Comprehend that congruent figures have equal corresponding side lengths, angle measures, and areas.
- Describe (orally and in writing) two figures that can be moved to one another using a sequence of rigid transformations as “congruent.”

Learning Targets

- I can decide visually whether or not two figures are congruent.

Lesson Narrative

In this lesson, students explore what it means for shapes to be “the same” and learn that the term **congruent** is a mathematical way to talk about figures being the same that has a precise meaning. Specifically, they learn that two figures are congruent if there is a sequence of translations, rotations, and reflections that moves one to the other. They learn that figures that are congruent can have different orientations, but corresponding lengths and angle measures are equal. Agreeing upon and formulating the definition of congruence requires careful use of precise language (MP6) and builds upon all of the student experiences thus far in this unit, moving shapes and trying to make them match up.

As they work to decide whether or not pairs of shapes are congruent, students will apply MP7. For shapes that are not congruent, what property can be identified in one that is not shared by the other? This could be an angle measure, a side length, or the size of the shape. For shapes that are congruent, is there any way to tell other than experimenting with tracing paper? In some cases, like the rectangles, students discover that looking at the length and width is enough to decide if they are congruent.

In elementary grades, deciding if two shapes are the “same” usually involves making sure that they are the same general shape (for example, triangles or circles) and that the size is the same. As shapes become more complex and as we develop new ways to measure them (angles for example), something more precise is needed. The definition of congruence here states that two shapes are congruent if there is a sequence of translations, rotations, and reflections that matches one shape up exactly with the other. This definition has many advantages:

- It does not require measuring all side lengths or angles.
- It applies equally well to all shapes, not just polygons.

- It is precise and unambiguous: certain moves are allowed and two shapes are congruent when one can be moved to align exactly with the other.

The material treated here will be taken up again in high school (G-CO.B) from a more abstract point of view. In grade 8, it is essential for students to gain experience executing rigid motions with a variety of tools (tracing paper, coordinates, technology) to develop the intuition that they will need when they study these moves (or transformations) in greater depth later.

Alignments

Addressing

- 8.G.A.1: Verify experimentally the properties of rotations, reflections, and translations:
- 8.G.A.2: Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

Building Towards

- 8.G.A.2: Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR1: Stronger and Clearer Each Time
- MLR2: Collect and Display
- Think Pair Share

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Required Preparation

Colored pencils are supposed to be a usual part of the geometry toolkit, but they are called out here because one activity asks students to shade rectangles using different colors.

Student Learning Goals

Let's decide whether shapes are the same.

10.1 Find the Right Hands

Warm Up: 5 minutes

In this activity, students get their first formal introduction to the idea of mirror orientation, sometimes called “handedness” because left and right hands are reflections of each other. The easiest way to decide which are the right hands is to hold one’s hands up and rotate them until they match a particular figure (or don’t). This prepares them for a discussion about whether figures with different mirror orientation are the same or not.

Addressing

- 8.G.A.1

Building Towards

- 8.G.A.2

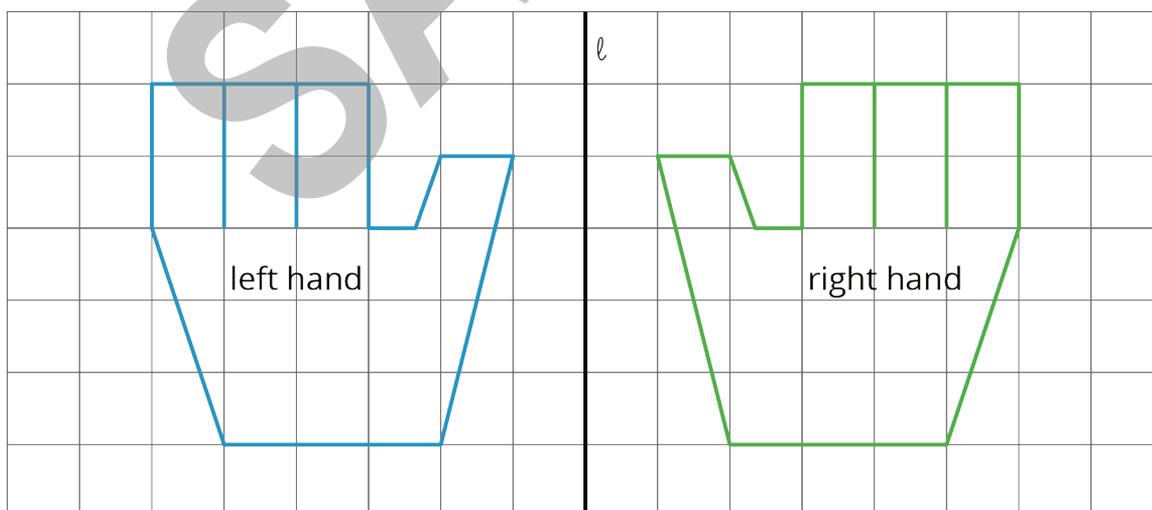
Instructional Routines

- Think Pair Share

Launch

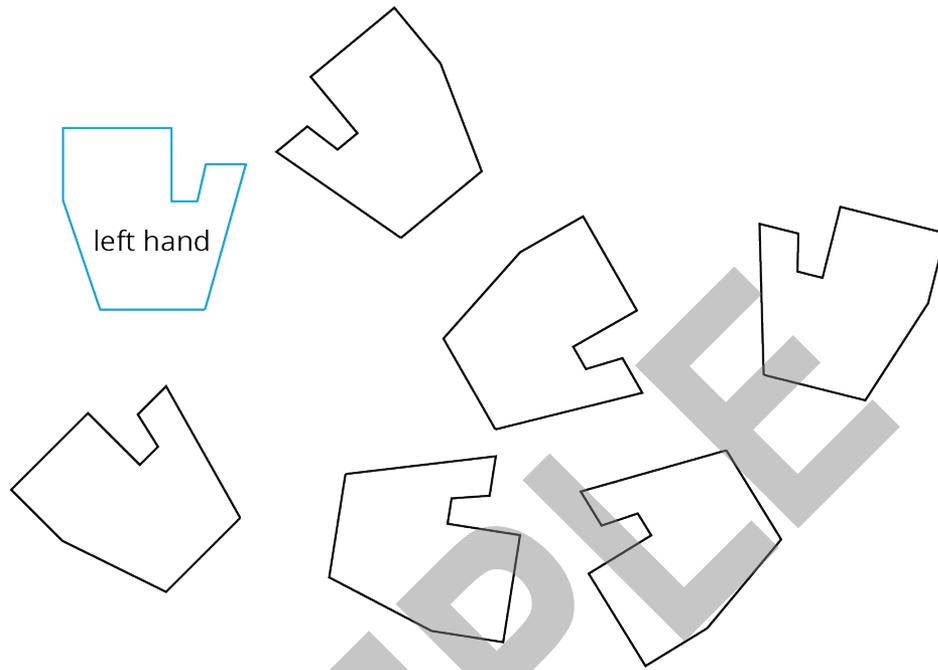
Arrange students in groups of 2, and provide access to geometry toolkits. Give 2 minutes of quiet work time, followed by time for sharing with a partner and a whole-class discussion.

Show students this image or hold up both hands and point out that our hands are mirror images of each other. These are hands shown from the back. If needed, clarify for students that all of the hands in the task are shown from the back.

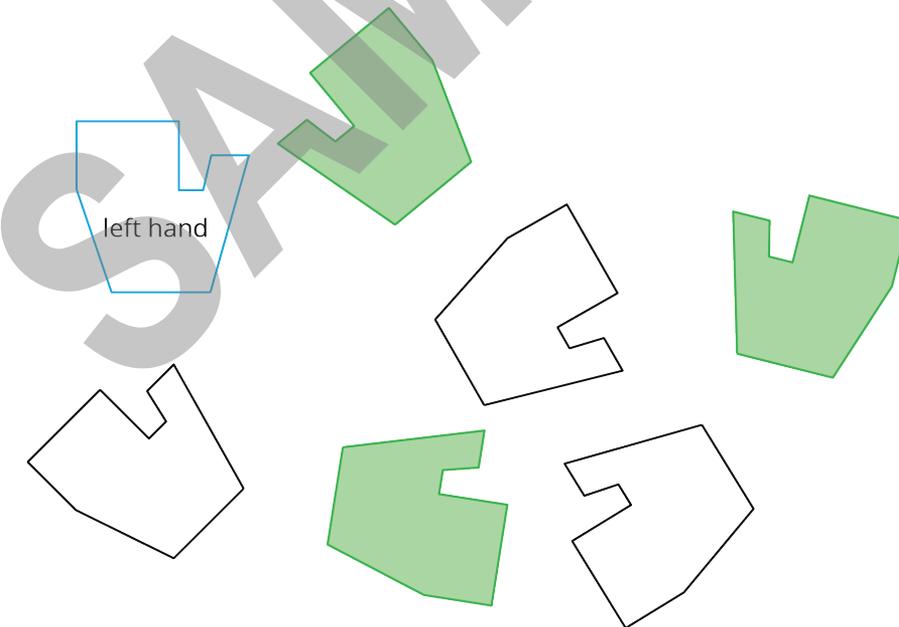


Student Task Statement

A person's hands are mirror images of each other. In the diagram, a left hand is labeled. Shade all of the right hands.



Student Response



Activity Synthesis

Ask students to think about the ways in which the left and right hands are the same, and the ways in which they are different.

Some ways that they are the same include:

- The side lengths and angles on the left and right hands match up with one another.
- If a left hand is flipped, it can match it up perfectly with a right hand (and vice versa).

Some ways that they are different include:

- They can not be lined up with one another *without* flipping one of the hands over.
- It is not possible to make a physical left and right hand line up with one another, except as “mirror images.”

10.2 Are They the Same?

15 minutes

In previous work, students learned to identify translations, rotations, and reflections. They started to study what happens to different shapes when these transformations are applied. They used sequences of translations, rotations, and reflections to build new shapes and to study complex configurations in order to compare, for example, vertical angles made by a pair of intersecting lines. Starting in this lesson, rigid transformations are used to formalize what it means for two shapes to be the same, a notion which students have studied and applied since the early grades of elementary school.

In this activity, students express what it means for two shapes to be the same by considering carefully chosen examples. Students work to decide whether or not the different pairs of shapes are the same. Then the class discusses their findings and comes to a consensus for what it means for two shapes to be the same: the word “same” is replaced by “congruent” moving forward.

There may be discussion where a reflection is required to match one shape with the other. Students may disagree about whether or not these should be considered the same and discussion should be encouraged. This activity encourages MP3 as students need to explain why they believe that a pair of figures is the same or is not the same.

Monitor for students who use these methods to decide whether or not the shapes are the same and invite them to share during the discussion:

- Observation (this is often sufficient to decide that they are not the same): Encourage students to articulate what feature(s) of the shapes help them to decide that they are not the same.
- Measuring side lengths using a ruler or angles using a protractor: Then use differences among these measurements to argue that two shapes are not the same.
- Cutting out one shape and trying to move it on top of the other: A variant of this would be to separate the two images and then try to put one on top of the other or use tracing paper to trace one of the shapes. This is a version of applying transformations studied extensively prior to this lesson.

Addressing

- 8.G.A.2

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR2: Collect and Display

Launch

Give 5 minutes of quiet work time followed by a whole-class discussion. Provide access to geometry toolkits.

Support for Students with Disabilities

Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, "This pair of shapes is/is not the same because..." or "If I translate/rotate/reflect, then..."

Supports accessibility for: Language; Organization

Support for English Language Learners

Conversing, Representing: MLR2 Collect and Display. As students work on comparing shapes, circulate and listen to students talk. Record common or important phrases (e.g., side length, rotated, reflected, etc.), together with helpful sketches or diagrams on a display. Pay particular attention to how students are using transformational language while determining whether the shapes are the same. Scribe students' words and sketches on a visual display to refer back to during whole-class discussions throughout this lesson and the rest of the unit. This will help students use mathematical language during their group and whole-class discussions.

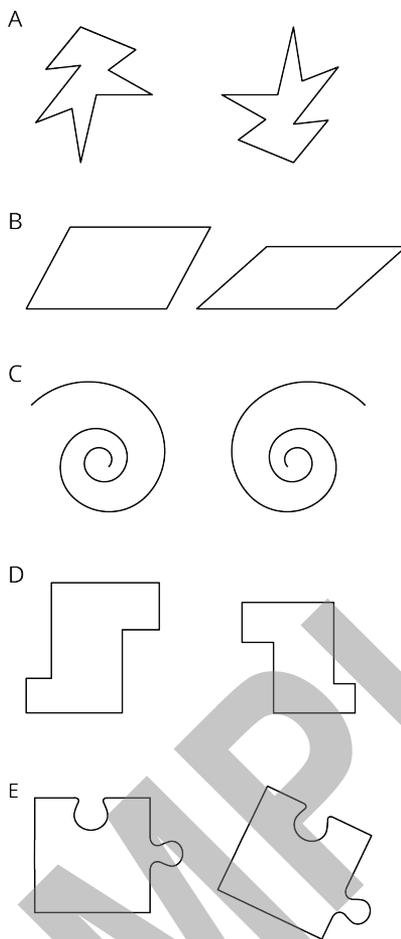
Design Principle(s): Support sense-making

Anticipated Misconceptions

Students may think all of the shapes are the same because they are the same general shape at first glance. Ask these students to look for any differences they can find among the pairs of shapes.

Student Task Statement

For each pair of shapes, decide whether or not they are the same.



Student Response

1. The two shapes are the same. Rotating the shape on the left (by 180 degrees) around the top point and moving it down and to the right it matches up perfectly with the shape on the right.
2. They are not the same. Possible strategy: The side lengths of the shapes are the same but the angles are not. The shape on the right is more squished down (and has less area) so they are not the same.
3. They are the same. Possible strategy: The shapes are both curly arrows and look like they are the same size. Reflecting over a vertical line halfway between the two shapes, they appear to match up perfectly with one another. *Or:* They are not the same. Possible strategy: The curly arrow on the left moves in a clockwise direction while the curly arrow on the right moves in a counterclockwise direction.
4. They are not the same. Possible strategy: The general shapes are the same and the angles match up but the side lengths are different. The shape on the left is bigger than the shape on the right.
5. They are not the same. Possible strategy: The part that sticks out of the right side is higher on the first piece and lower on the second piece. Building a puzzle, both shapes would not fit in the same spot.

Activity Synthesis

For each pair of shapes, poll the class. Count how many students decided each pair was the same or not the same. Then for each pair of shapes, select at least one student to defend their reasoning. (If there is unanimous agreement over any of the pairs of shapes, these can be dealt with quickly, but allow the class to hear at least one argument for each pair of shapes.)

Sequence these explanations in the order suggested in the Activity Narrative: general observations, taking measurements, and applying rigid transformations with the aid of tracing paper.

The most general and precise of these criteria is the third which is the foundation for the mathematical definition of congruence: The other two are consequences. The moves allowed by rigid transformations do not change the shape, size, side lengths, or angle measures.

There may be disagreement about whether or not to include reflections when deciding if two shapes are the same. Here are some reasons to include reflections:

- A shape and its reflected image can be matched up perfectly (using a reflection).
- Corresponding angles and side lengths of a shape and its reflected image are the same.

And here are some reasons against including reflections:

- A left foot and a right foot (for example) do not work exactly the same way. If we literally had two left feet it would be difficult to function normally!
- Translations and rotations can be enacted, for example, by putting one sheet of tracing paper on top of another and physically translating or rotating it. For a reflection the typical way to do this is to lift one of the sheets and flip it over.

If this disagreement doesn't come up, ask students to think about why someone might conclude that the pair of figures in C were not the same. Explain to students that people in the world can mean many things when they say two things are "the same." In mathematics there is often a need to be more precise, and one *kind of* "the same" is **congruent**. (Two figures *are* congruent if one is a reflection of the other, but one could, if one wanted, define a different term, a different kind of "the same," where flipping was not allowed!)

Explain that Figure A is **congruent** to Figure B if there is a sequence of translations, rotations, and reflections which make Figure A match up exactly with Figure B.

Combining this with the earlier discussion a few general observations about congruent figures include

- Corresponding sides of congruent figures are congruent.
- Corresponding angles of congruent figures are congruent.
- The area of congruent figures are equal.

What can be “different” about two congruent figures? The location (they don't have to be on top of each other) and the orientation (requiring a reflection to move one to the other) can be different.

10.3 Area, Perimeter, and Congruence

10 minutes

Sometimes people characterize congruence as “same size, same shape.” The problem with this is that it isn't clear what we mean by “same shape.” All of the figures in this activity have the same shape because they are all rectangles, but they are not all congruent. Students examine a set of rectangles and classify them according to their area and perimeter. Then they identify which ones are congruent. Because congruent shapes have the same side lengths, congruent rectangles have the same perimeter. But rectangles with the same perimeter are *not* always congruent. Congruent shapes, including rectangles, also have the same area. But rectangles with the same area are *not* always congruent. Highlighting important features, like perimeter and area, which can be used to quickly establish that two shapes are not congruent develops MP7, identifying fundamental properties shared by any pair of congruent shapes.

Addressing

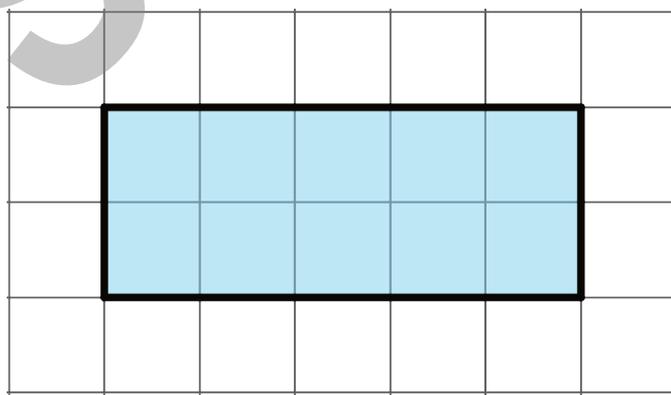
- 8.G.A.2

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- Think Pair Share

Launch

Tell students that they will investigate further how finding the area and perimeter of a shape can help show that two figures are not congruent. It may have been a while since students have thought about the terms area and perimeter. If necessary, to remind students what these words mean and how they can be computed, display a rectangle like this one for all to see. Ask students to explain what perimeter means and how they can find the perimeter and area of this rectangle.



Arrange students in groups of 2. Provide access to geometry toolkits (colored pencils are specifically called for). Give 2 minutes for quiet work time followed by sharing with a partner and a whole-class discussion.

Support for Students with Disabilities

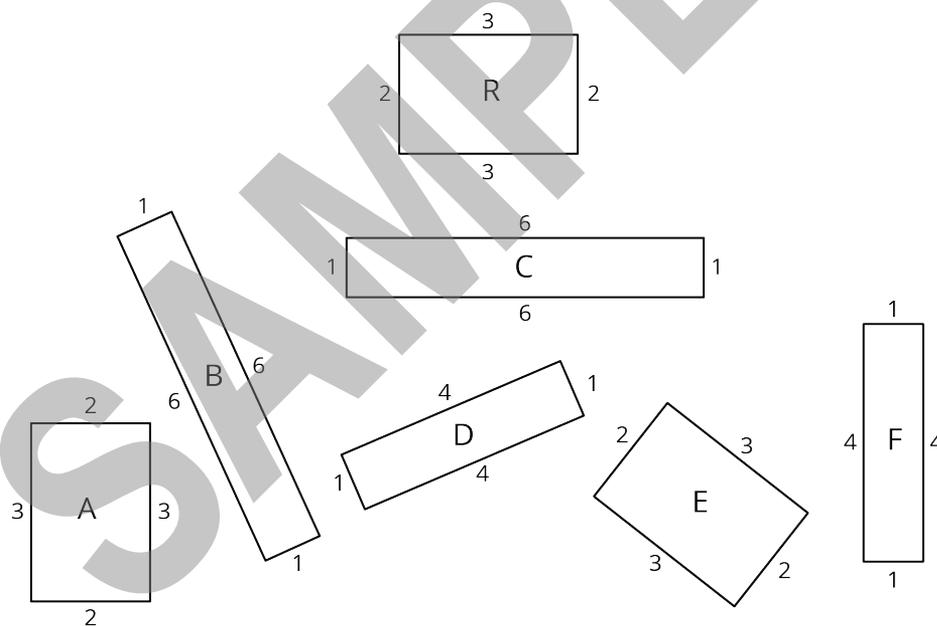
Representation: Internalize Comprehension. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. For example, provide students with a subset of the rectangles to start with and introduce the remaining remaining rectangles once students have completed their initial set of comparisons.

Supports accessibility for: Conceptual processing; Organization

Anticipated Misconceptions

Watch for students who think about the final question in terms of “same shape and size.” Remind them of the definition of congruence introduced in the last activity.

Student Task Statement

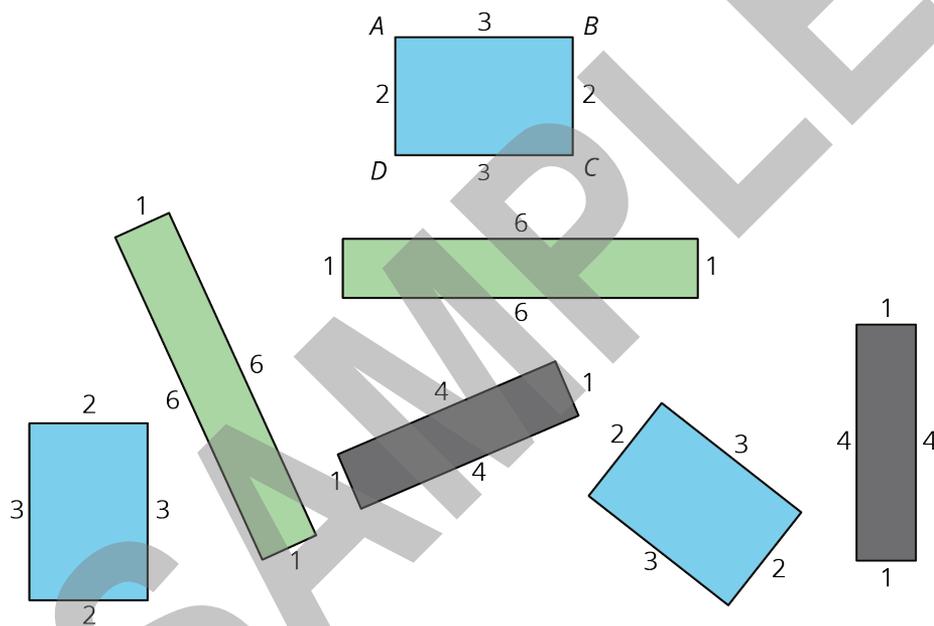


1. Which of these rectangles have the same area as Rectangle R but different perimeter?
2. Which rectangles have the same perimeter but different area?
3. Which have the same area *and* the same perimeter?
4. Use materials from the geometry tool kit to decide which rectangles are **congruent**. Shade congruent rectangles with the same color.

Student Response

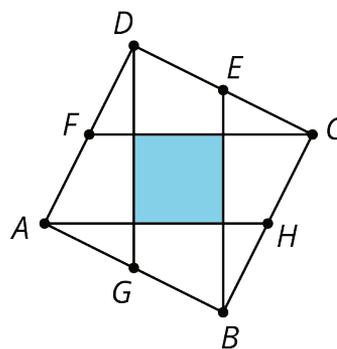
The perimeter of Rectangle R is 10 units since $3 + 2 + 3 + 2 = 10$ while its area is 6 square units since $2 \cdot 3 = 6$. All of the rectangles in the picture share at least one of these properties (either the perimeter or the area) but only the 2 unit by 3 unit rectangles share both:

1. Rectangles B and C have the same area (6 square units) but different perimeter (14 units)
2. Rectangles D and F have the same perimeter (10 units) but different area (4 square units)
3. Rectangles A and E have the same area and perimeter: only their position and orientation on the page is different.
4. The 2 by 3 rectangles are congruent to Rectangle R. In each case, Rectangle R can be translated and rotated so that it matches up perfectly with the 2-by-3 rectangle. The same argument shows that Rectangles B and C are congruent as are Rectangles D and F.



Are You Ready for More?

In square $ABCD$, points E , F , G , and H are midpoints of their respective sides. What fraction of square $ABCD$ is shaded? Explain your reasoning.



Student Response

$\frac{1}{5}$ of square $ABCD$ is shaded. Reasoning varies. Sample reasoning: Transform the unshaded pieces into four congruent squares that are each congruent to the shaded square.

It is interesting to generalize this problem such that points $E, F, G,$ and H partition the sides of $ABCD$ in a ratio other than $1 : 1$.

Activity Synthesis

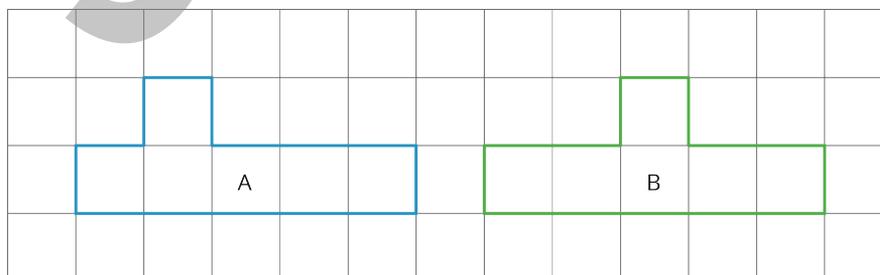
Invite students who used the language of transformations to answer the final question to describe how they determined that a pair of rectangles are congruent.

Perimeter and area are two different ways to measure the size of a shape. Ask the students:

- "Do congruent rectangles have the same perimeter? Explain your reasoning." (Yes. Rigid motions do not change distances, and so congruent rectangles have the same perimeter.)
- "Do congruent rectangles have the same area? Explain your reasoning." (Yes. Rigid motions do not change area *or* rigid motions do not change distances and so do not change the length times the width in a rectangle.)
- "Are rectangles with the same perimeter always congruent?" (No. Rectangles D and F have the same perimeter but they are not congruent.)
- "Are rectangles with the same area always congruent?" (No. Rectangles B and C have the same area but are not congruent.)

One important take away from this lesson is that measuring perimeter and area is a good method to show that two shapes are *not* congruent if these measurements differ. When the measurements are the same, more work is needed to decide whether or not two shapes are congruent.

A risk of using rectangles is that students may reach the erroneous conclusion that if two figures have both the same area and the same perimeter, then they are congruent. If this comes up, challenge students to think of two shapes that have the same area and the same perimeter, but are not congruent. Here is an example:



Support for English Language Learners

Writing, Speaking: MLR1 Stronger and Clearer Each Time. Use this routine with to give students a structured opportunity to revise their written strategies for deciding which rectangles are congruent. Give students time to meet with 2–3 partners to share and get feedback on their responses. Display prompts for feedback that will help individuals strengthen their ideas and clarify their language. For example, “How was a sequence of transformations used to...?”, “What properties do the shapes share?”, and “What was different and what was the same about each pair?” Students can borrow ideas and language from each partner to strengthen their final product.

Design Principle(s): Optimize output (for explanation)

Lesson Synthesis

Ask students to state their best definition of **congruent**. (Two shapes are congruent when there is a sequence of translations, rotations, and reflections that take one shape to the other.)

Some important concepts to discuss:

- "How can you check if two shapes are congruent?" (For rectangles, the side lengths are enough to tell. For more complex shapes, experimenting with transformations is needed.)
- "Are a shape and its mirror image congruent?" (Yes, because a reflection takes a shape to its mirror image.)
- "What are some ways to know that two shapes are not congruent?" (Two shapes are not congruent if they have different areas, side lengths, or angles.)
- "What are some properties that are shared by congruent shapes?" (They have the same number of sides, same length sides, same angles, same area.)

10.4 Mirror Images

Cool Down: 5 minutes

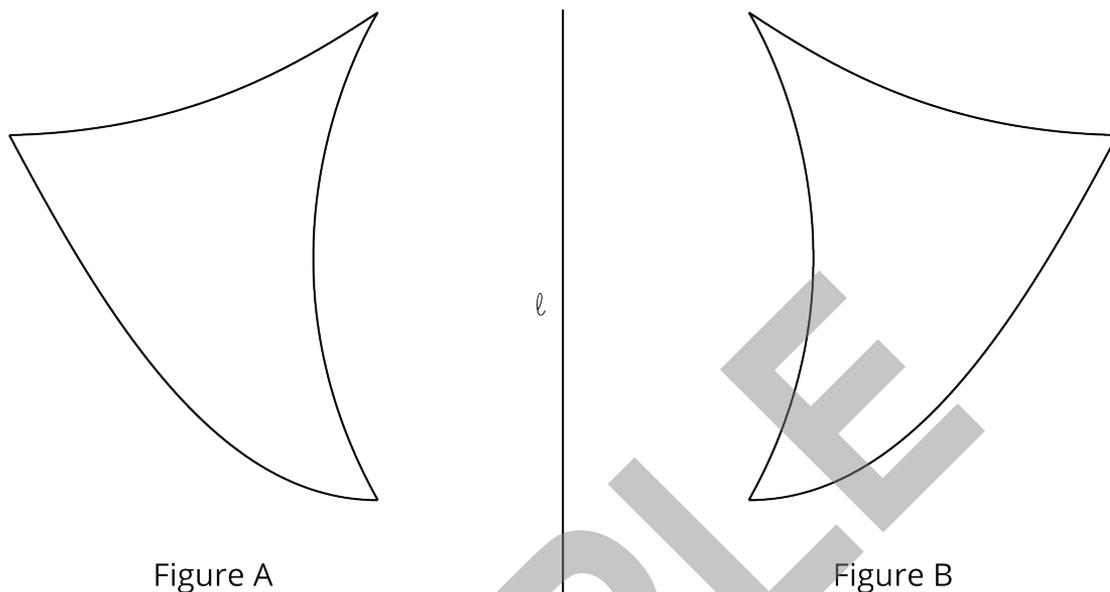
Throughout this unit, students have been using translations, rotations, and reflections to move figures in the plane. In this lesson, students have learned that Figure A is congruent to Figure B when there is a sequence of translations, rotations, and reflections that take Figure A to Figure B. Here they apply this to two non-polygonal figures, one of which is a reflection of the other.

Addressing

- 8.G.A.2

Student Task Statement

Figure B is the image of Figure A when reflected across line ℓ . Are Figure A and Figure B congruent? Explain your reasoning.

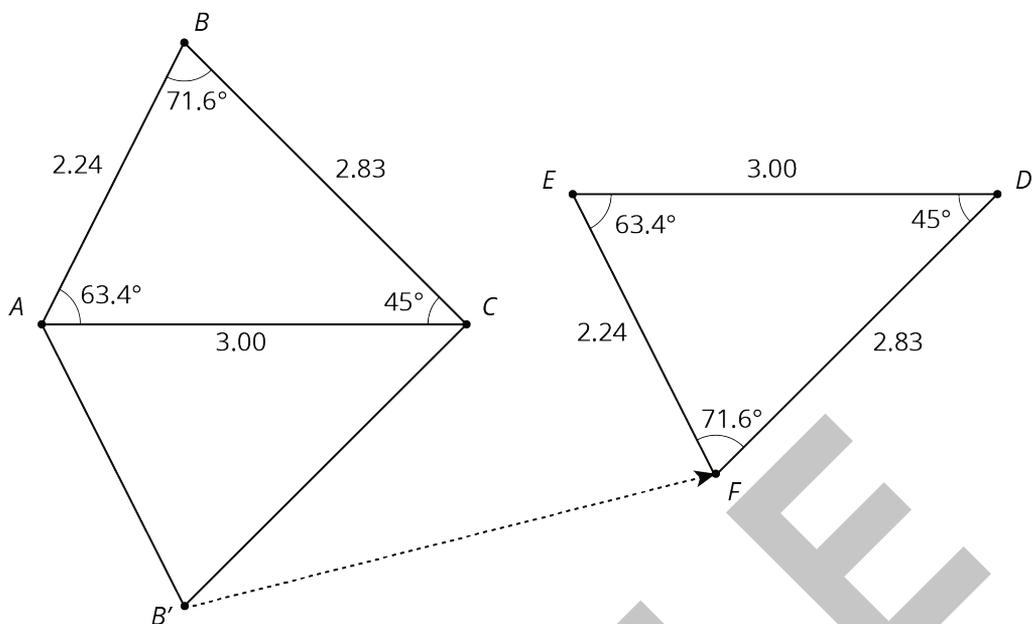


Student Response

Yes, they are congruent. There is a rigid transformation that takes one figure to the other, so they are congruent.

Student Lesson Summary

Congruent is a new term for an idea we have already been using. We say that two figures are congruent if one can be lined up exactly with the other by a sequence of rigid transformations. For example, triangle EFD is congruent to triangle ABC because they can be matched up by reflecting triangle ABC across AC followed by the translation shown by the arrow. Notice that all corresponding angles and side lengths are equal.



Here are some other facts about congruent figures:

- We don't need to check all the measurements to prove two figures are congruent; we just have to find a sequence of rigid transformations that match up the figures.
- A figure that looks like a mirror image of another figure can be congruent to it. This means there must be a reflection in the sequence of transformations that matches up the figures.
- Since two congruent polygons have the same area and the same perimeter, one way to show that two polygons are *not* congruent is to show that they have a different perimeter or area.

Glossary

- congruent

Lesson 10 Practice Problems

Problem 1

Statement

If two rectangles have the same perimeter, do they have to be congruent? Explain how you know.

Solution

No. Two non-congruent rectangles can have the same perimeter. For example, a rectangle with side lengths 3 inches and 4 inches is not congruent to a rectangle with side lengths 2 inches and 5 inches. Even though the angles of all rectangles have the same measure, when two figures are congruent all side lengths and angle measures are the same.

Problem 2

Statement

Draw two rectangles that have the same area, but are *not* congruent.

Solution

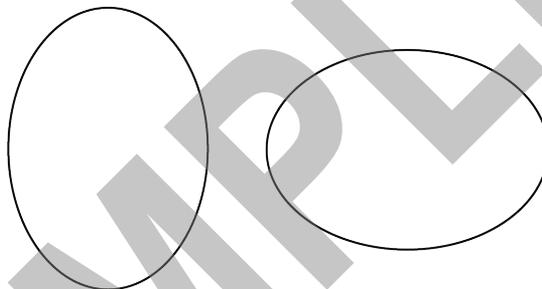
Answers vary. Sample response: a 2-by-6 rectangle and a 3-by-4 rectangle.

Problem 3

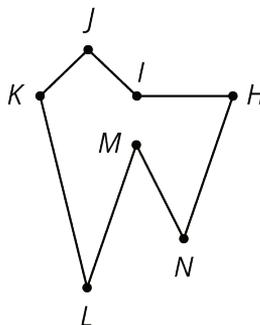
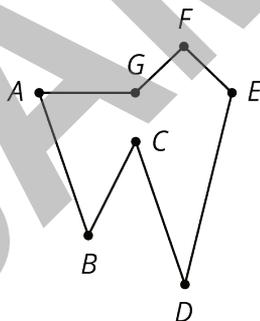
Statement

For each pair of shapes, decide whether or not the two shapes are congruent. Explain your reasoning.

a.



b.



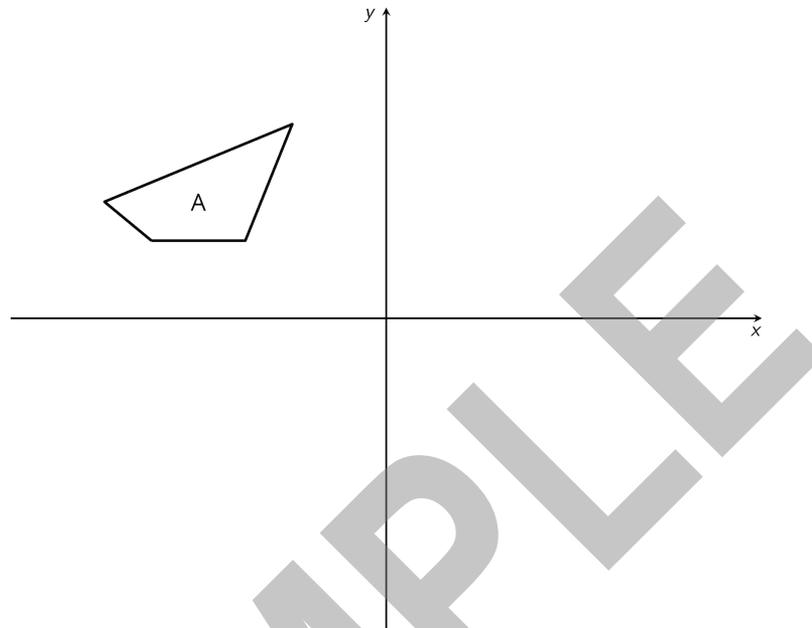
Solution

- These appear to be congruent. If the shape on the right is traced, it can be moved over and it appears to match up perfectly with the shape on the left. This can be done with a rotation (90 degrees clockwise) and then a translation.
- These appear to be congruent. If $ABCDEF$ is reflected about a vertical line and then translated, it appears to land on top of $HNMLKJI$.

Problem 4

Statement

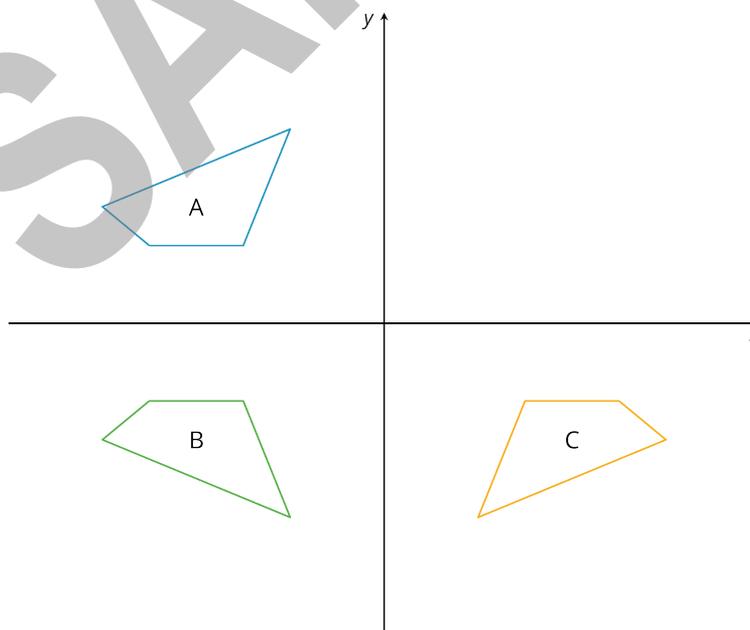
- a. Reflect Quadrilateral A over the x -axis. Label the image quadrilateral B. Reflect Quadrilateral B over the y -axis. Label the image C.



- b. Are Quadrilaterals A and C congruent? Explain how you know.

Solution

a.



- b. Yes, because there is a rigid transformation taking A to C, the two shapes are congruent.

Problem 5

Statement

The point $(-2, -3)$ is rotated 90 degrees counterclockwise using center $(0, 0)$. What are the coordinates of the image?

- A. $(-3, -2)$
- B. $(-3, 2)$
- C. $(3, -2)$
- D. $(3, 2)$

Solution

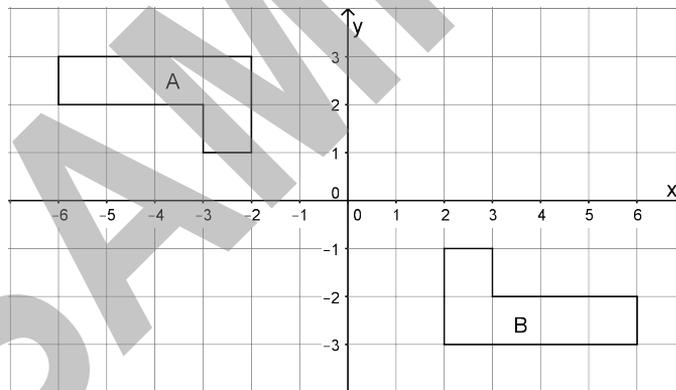
C

(From Unit 1, Lesson 5.)

Problem 6

Statement

Describe a rigid transformation that takes Polygon A to Polygon B.



Solution

Answers vary. Sample response: Rotate Polygon A 180 degrees around $(0, 0)$.

(From Unit 1, Lesson 6.)

Lesson 11: Congruence

Goals

- Determine whether shapes are congruent by measuring corresponding points.
- Draw and label corresponding points on congruent figures.
- Justify (orally and in writing) that congruent figures have equal corresponding distances between pairs of points.
- Justify (orally and in writing) that two polygons on a grid are congruent using the definition of congruence in terms of transformations.

Learning Targets

- I can decide using rigid transformations whether or not two figures are congruent.
- I can use distances between points to decide if two figures are congruent.

Lesson Narrative

In this lesson, students find rigid transformations that show two figures are congruent and make arguments for why two figures are not congruent. They learn that, for many shapes, simply having corresponding side lengths that are equal will not guarantee the figures are congruent.

In a previous lesson, students defined what it means for two shapes to be congruent and started to apply the definition to determine if a pair of shapes are congruent. Now, they determine whether or not pairs of shapes are congruent with the structure of a grid. With this structure, students attend to precision (MP6) when describing translations, reflections, and rotations by using language like:

- “Translate 3 units down and 2 units to the left,” instead of “translate down and to the left.”
- “Reflect the shape over this vertical line,” instead of “reflect the shape.”

This lesson also asks students to consider more complex shapes with curved sides. The focus here is on the fact that the distance between any pair of corresponding points of congruent figures must be the same. Because there are too many pairs of points to consider, this is mainly a criterion for showing that two figures are not congruent: that is, if there is a pair of points on one figure that are a different distance apart than the corresponding points on another figure, then those figures are not congruent.

Throughout the lesson, students have to be careful how they name congruent shapes, making sure that corresponding points are listed in the proper order.

Alignments

Addressing

- 8.G.A.1.a: Lines are taken to lines, and line segments to line segments of the same length.

- 8.G.A.2: Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's decide if two figures are congruent.

11.1 Translated Images

Warm Up: 5 minutes

This task helps students think strategically about what kinds of transformations they might use to show two figures are congruent. Being able to recognize when two figures have either a mirror orientation or rotational orientation is useful for planning out a sequence of transformations.

Addressing

- 8.G.A.2

Launch

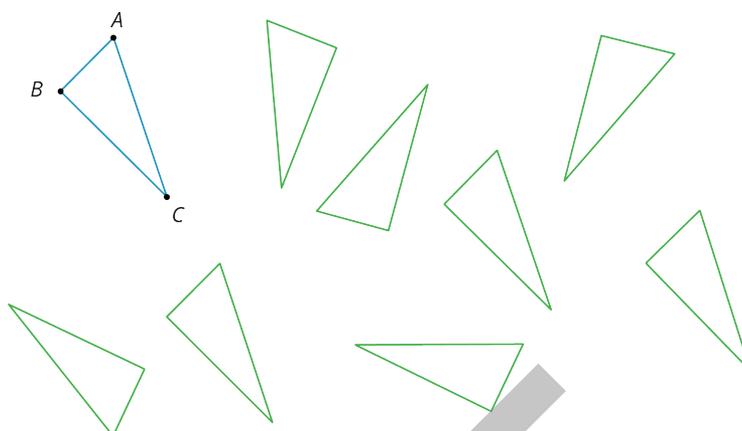
Provide access to geometry toolkits. Allow for 2 minutes of quiet work time followed by a whole-class discussion.

Anticipated Misconceptions

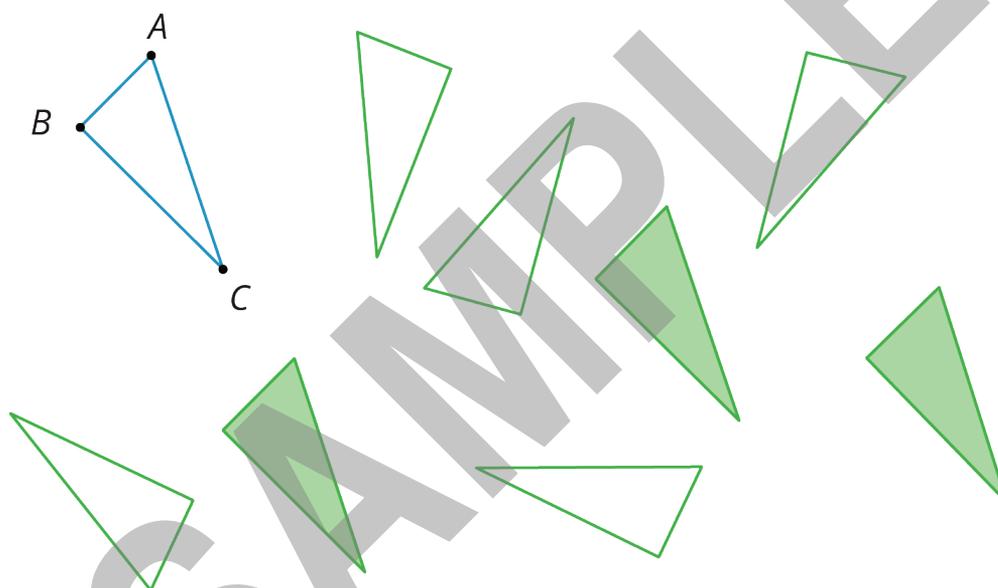
If any students assert that a triangle is a translation when it isn't really, ask them to use tracing paper to demonstrate how to translate the original triangle to land on it. Inevitably, they need to rotate or flip the paper. Remind them that a translation consists only of sliding the tracing paper around without turning it or flipping it.

Student Task Statement

All of these triangles are congruent. Sometimes we can take one figure to another with a translation. Shade the triangles that are images of triangle ABC under a translation.



Student Response



Activity Synthesis

Point out to students that if we just translate a figure, the image will end up pointed in the same direction. (More formally, the figure and its image have the same mirror and rotational orientation.) Rotations and reflections usually (but not always) change the orientation of a figure.

For a couple of the triangles that are not translations of the given figure, ask what sequence of transformations would show that they are congruent, and demonstrate any rotations or reflections required.

11.2 Congruent Pairs

15 minutes

In the previous lesson, students formulated a precise mathematical definition for congruence and began to apply transformations to determine whether or not pairs of figures are congruent. This

activity is a direct continuation of that work with the extra structure of a square grid. The square grid can be a helpful structure for describing the different transformations in a precise way. For example, with translations we can talk about translating up or down or to the left or right by a specified number of units. Similarly, we can readily reflect over horizontal and vertical lines and perform some simple rotations. Students may also wish to use tracing paper to help execute these transformations (MP5).

Students are given several pairs of shapes on grids and asked to determine if the shapes are congruent. The congruent shapes are deliberately chosen so that more than one translation, rotation, or reflection is required to show the congruence. In these cases, students will likely find different ways to show the congruence. Monitor for different sequences of transformations that show congruence. For example, for the first pair of quadrilaterals, some different ways are:

- translate $EFGH$ 1 unit to the right, and then rotate its image 180° about $(0, 0)$
- reflect $ABCD$ over the x -axis, then reflect its image over the y -axis, and then translate this image 1 unit to the left

For the pairs of shapes that are not congruent, students need to identify a feature of one shape not shared by the other in order to argue that it is not possible to move one shape on top of another with rigid motions. At this early stage, arguments can be informal. Monitor for students who identify that:

- the side lengths are different so it is not possible to make them match up
- the angles are different so the two shapes can not be made to match up
- the areas of the shapes are different

Addressing

- 8.G.A.2

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- MLR7: Compare and Connect

Launch

Provide access to geometry toolkits. Allow for 5–10 minutes of quiet work time followed by a whole-class discussion.

Support for Students with Disabilities

Engagement: Internalize Self Regulation. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. Invite students to choose and respond to 3 out of 5 questions. Once students have successfully completed them, invite them to share with a partner prior to the whole class discussion.

Supports accessibility for: Organization; Attention

Anticipated Misconceptions

Students may want to visually determine congruence each time or explain congruence by saying, “They look the same.” Encourage those students to explain congruence in terms of translations, rotations, reflections, and side lengths. For students who focus on features of the shapes such as side lengths and angles, ask them how they could show the side lengths or angle measures are the same or different using the grid or tracing paper.

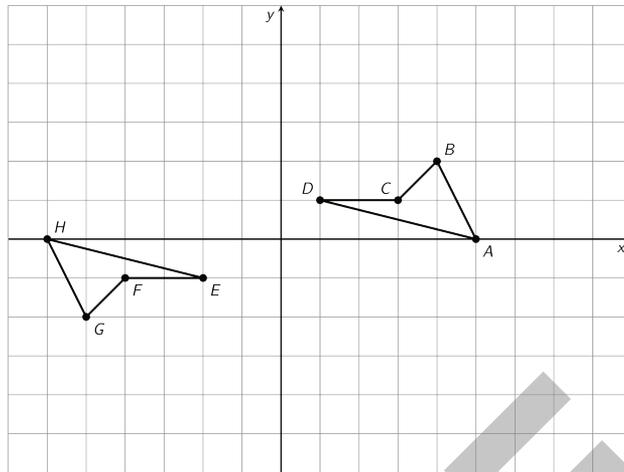
When discussing congruence for the second to last question, students may say that quadrilateral $GHIJ$ is congruent to quadrilateral $PQRS$ due to wanting to use an alphabetical ordering of the letters, but this is not correct. After a set of transformations is applied to quadrilateral $GHIJ$, it corresponds to quadrilateral $QRSP$. The vertices must be listed in this order to accurately communicate the correspondence between the two congruent quadrilaterals.

For the last question, students may be correct in saying the shapes are not congruent but for the wrong reason. They may say one is a 3-by-3 square and the other is a 2-by-2 square, counting the diagonal side lengths as one unit. If so, have them compare lengths by marking them on the edge of a card, or measuring them with a ruler.

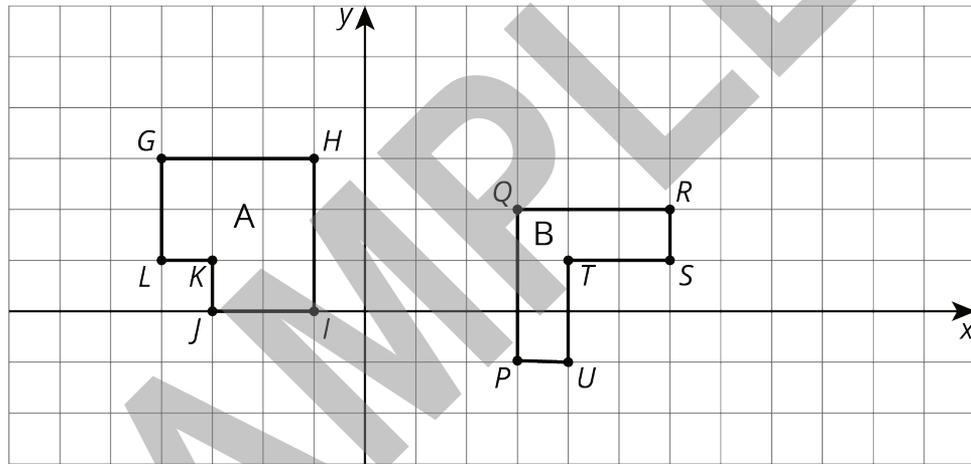
Student Task Statement

For each of the following pairs of shapes, decide whether or not they are congruent. Explain your reasoning.

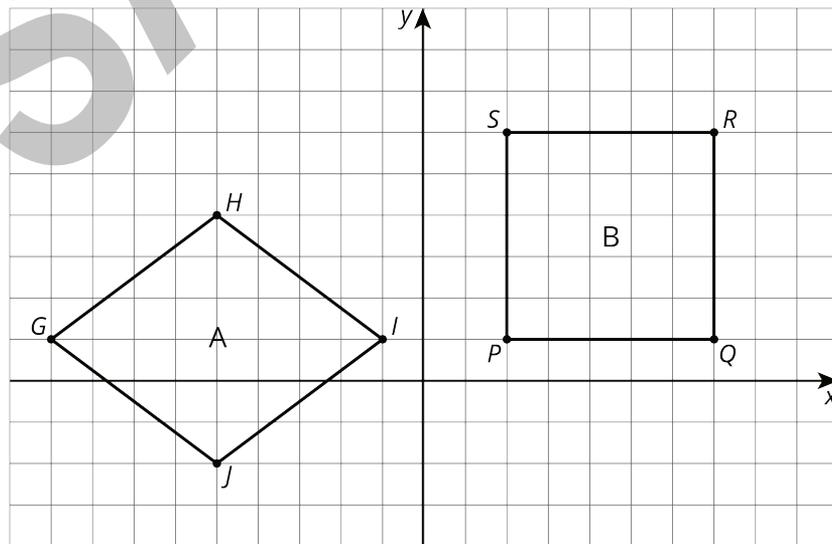
1.



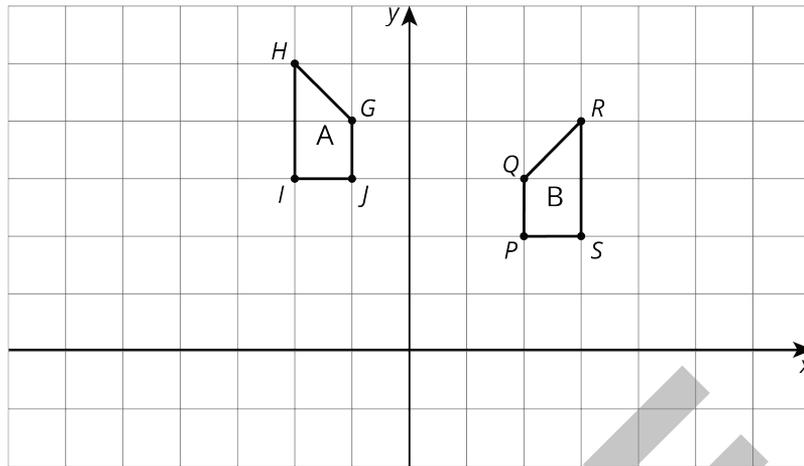
2.



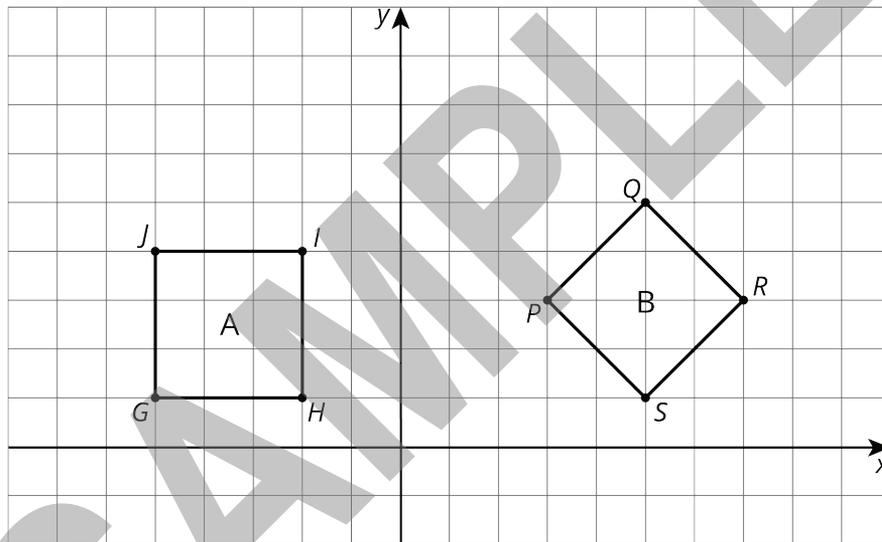
3.



4.



5.



Student Response

1. These are congruent. Sample response: Rotate quadrilateral $ABCD$ around D by 180 degrees, and then translate left 3 units and down 2 units. It matches up perfectly with $HGFE$.
2. No, they are not congruent. Sample response: Hexagon A has greater area (8 square units) than Hexagon B (6 square units). They are not congruent because translations, rotations, and reflections do *not* change the area of a figure.
3. No, they are not congruent. Sample response: Both shapes are quadrilaterals, and the side lengths all appear to be 5 units in length. But the angles are not the same. Quadrilateral B is a square with 4 right angles. Quadrilateral A is a rhombus. Angles G and I are acute while Angles H and J are obtuse. Since translations, rotations, and reflections do not change angle measures, there is no way to match up any of the angles of these quadrilaterals.

4. Yes, they are congruent. Sample response: Reflect Quadrilateral A over the y -axis, and then translate one unit to the right and one unit down. It matches up perfectly with Quadrilateral B.
5. No, they are not congruent. Sample response 1: Rotate Quadrilateral B about S by 45 degrees counterclockwise and then translate to the left by 7 units. Angle PSR matches up with angle GHI , but the sides of Quadrilateral B are a little shorter than those of Quadrilateral A, so the two shapes are not congruent. Sample response 2: The area of square Quadrilateral A is 9 square units. The area of Quadrilateral B (which is also a square) is 8 square units because it contains 4 whole unit squares and then 8 half unit squares that make 4 more unit squares. Congruent shapes have the same area so these two shapes are not congruent.

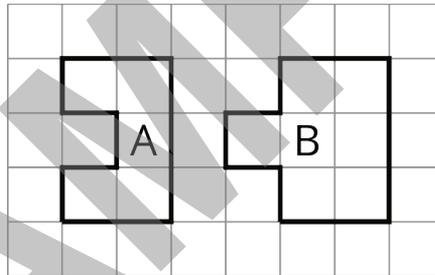
Are You Ready for More?

A polygon has 8 sides: five of length 1, two of length 2, and one of length 3. All sides lie on grid lines. (It may be helpful to use graph paper when working on this problem.)

1. Find a polygon with these properties.
2. Is there a second polygon, not congruent to your first, with these properties?

Student Response

Here are two non-congruent shapes that meet the conditions.



Activity Synthesis

Survey the class, perhaps using a show of hands, to identify which shapes are congruent (1 and 4) and which ones are not (2, 3, and 5). For the congruent shapes, ask which motions (translations, rotations, or reflections) students used, and select previously identified students to show different methods. Sequence the methods from most steps to fewest steps when possible.

For the shapes that are not congruent, invite students to identify features that they used to show this and ask students if they tried to move one shape on top of the other. If so, what happened? It is important for students to notice typical reasoning when identifying congruent or non-congruent figures.

The purpose of the discussion is to understand that when two shapes are congruent, there is a rigid transformation that matches one shape up perfectly with the other. Choosing the right sequence takes practice. Students should be encouraged to experiment, using technology and tracing paper when available. When two shapes are not congruent, there is no rigid transformation that matches one shape up perfectly with the other. It is not possible to perform every possible sequence of

transformations in practice, so to show that one shape is not congruent to another, we identify a property maintained by rigid transformations of one shape that is not shared by the other. For the shapes in this problem set, students can focus on side lengths or angles. Since transformations do not change side lengths or angles, this is enough to conclude that the two shapes are not congruent.

Support for English Language Learners

Representing, Conversing, Listening: MLR7 Compare and Connect. As students prepare their work for discussion, look for approaches that focus on visually determining congruence and on approaches that focus on features of the shapes such as side lengths and angles. Encourage students to explain congruence in terms of translations, rotations, reflections, and side lengths and to show physical representations of congruence of side lengths and angle measures using grid or tracing paper. Emphasize transformational language used to make sense of strategies to identify congruent and non-congruent figures.

Design Principle(s): Maximize meta-awareness; Support sense-making

11.3 Corresponding Points in Congruent Figures

15 minutes

Corresponding sides of congruent polygons have the same length. For shapes like ovals, examined in the previous activity, there are no “sides.” However, if points A and B on one figure correspond to points A' and B' on a congruent figure, then the length of segment AB is equal to the length of segment $A'B'$ because translations, rotations, and reflections do not change distances between points. Students have seen and worked with this idea in the context of polygons and their sides. This remains true for other shapes as well.

Because rigid motions do not change distances between points, corresponding points on congruent figures (even oddly shaped figures!) are the same distance apart. This is one more good example MP7 as the fundamental mathematical property of rigid motions is that they do not change distances between corresponding points: this idea holds for *any* points on *any* congruent figures.

There are two likely strategies for identifying corresponding points on the two corresponding figures:

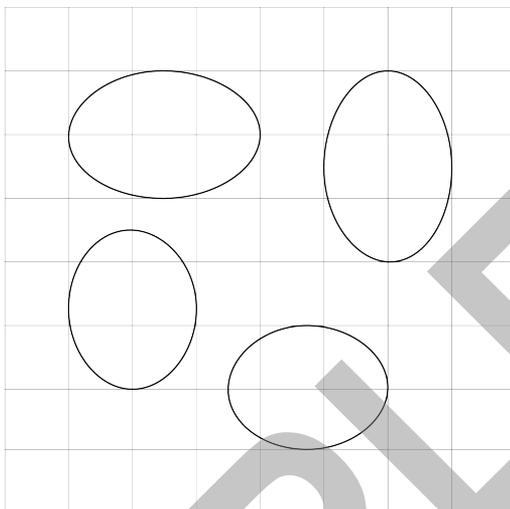
- Looking for corresponding parts of the figures such as the line segments
- Performing rigid motions with tracing paper to match the figures up

Both are important. Watch for students using each technique and invite them to share during the discussion.

Teacher Notes for IM 6–8 Math Accelerated

This activity originally followed one that examined four ovals on a grid and asked students to decide if any of the ovals were congruent. Since the oval activity is not used in IM 6–8 Math Accelerated, Replace the Launch with this shortened version of it to help prepare students for the main activity:

Display the image for all to see:



Ask, “Are any of the ovals congruent to one another? Be prepared to explain how you know.” (All four shapes are ovals. For the top two shapes, they can be surrounded by rectangles that measure 3 units by 2 units. For the bottom two shapes, they measure about $2\frac{1}{2}$ (possibly a little less) units by 2 units. This means that the top shapes are not congruent to the bottom shapes.) Emphasize that showing that two oval shapes are congruent requires using the definition of congruence: Is it possible to move one shape so that it matches up perfectly with the other using only rigid transformations? Experimentation with transformations is essential when showing that two of the ovals match up because, unlike polygons, these shapes are not determined by a finite list of vertices and side lengths.

Arrange students in groups of 2. Provide access to geometry toolkits. Allow for 5 minutes of quiet work time followed by sharing with a partner and a whole-class discussion.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Connect a new concept to one with which students have experienced success. For example, review the criteria used to determine congruence for polygons so that students can transfer these strategies in determining congruence for curved shapes.

Supports accessibility for: Social-emotional skills; Conceptual processing

Addressing

- 8.G.A.1.a

- 8.G.A.2

Instructional Routines

- MLR7: Compare and Connect

Launch

Keep students in the same groups. Allow for 5 minutes of quiet work time followed by sharing with a partner and a whole-class discussion. Provide access to geometry toolkits (rulers are needed for this activity).

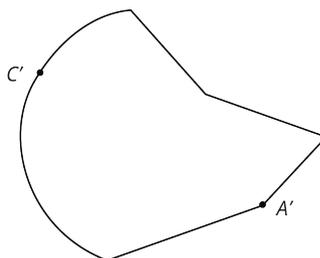
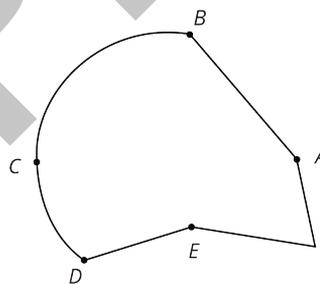
Support for Students with Disabilities

Action and Expression: Internalize Executive Functions. Provide students with the images on grid or graph paper to assist in labeling corresponding parts and taking measurements of line segments.

Supports accessibility for: Language; Organization

Student Task Statement

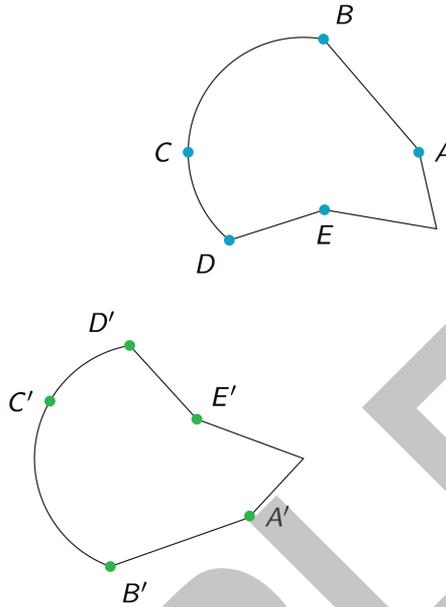
Here are two congruent shapes with some corresponding points labeled.



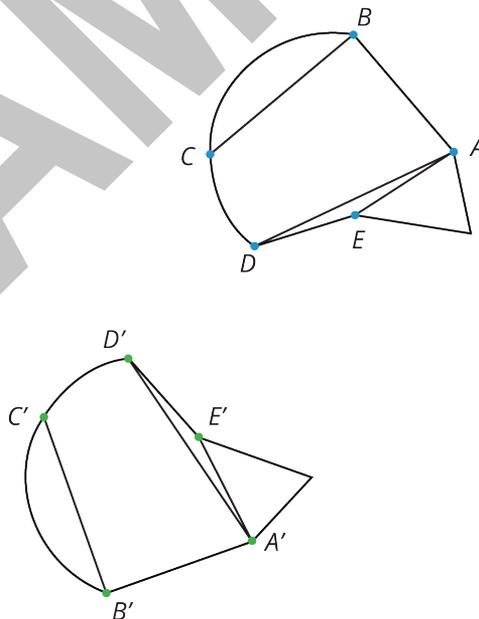
1. Draw the points corresponding to B , D , and E , and label them B' , D' , and E' .
2. Draw line segments AD and $A'D'$ and measure them. Do the same for segments BC and $B'C'$ and for segments AE and $A'E'$. What do you notice?
3. Do you think there could be a pair of corresponding segments with different lengths? Explain.

Student Response

1.



2. The lengths are the same. The rigid transformations of the plane used to show the congruence of these shapes do not change distances between points. So, the distance between A and D , for example, is the same as the distance between A' and D' . The same is true for the other pairs of corresponding points. The segments connecting these points are all shown here:



3. Sample response: No, rigid transformations do not change distances between points. Corresponding segments in the two congruent figures *must* have the same length.

Activity Synthesis

Ask selected students to show how they determined the points corresponding to B , D , and E , highlighting different strategies (identifying key features of the shapes and performing rigid motions). Ask students if these strategies would work for finding C' if it had not been marked. Performing rigid motions matches the shapes up perfectly, and so this method allows us to find the corresponding point for *any* point on the figure. Identifying key features only works for points such as A , B , D , and E , which are essentially like vertices and can be identified by which parts of the figures are “joined” at that point.

While it is challenging to test “by eye” whether or not complex shapes like these are congruent, the mathematical meaning of the word “congruent” is the same as with polygons: two shapes are congruent when there is a sequence of translations, reflections, and rotations that match up one shape exactly with the other. Because translations, reflections, and rotations do not change distances between points, *any* pair of corresponding segments in congruent figures will have the same length.

If time allows, have students use tracing paper to make a new figure that is either congruent to the shape in the activity or *slightly* different. Display several for all to see and poll the class to see if students think the figure is congruent or not. Check to see how the class did by lining up the new figure with one of the originals. Work with these complex shapes is important because we tend to rely heavily on visual intuition to check whether or not two polygons are congruent. This intuition is usually reliable unless the polygons are complex or have very subtle differences that cannot be easily seen. The meaning of congruence in terms of rigid motions and our visual intuition of congruence can effectively reinforce one another:

- If shapes look congruent, then we can use this intuition to find the right motions of the plane to demonstrate that they are congruent.
- Through experimenting with rigid motions, we increase our visual intuition about which shapes are congruent.

Support for English Language Learners

Representing, Conversing, Listening: MLR7 Compare and Connect. As students work, look for students who perform rigid motions with tracing paper to test congruence of two figures. Call students’ attention to the different ways they match up figures to identify corresponding points, and to the different ways these operations are made visible in each representation (e.g., lengths of segments that are equal, translations, rotations and reflections do not change distances between points). Emphasize and amplify the mathematical language students use when determining if two figures are congruent.

Design Principle(s): Maximize meta-awareness; Support sense-making

Lesson Synthesis

This lesson wraps up work on congruence. Important points to highlight include:

- Two figures are congruent when there is a sequence of translations, rotations, and reflections that match one figure up perfectly with the other. (This is from the previous lesson but it is vital to thinking in this lesson as well.)
- When showing that two figures are congruent on a grid, we use the structure of the grid to describe each rigid motion. For example, translations can be described by indicating how many grid units to move left or right and how many grid units to move up or down. Reflections can be described by indicating the line of reflection. (An axis or a particular grid line are readily available.)
- To show that two figures are not congruent it is enough to find corresponding points on the figures which are not the same distance apart, or corresponding angles that have different measure.
- The distance between pairs of corresponding points in congruent figures is the same. (This says that corresponding side lengths on polygons have the same length but it applies to curved figures also or to any pair of points, not necessarily vertices, on polygons.)

11.4 Explaining Congruence

Cool Down: 5 minutes

Students decide whether or not two ovals are congruent. These particular ovals are visually quite distinct so expect students to use one of these methods:

- Identify a distance on one oval that is different than the corresponding distance on the other oval.
- Trace one of the ovals and observe that it does not match up with the other one.

Addressing

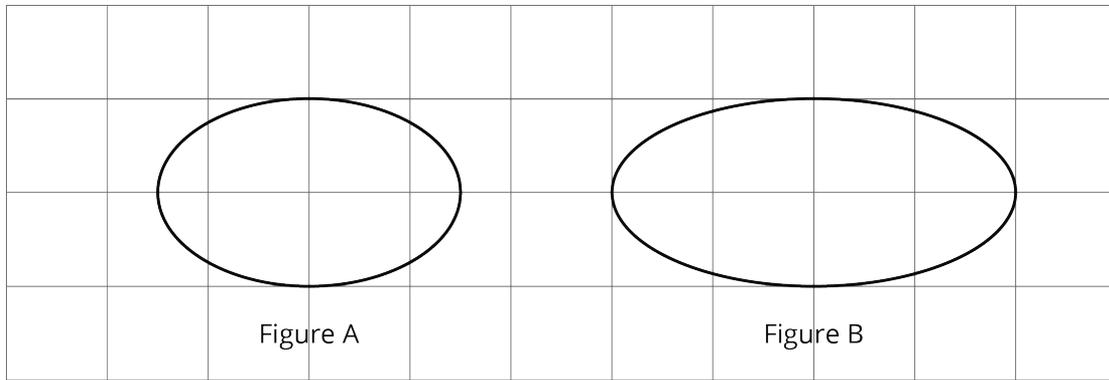
- 8.G.A.2

Launch

Make tracing paper available.

Student Task Statement

Are Figures A and B congruent? Explain your reasoning.



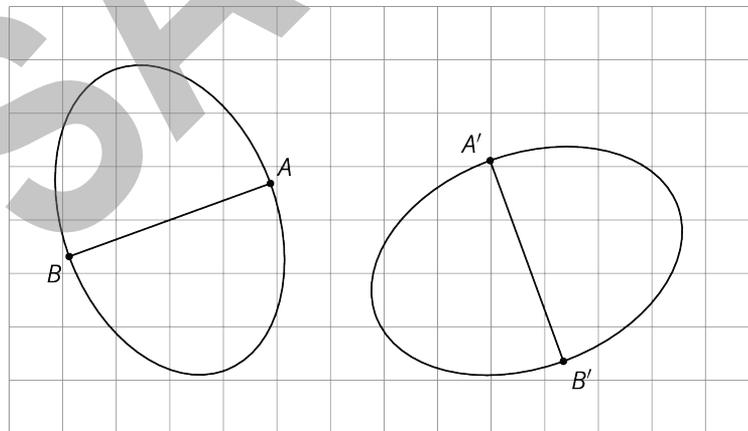
Student Response

Answers vary. Sample response: These figures are not congruent because if they were, the longest horizontal distances between two points would be the same. However, for A it is less than 4 units, and for B it is about 4 units.

Student Lesson Summary

How do we know if two figures are congruent?

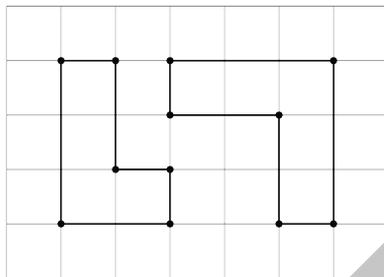
- If we copy 1 figure on tracing paper and move the paper so the copy covers the other figure exactly, then that suggests they are congruent.
- We can prove that 2 figures are congruent by describing a sequence of translations, rotations, and reflections that move 1 figure onto the other so they match up exactly.
- Distances between corresponding points on congruent figures are always equal, even for curved shapes. For example, corresponding segments AB and $A'B'$ on these congruent ovals have the same length:



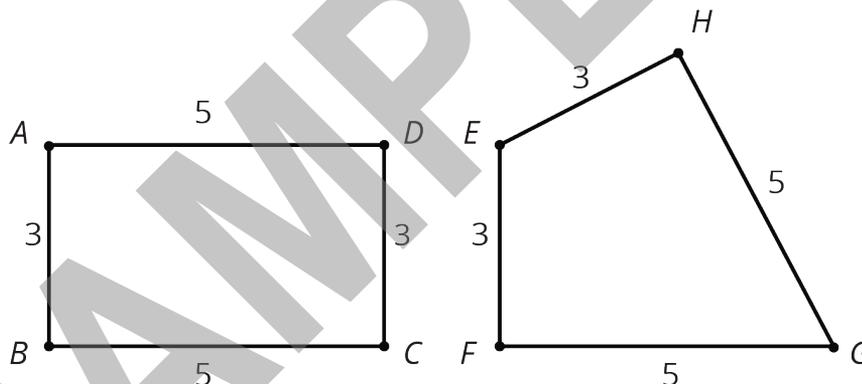
How do we know that 2 figures are not congruent?

- If there is no correspondence between the figures where the parts have equal measure, that proves that the two figures are not congruent. In particular,

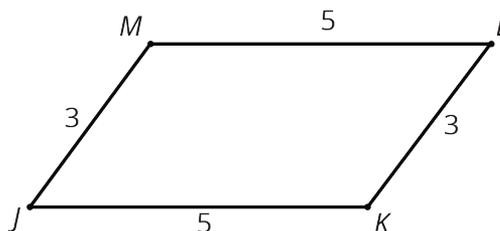
- If two polygons have different sets of side lengths, they can't be congruent. For example, the figure on the left has side lengths 3, 2, 1, 1, 2, 1. The figure on the right has side lengths 3, 3, 1, 2, 2, 1. There is no way to make a correspondence between them where all corresponding sides have the same length.



- If two polygons have the same side lengths, but their orders can't be matched as you go around each polygon, the polygons can't be congruent. For example, rectangle $ABCD$ can't be congruent to quadrilateral $EFGH$. Even though they both have two sides of length 3 and two sides of length 5, they don't correspond in the same order. In $ABCD$, the order is 3, 5, 3, 5 or 5, 3, 5, 3; in $EFGH$, the order is 3, 3, 5, 5 or 3, 5, 5, 3 or 5, 5, 3, 3.

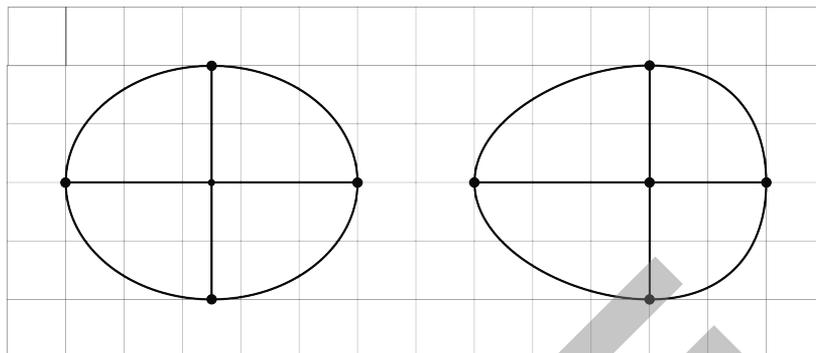


- If two polygons have the same side lengths, in the same order, but different corresponding angles, the polygons can't be congruent. For example, parallelogram $JKLM$ can't be congruent to rectangle $ABCD$. Even though they have the same side lengths in the same order, the angles are different. All angles in $ABCD$ are **right angles**. In $JKLM$, angles J and L are less than 90 degrees and angles K and M are more than 90 degrees.



- If you have curved figures, like these 2 ovals, you can find parts of the figures that should correspond but that have different measurements. On both, the longest

distance from left to right is 5 units across, and the longest distance from top to bottom is 4 units. The line segment from the highest to lowest point is in the middle of the left oval, but in the right oval, it's 2 units from the right end and 3 units from the left end. This proves they are not congruent.



Glossary

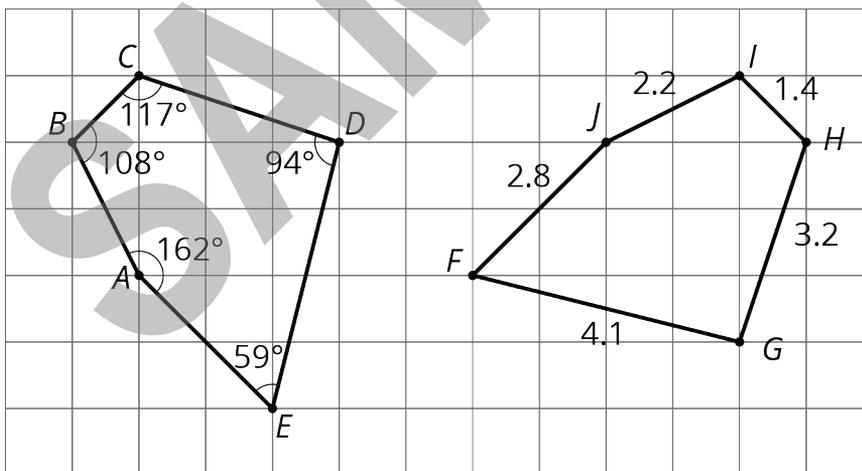
- right angle

Lesson 11 Practice Problems

Problem 1

Statement

- Show that the two pentagons are congruent.
- Find the side lengths of $ABCDE$ and the angle measures of $FGHIJ$.



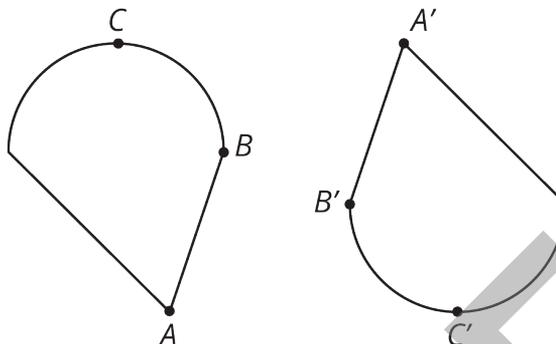
Solution

- After performing a 90-degree clockwise rotation with center D , then translating 3 units down and 6 units to the right, $ABCDE$ matches up perfectly with $JIHGF$. The rotation and translation do not change side lengths or angle measures.
- $AB = 2.2$, $BC = 1.4$, $CD = 3.2$, $DE = 4.1$, and $EA = 2.8$. $m\angle F = 59^\circ$, $m\angle G = 94^\circ$, $m\angle H = 117^\circ$, $m\angle I = 108^\circ$, and $m\angle J = 162^\circ$.

Problem 2

Statement

These two figures are congruent, with corresponding points marked.



- Are angles ABC and $A'B'C'$ congruent? Explain your reasoning.
- Measure angles ABC and $A'B'C'$ to check your answer.

Solution

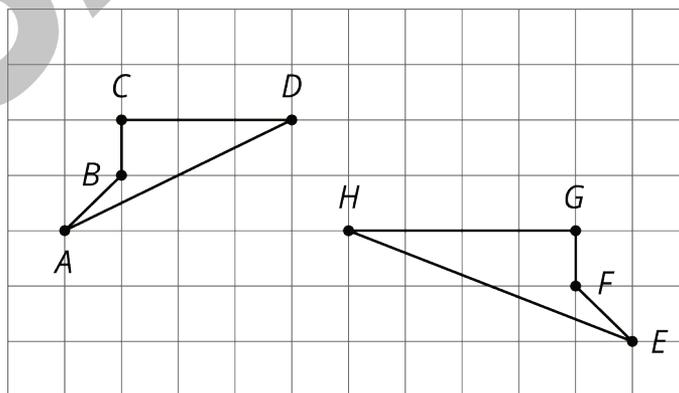
- Yes they are angles made by corresponding points on congruent figures so they are congruent.
- Both angles measure about 110 degrees.

Problem 3

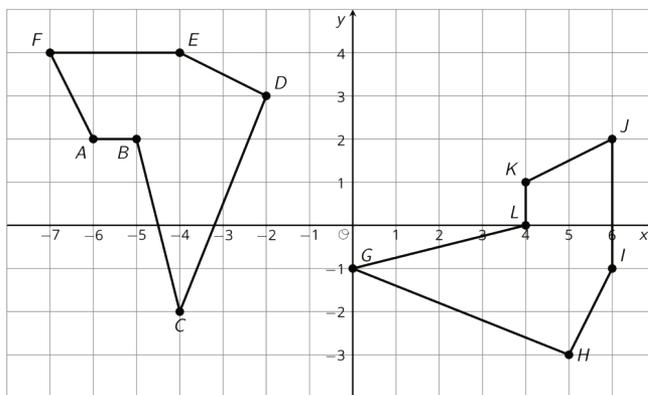
Statement

For each pair of shapes, decide whether or not the two shapes are congruent. Explain your reasoning.

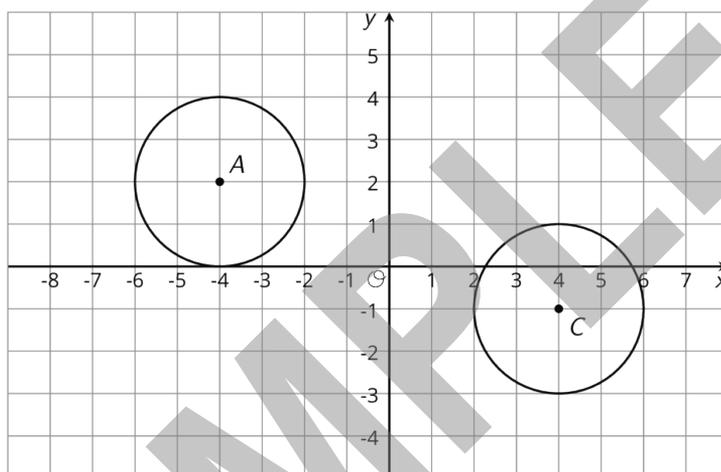
a.



b.



c.



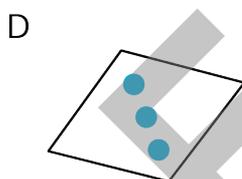
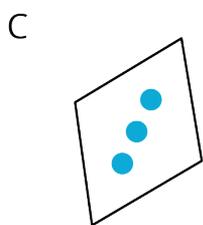
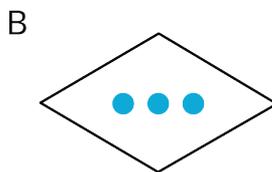
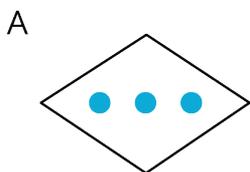
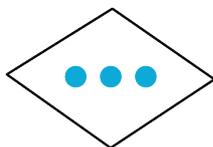
Solution

- Not congruent. Segment \overline{EH} in polygon $EFGH$ is longer than any of the sides in polygon $ABCD$. A , B , and C can be matched up with vertices E , F , and G , but H does not match up with D .
- Congruent. If $ABCDEF$ is rotated 90 degrees clockwise about C and then moved 4 units to the right and 1 unit up, it matches up perfectly with $GHIJKL$.
- Congruent. If the circle on the top left is translated to the right by 8 units and down 3 units, it lands on top of the other circle.

Problem 4

Statement

Which of these four figures are congruent to the top figure?



A. A

B. B

C. C

D. D

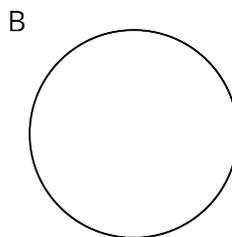
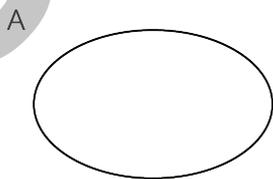
Solution

C

Problem 5

Statement

Here are two figures.



Show, using measurement, that these two figures are *not* congruent.

Solution

Answers vary. Sample response: The rightmost and leftmost points on Figure A are further apart than any pair of points on Figure B. So these two points can not correspond to any pair of points on Figure B and the two figures are not congruent.

SAMPLE

Section: Angles in a Triangle

Lesson 12: Alternate Interior Angles

Goals

- Calculate angle measures using alternate interior, adjacent, vertical, and supplementary angles to solve problems.
- Comprehend the terms “complementary” and “supplementary” (in spoken and written language) as they describe pairs of angles.
- Justify (orally and in writing) that “alternate interior angles” made by a “transversal” connecting two parallel lines are congruent using properties of rigid motions.

Learning Targets

- I can find unknown angle measures by reasoning about complementary or supplementary angles.
- If I have two parallel lines cut by a transversal, I can identify alternate interior angles and use that to find missing angle measurements.

Lesson Narrative

In this lesson, students justify that **alternate interior angles** are congruent, and use this and the vertical angle theorem, previously justified, to solve problems. They are also introduced to the terms **complementary**, for describing two angles whose measures add to 90° , and **supplementary**, for describing two angles whose measures add to 180° . They practice finding an unknown angle given the measure of another angle that is complementary or supplementary.

Thus far in this unit, students have studied different types of rigid motions, using them to examine and build different figures. This work continues here, with an emphasis on examining angles. In a previous lesson, 180° rotations were employed to show that vertical angles, made by intersecting lines, are congruent. The warm-up recalls previous facts about angles made by intersecting lines, including both vertical and adjacent angles. Next a third line is added, parallel to one of the two intersecting lines. There are now 8 angles, 4 each at the 2 intersection points of the lines. At each vertex, vertical and adjacent angles can be used to calculate all angle measures once one angle is known. But how do the angle measures compare at the two vertices? It turns out that each angle at one vertex is congruent to the corresponding angle (via translation) at the other vertex and this can be seen using rigid motions: translations and 180° rotations are particularly effective at revealing the relationships between the angle measures.

During the lesson, students will notice as they calculate angles that they are repeatedly using vertical and adjacent angles and that often they have a choice which method to apply (MP8). They will also notice that the angles made by parallel lines cut by a **transversal** are the same and this observation is the key structure in this lesson.

Teacher Notes for IM 6–8 Math Accelerated

This lesson is the first time students see the terms **complementary** and **supplementary angles** in IM 6–8 Math Accelerated. Make sure to highlight their use during the lesson synthesis in addition to the focus on alternate interior angles. Here are some possible questions to include in the discussion:

- “What does it mean for two angles to be supplementary?” (Their measures sum to 180° .)
- “What does it mean for two angles to be complementary?” (Their measures sum to 90° .)
- “If you know two angles are supplementary and you know the measure of one angle, how can you find the other?” (Subtract the known one from 180° .)

Alignments

Addressing

- 7.G.B.5: Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.
- 8.G.A.1: Verify experimentally the properties of rotations, reflections, and translations:
- 8.G.A.5: Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

Instructional Routines

- MLR2: Collect and Display
- MLR4: Information Gap Cards
- MLR8: Discussion Supports
- Think Pair Share

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Required Preparation

Students need rulers and tracing paper from the geometry toolkits.

Student Learning Goals

Let's explore why some angles are always equal.

12.1 Angle Pairs

Warm Up: 10 minutes

In this activity, students practice determining the measurement of unknown angles using their relationship to angles with known angle measure and what students already know about the measurement of right angles and straight angles. After they have worked on the activity and shared their solutions, they are introduced to the vocabulary terms complementary and supplementary. Complementary describes angles whose measures sum to 90 degrees and supplementary describes angles whose measures sum to 180 degrees.

Some students may wish to use protractors, either to double check work or to investigate the different angle measures. This is an appropriate use of technology (MP5), but ask these students what other methods they could use instead.

Addressing

- 7.G.B.5

Launch

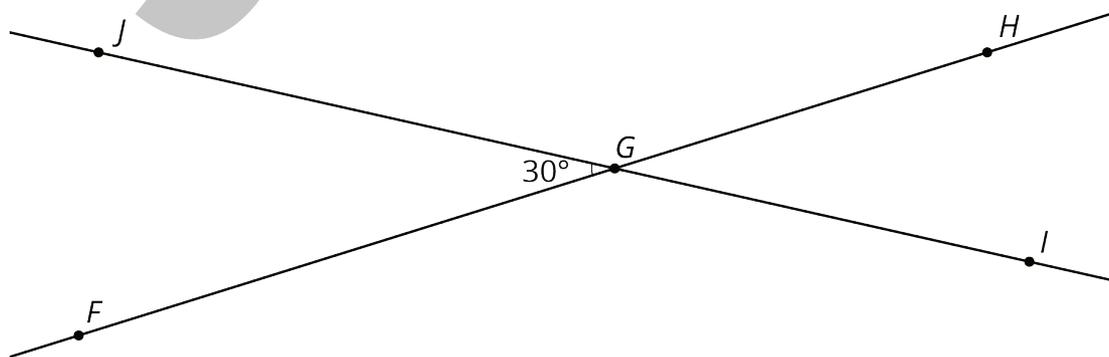
Provide access to geometry toolkits.

Anticipated Misconceptions

If needed, remind students that right angles have a measure of 90° and straight angles have a measure of 180° .

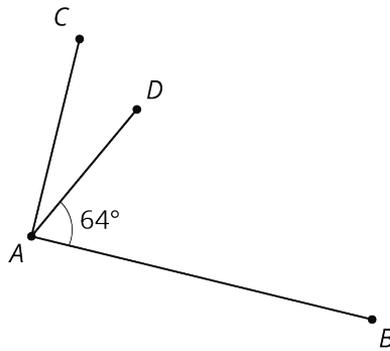
Student Task Statement

1. Find the measure of angle JGH . Explain or show your reasoning.



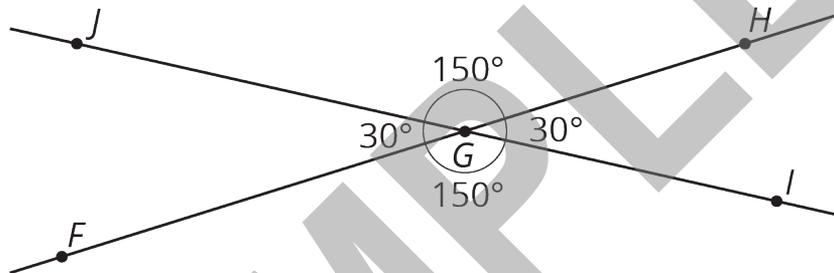
2. Find and label a second 30° angle in the diagram. Find and label an angle congruent to angle JGH .

3. Angle BAC is a right angle. Find the measure of angle CAD .



Student Response

1. 150° . Sample response: In the diagram, the given 30° angle and angle JGH make a straight angle, so they add up to 180° .



2. Angles are labeled as shown using reasoning about vertical or supplementary angles.
3. 26° since $90 - 64 = 26$.

Activity Synthesis

The goal of this discussion is to introduce students to the terms complementary and supplementary for describing relationships between pairs of angles. Display the images for all to see and select students to briefly share their response to each question, recording any calculations used.

Explain to students that the term **complementary** describes a pair of angles whose measures sum to 90 degrees, and the term **supplementary** describes a pair of angles whose measures sum to 180 degrees. This is true even for pairs of angles that are not adjacent. Conclude the discussion by inviting students to identify pairs of angles that are complementary (CAD and DAB), supplementary (FGJ and JGH , JGH and HGI , HGI and IGF , or IGF and FGJ), or vertical (FGJ and HGI or JGH and IGF) from the activity.

12.2 Cutting Parallel Lines with a Transversal

15 minutes

In this task, students explore the relationship between angles formed when two parallel lines are cut by a transversal line. Students investigate whether knowing the measure of one angle is

sufficient to figure out all the angle measures in this situation. They also consider whether the relationships they found hold true for any two lines cut by a transversal.

The last two questions in this activity are optional, to be completed if time allows. Make sure to leave enough time for the next activity, “Alternate Interior Angles are Congruent.”

As students work with their partners, they begin to fill in supplementary angles and vertical angles. To find the measures of corresponding and alternate interior, students may use tracing paper and some of the strategies found earlier in the unit. For example, they may use tracing paper to translate vertex B to vertex E . They might try to translate B' to E' in the second picture and observe that the angles at those two vertices are not congruent. Similarly, to find measures of vertical angles students may use a 180° rotation like they did earlier in this unit when showing that vertical angles are congruent. Monitor for students who use these different strategies and select them to share during the discussion.

For students who finish early, ask them to think of *different* methods they could use to determine the angles: For example, all of the congruent angles can be shown to be congruent with transformations.

Addressing

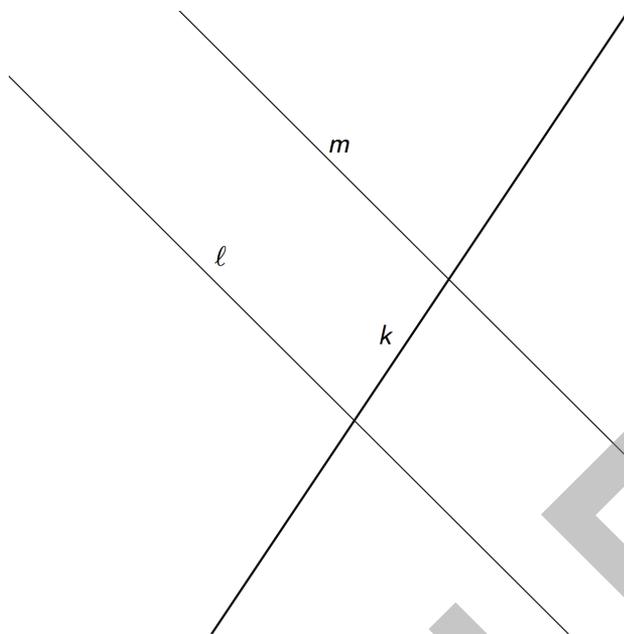
- 8.G.A.1
- 8.G.A.5

Instructional Routines

- MLR2: Collect and Display
- Think Pair Share

Launch

A **transversal** (or *transversal line*) for a pair of parallel lines is a line that meets each of the parallel lines at exactly one point. Introduce this idea and provide a picture such as this picture where line k is a transversal for parallel lines ℓ and m :



Arrange students in groups of 2. Provide access to geometry toolkits. Give students 1 minute of quiet think time to plan on how to find the angle measures in the picture then time to share their plan with their partner. Give partners time for the rest of the task, followed by a whole-class discussion. Instruct students to stop after the third question if you've decided to skip the last two questions.

Support for Students with Disabilities

Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. Invite students to suggest language or diagrams to include that will support their understanding of these terms. Include the following term and maintain the display for reference throughout the unit: transversal.

Supports accessibility for: Conceptual processing; Language

Support for English Language Learners

Conversing, Representing: MLR2 Collect and Display. As students discuss with a partner, listen for and collect vocabulary, gestures, and diagrams students use to describe the relationships they notice between the angles formed when two parallel lines are cut by a transversal. Capture student language that reflects a variety of ways to determine angle congruence. Record students' words, phrases, and diagrams onto a visual display and update it throughout the lesson. Remind students to borrow language from the display as needed. This will help students use mathematical language during small-group and whole-class discussions.

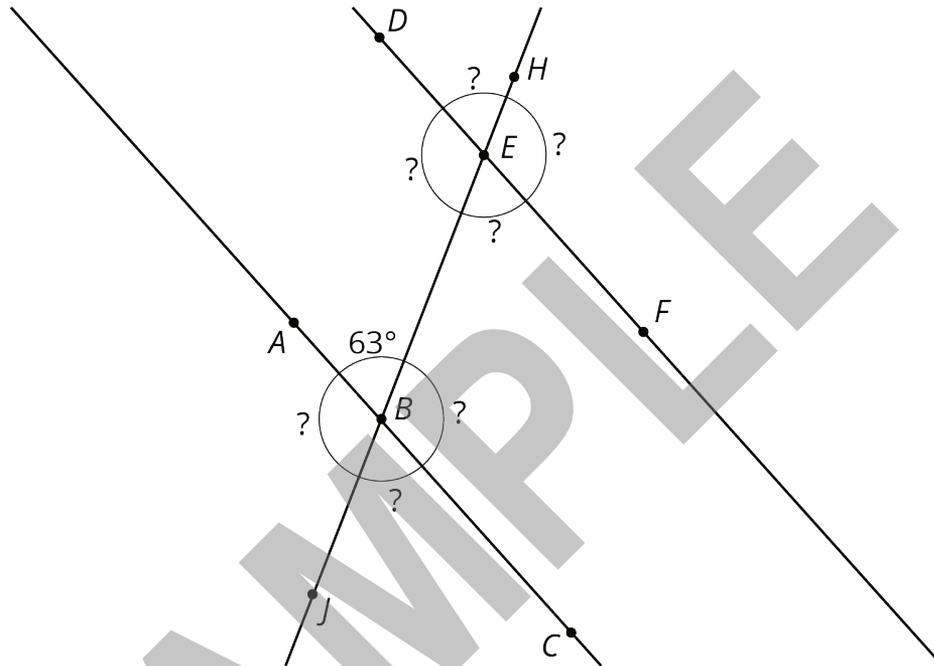
Design Principle(s): Support sense-making

Anticipated Misconceptions

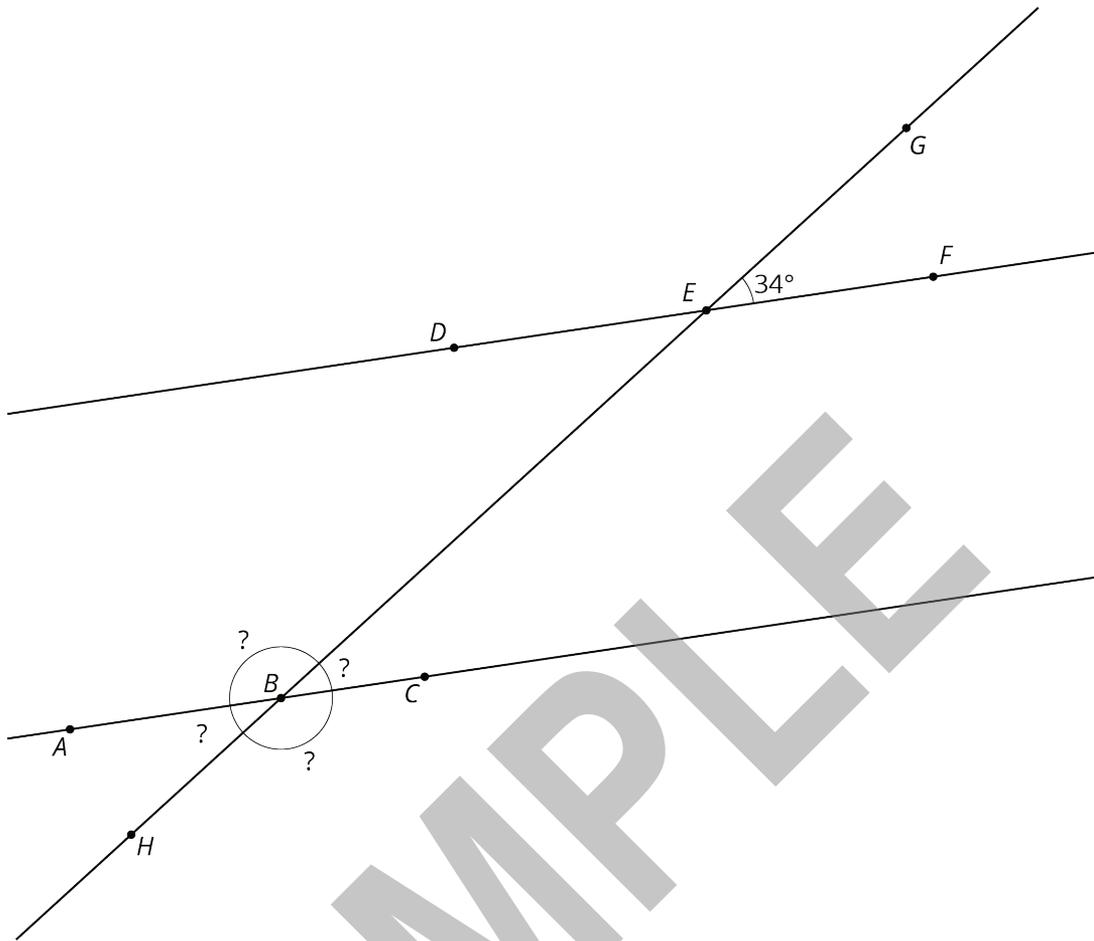
In the first image, students may fill in congruent angle measurements based on the argument that they look the same size. Ask students how they can be certain that the angles don't differ in measure by 1 degree. Encourage them a way to explain how we can know for sure that the angles are exactly the same measure.

Student Task Statement

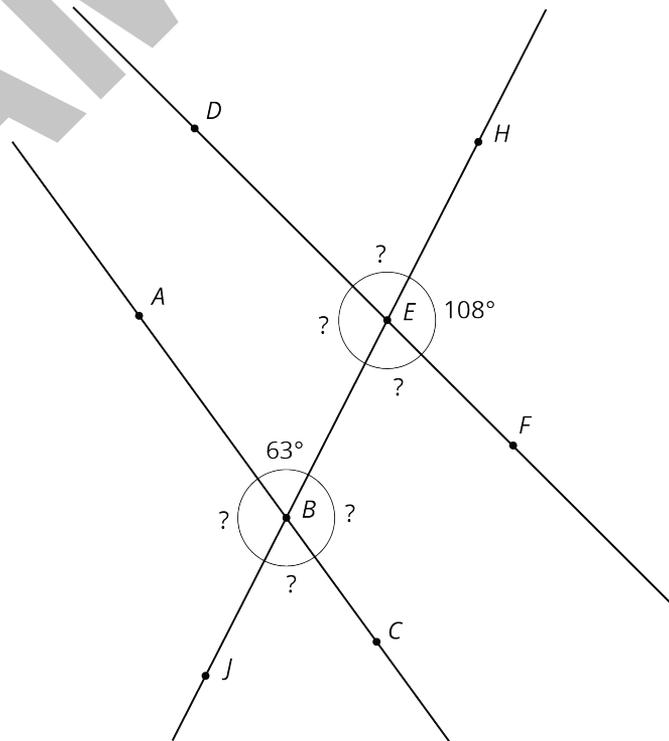
Lines AC and DF are parallel. They are cut by transversal HJ .



1. With your partner, find the seven unknown angle measures in the diagram. Explain your reasoning.
2. What do you notice about the angles with vertex B and the angles with vertex E ?
3. Using what you noticed, find the measures of the four angles at point B in the second diagram. Lines AC and DF are parallel.



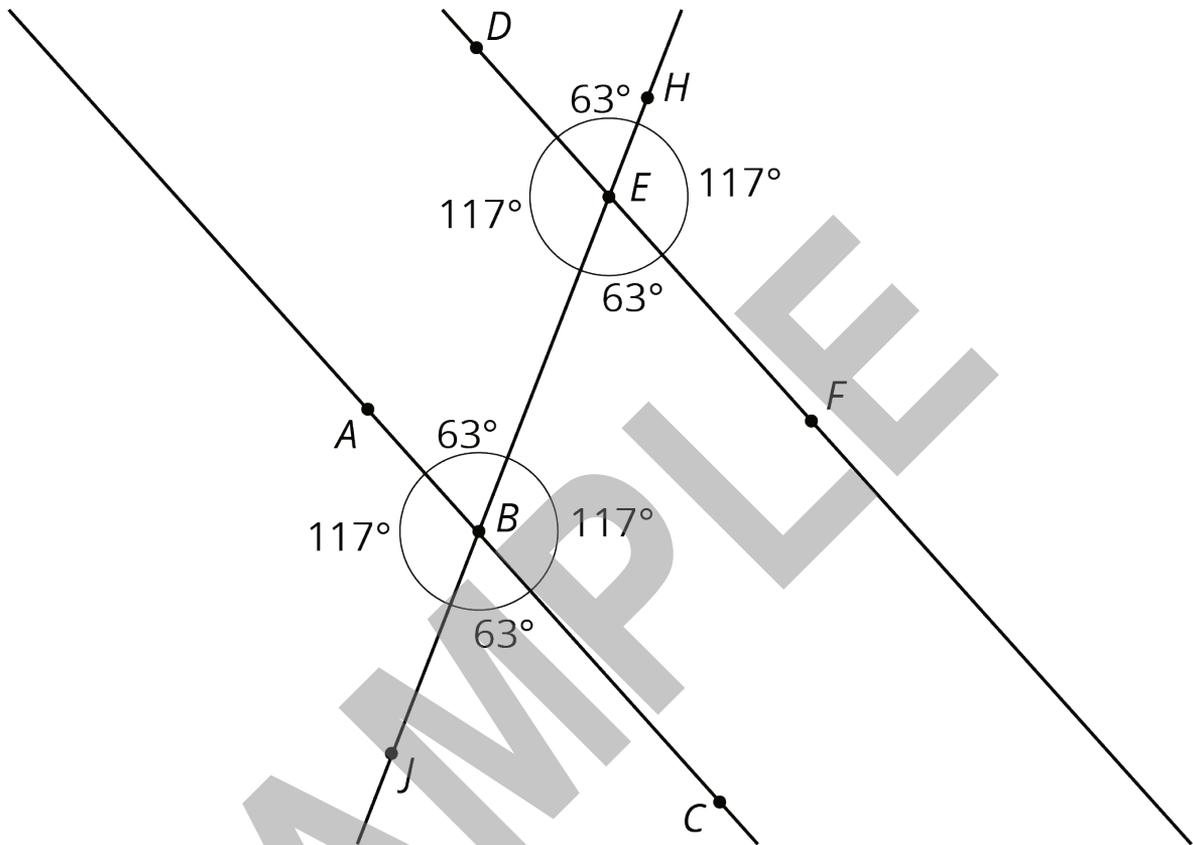
4. The next diagram resembles the first one, but the lines form slightly different angles. Work with your partner to find the six unknown angles with vertices at points B and E .



5. What do you notice about the angles in this diagram as compared to the earlier diagram? How are the two diagrams different? How are they the same?

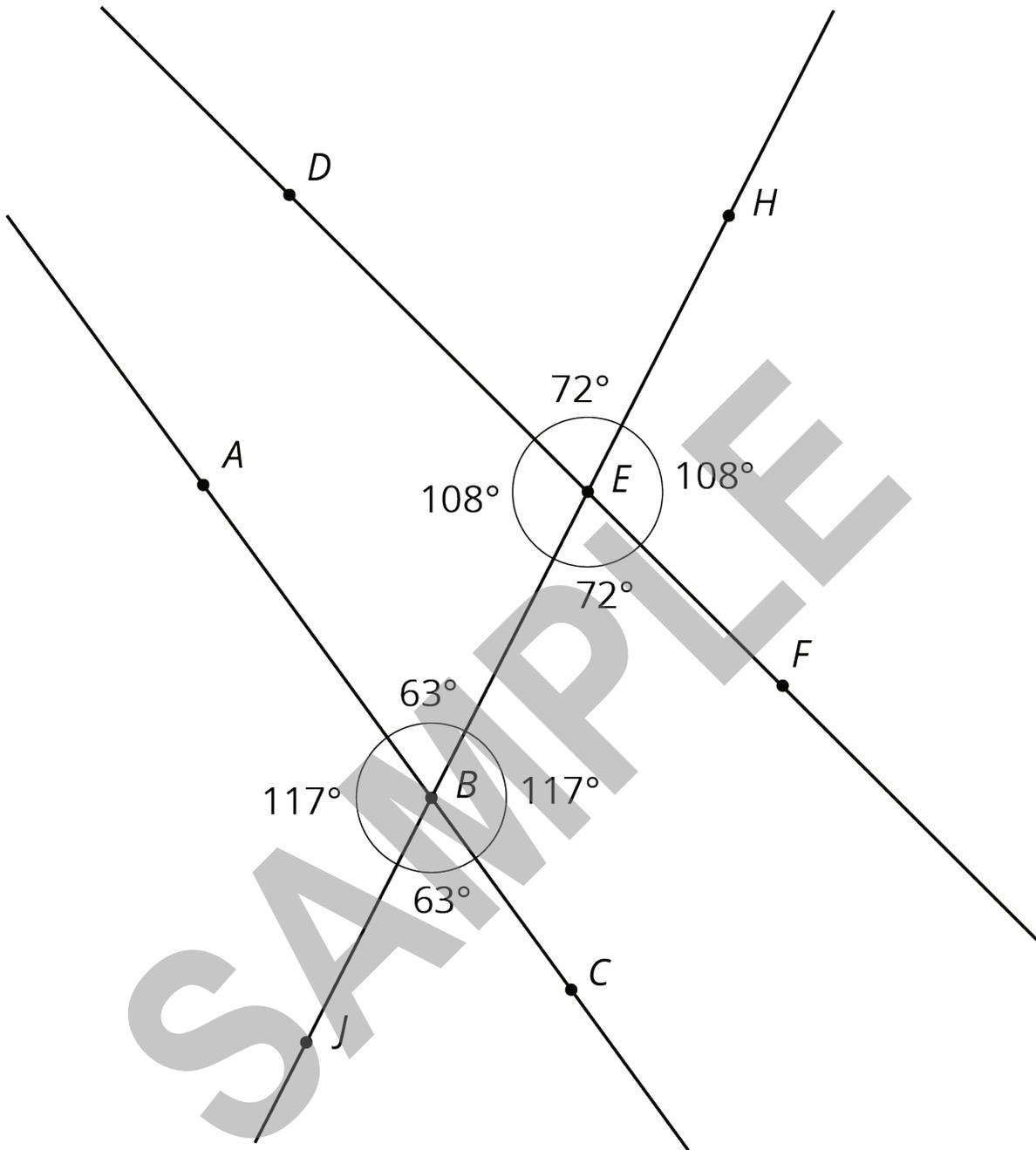
Student Response

1.



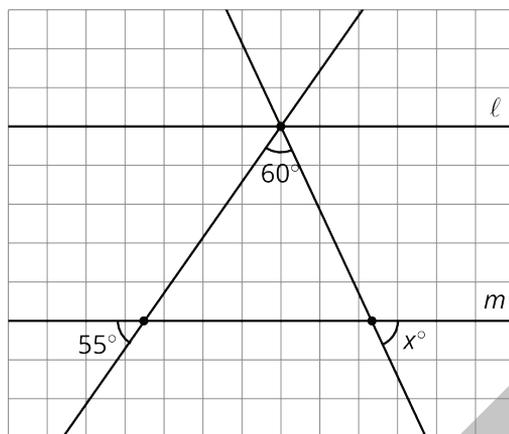
Explanations vary. Sample strategy 1: Tracing paper helped find the three 117° angles. Each of the other four angles is supplementary to a 117° angle, so they are all 63° angles. Sample strategy 2: Using pairs of vertical angles shows that angle CBJ is a 63° angle. The other angles at vertex B can be found using supplementary angles. The angles at vertex E can be found the same way after using tracing paper to find one of them.

2. Answers vary. Sample response: The angles in the same place relative to the transversal have the same measure.
3. Answers vary. Sample response: Angle ABH is a 34° angle because it forms a vertical pair with the marked 34° angle after translating E to B . Angle HBC is a 146° angle because it is supplementary to the 34° angle found by translating E to B .
- 4.



5. Answers vary. Sample response: In both pictures, the two pair of vertical angles at each vertex are congruent. Also adjacent angles at each vertex are supplementary. In the first picture, the angle measures at the two vertices are the same while in the second picture they are different.

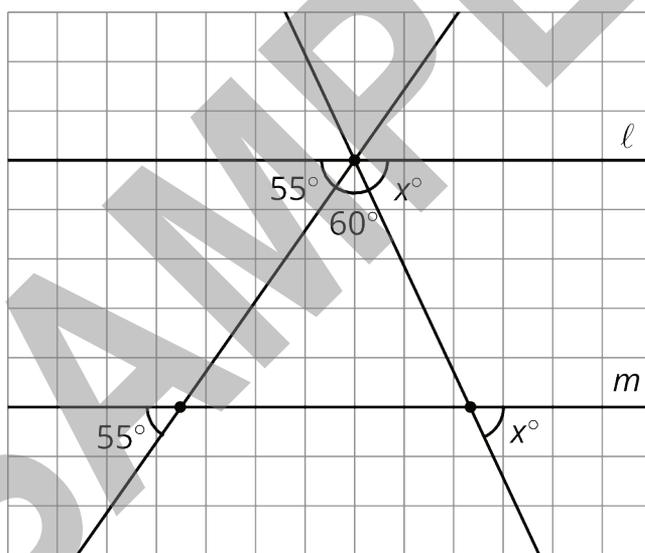
Are You Ready for More?



Parallel lines ℓ and m are cut by two transversals which intersect ℓ in the same point. Two angles are marked in the figure. Find the measure x of the third angle.

Student Response

$$x = 65^\circ.$$



Adding in two alternate interior angles, we see that the angles marked 55° , 60° , and x° make a straight angle, so $55 + 60 + x = 180$.

Activity Synthesis

The purpose of this discussion is to make sure students noticed relationships between the angles formed when two parallel lines are cut by a transversal and to introduce the term alternate interior angles to students. Display the images from the Task Statement for all to see one at a time and invite groups to share their responses. Encourage students to use precise vocabulary, such as supplementary and vertical angles, when describing how they figured out the different angle measurements. After students point out the matching angles at the two vertices, define the term **alternate interior angles** and ask a few students to identify some pairs of angles from the activity.

Consider asking some of the following questions:

- "What were some tools you used to find or confirm angle measures?" (Tracing paper, protractor, transformations)
- "What were some angle relationships you used to find missing measures?" (Vertical angles, supplementary angles)
- "What do you notice about the angles at vertex B compared to the angles at vertex E ?" (They have the same angle measures for angles in the same position relative to the transversal.)
- "Which angle relationships were true for all three pictures? Which were true for only one or two of the pictures?" (Congruent vertical and supplementary angles around a vertex were always true. Congruent angles in corresponding positions at the two vertices were only true in the first two pictures, which had parallel lines.)

12.3 Alternate Interior Angles Are Congruent

15 minutes

The goal of this task is to experiment with rigid motions to help visualize why alternate interior angles (made by a transversal connecting two parallel lines) are congruent. This result will be used in a future lesson to establish that the sum of the angles in a triangle is 180 degrees. The second question is optional if time allows. This provides a deeper understanding of the relationship between the angles made by a pair of (not necessarily parallel) lines cut by a transversal.

Expect informal arguments as students are only beginning to develop a formal understanding of parallel lines and rigid motions. They have, however, studied the idea of 180 degree rotations in a previous lesson where they used this technique to show that a pair of vertical angles made by intersecting lines are congruent. Consider recalling this technique especially to students who get stuck and suggesting the use of tracing paper.

Given the diagram, the tracing paper, and what they have learned in this unit, students should be looking for ways to demonstrate that alternate interior angles are congruent using transformations. Make note of the different strategies (including different transformations) students use to show that the angles are congruent and invite them to share their strategies during the discussion. Approaches might include:

- A 180 degree rotation about M
- First translating P to Q and then applying a 180-degree rotation with center Q

Teacher Notes for IM 6–8 Math Accelerated.

Adjust this activity to 10 minutes.

Addressing

- 8.G.A.1

- 8.G.A.5

Instructional Routines

- MLR8: Discussion Supports

Launch

Provide access to geometry toolkits. Tell students that in this activity, we will try to figure out why we saw all the matching angles we did in the last activity.

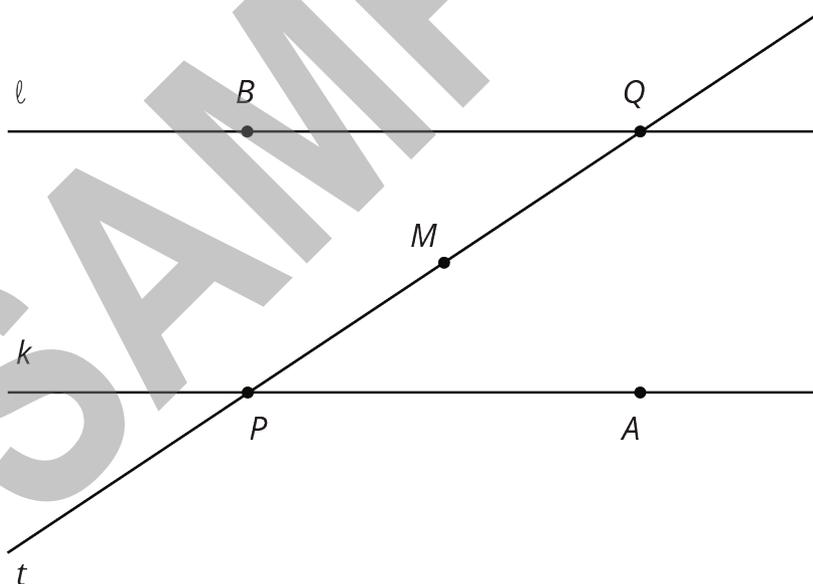
Support for Students with Disabilities

Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, "First, I ____ because...", "I noticed ____ so I...", "Why did you...?", "I agree/disagree because..."

Supports accessibility for: Language; Organization

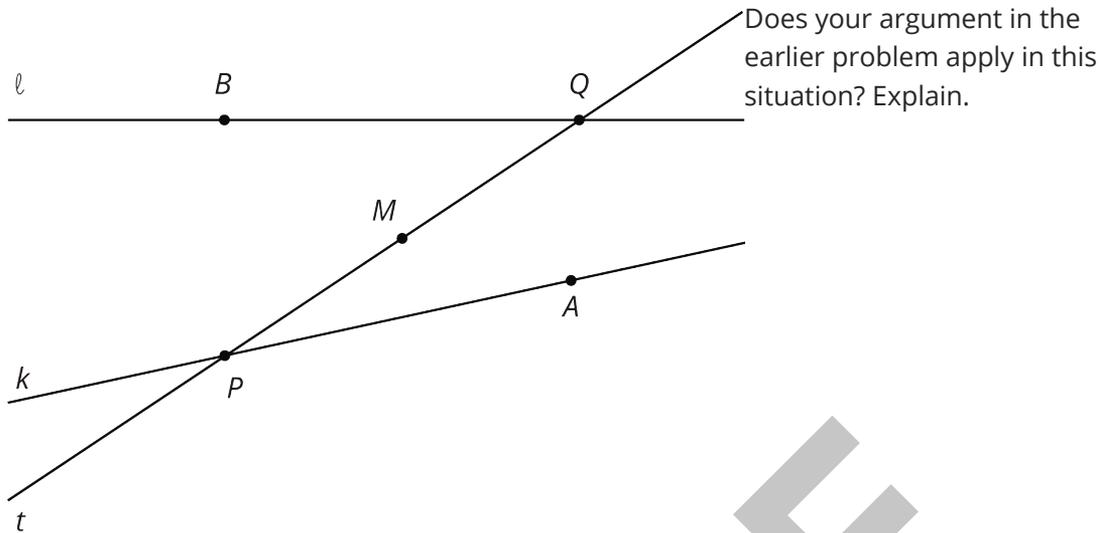
Student Task Statement

1. Lines ℓ and k are parallel and t is a transversal. Point M is the midpoint of segment PQ .



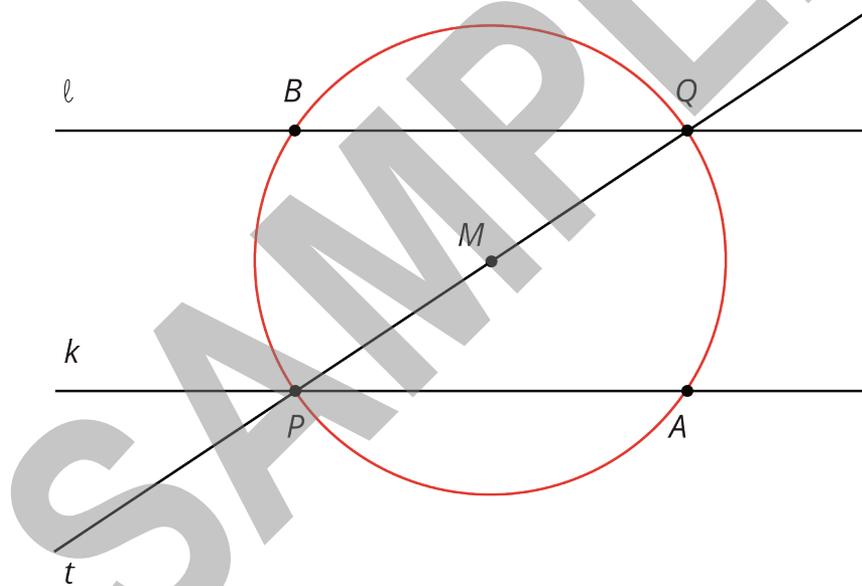
Find a rigid transformation showing that angles MPA and MQB are congruent.

2. In this picture, lines ℓ and k are no longer parallel. M is still the midpoint of segment PQ .



Student Response

1. Rotate the picture 180° with center M .



Since 180 degrees is half of a circle this takes each point on the circle to its "opposite." Point A maps to point B and B maps to A . So the 180 degree rotation will interchange P and Q . The rotation interchanges lines ell and m and also angles MQB and MPA so the angles are congruent.

2. If l and m are not parallel, a 180 degree rotation around M still takes P to Q and Q to P . The problem is that it does not take l to m , and it does not take m to l because m is not parallel to l . So this rotation does *not* take angle MQB to angle MPA and vice versa. The argument from 1 does not apply unless l and m are parallel.

Activity Synthesis

Select students to share their explanations. Pay close attention to which transformations students use in the first question and make sure to highlight different possibilities if they arise. Ask students to describe and demonstrate the transformations they used to show that alternate interior angles are congruent.

Highlight the fact that students are using many of the transformations from earlier sections of this unit. The argument here is especially close to the one used to show that vertical angles made by intersecting lines are congruent. In both cases a 180 degree rotation exchanges pairs of angles. For vertical angles, the center of rotation is the common point of intersecting lines. For alternate interior angles, the center of rotation is the midpoint of the transverse between the two parallel lines. But the structure of these arguments is identical.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to amplify students' mathematical uses of language when describing and demonstrating transformations used for showing alternate interior angles are congruent. After a student shares their response, invite another student to repeat the reasoning using the following mathematical language: vertical angles, 180 degree rotation, center of rotation, intersecting lines, midpoint, parallel lines. Invite all students to chorally repeat the phrases that include these words in context.

Design Principle(s): Support sense-making, Optimize output (for explanation)

12.4 Info Gap: Angle Finding

20 minutes

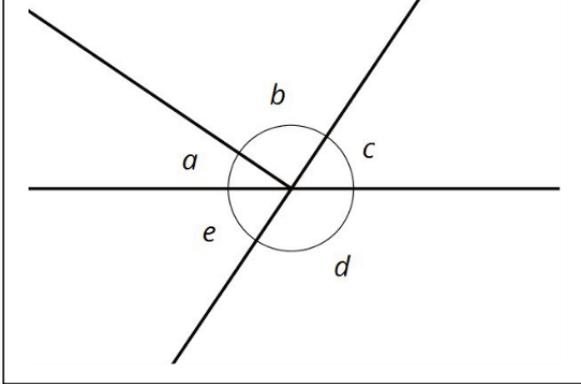
The purpose of this info gap activity is for students to see how they can use different pieces of information to solve for an unknown angle measure in a multi-step problem. During the whole-class discussion, students are introduced to writing and solving equations to represent the relationships between angles.

The info gap structure requires students to make sense of problems by determining what information is necessary. This may take several rounds of discussion (MP1). Since there are many different pieces of information that *could* be used to solve the problem but are not given on the data card, consider using a variation on the typical info gap structure: Instead of the student with the problem card asking for specific pieces of information, the student with the data card chooses a piece of information to share, and the student with the problem card explains how they use that piece of information. If enough information hasn't been given, the student with the data card chooses another piece of information to share.

You will need the blackline master for this activity. Here is the text of the cards for reference and planning:

Angle Finding Info Gap - **Problem Card 1:**

Find the measure of angle b .

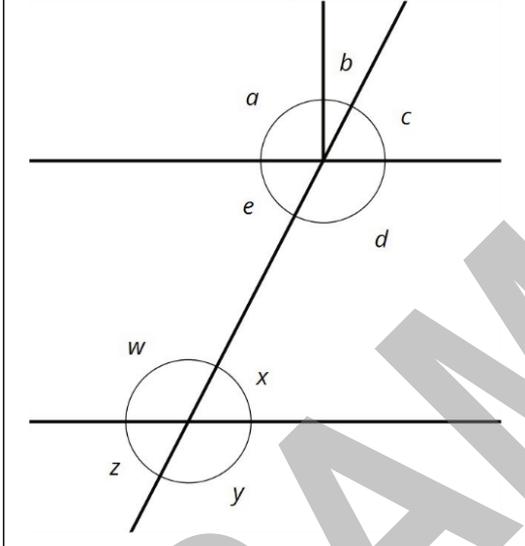


Angle Finding Info Gap - **Data Card 1:**

- Angles c and e are vertical angles.
- Angles a and c are complementary angles.
- The measure of angle $d = 124^\circ$.
- The measure of angle $c = 56^\circ$.

Angle Finding Info Gap - **Problem Card 2:**

Find the measure of angle b .



Angle Finding Info Gap - **Data Card 2:**

- The measure of angle $a = 90^\circ$.
- The measure of angle $z = 63^\circ$.
- Angles c and z are not vertical angles.
- Angles d and x are supplementary angles.

As students work, monitor for those who use different strategies or start with different pieces of information. Also, monitor for students who choose to show their reasoning by writing and solving equations.

Teacher Notes for IM 6–8 Math Accelerated.

This Info Gap activity is optional. If time allows, use this activity to give students additional practice calculating angle measurements using their understanding of complementary, supplementary, and vertical angles.

Addressing

- 7.G.B.5

Instructional Routines

- MLR4: Information Gap Cards

Launch

If desired, explain this variation from the typical info gap: instead of the student with the problem card asking for each piece of information, the student with the data card chooses a piece of information to share. The student with the problem card still needs to explain how they can use each piece of information. If more information is needed to solve the problem, the student with the data card chooses another piece of information to share. Also, students need to listen to their partner carefully because they may be asked to explain their partner's reasoning to the class.

Arrange students in groups of 2. Distribute a problem card to one student and a data card to the other student in each group.

Action and Expression: Internalize Executive Functions. Begin with a small-group or whole-class demonstration and think aloud of a sample situation problem card and data card to remind students how to use the info gap structure. Keep the worked-out equations and angle drawing on display for students to reference as they work.

Supports accessibility for: Memory; Conceptual processing

Support for English Language Learners

Conversing: Use this modified version of *MLR4 Information Gap* to give students an opportunity to discuss the information necessary to solve for an unknown angle measure. Display the follow sentence frames to support student discussion: "Can you tell me . . . (specific piece of information)", "Why do you need to know . . . (that piece of information)?", and "I can use this information to . . ."

Design Principle(s): Cultivate conversation

Anticipated Misconceptions

For the second set of cards, students may struggle to find the connection between the lower half of the figure and the upper half. Remind them that supplementary angles do not need to be next to one another, but they can be.

Student Task Statement

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.

Continue to ask questions until you have enough information to solve the problem.

4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner “*What specific information do you need?*” and wait for them to *ask* for information.

If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.

3. Before sharing the information, ask “*Why do you need that information?*” Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Student Response

1. $b = 90^\circ$. Possible strategies:

- $c = 56, a = 90 - 56 = 34, b = 180 - 34 - 56 = 90$
- $d = 124, e = 180 - 124 = 56, c = 56, a = 90 - 56 = 34, b = 180 - 34 - 56 = 90$
- $a + c = 90, a + b + c = 180, b = 180 - 90 = 90$

2. $b = 27^\circ$. Possible strategies:

- $z = 63, x = 63, d = 180 - 63 = 117, c = 180 - 117 = 63, a = 90, b = 180 - 90 - 63 = 27$
- $z = 63, w = 180 - 63 = 117, x = 180 - 117 = 63, d = 180 - 63 = 117, e = 180 - 117 = 63, c = 63, a = 90, b = 360 - 117 - 63 - 90 - 63 = 27$

Activity Synthesis

The goal of this discussion is for students to see that writing and solving equations is an efficient strategy to show their reasoning about multi-step angle problems.

For each problem, select a student who had the information card to share their partner's reasoning. Record their reasoning using equations and display for all to see. Listen carefully and make sure the

equation matches how they explain their reasoning. Here are some sample equations for the first problem.

solving for e given d

$$\begin{aligned} e + d &= 180 \\ e + 124 &= 180 \\ e &= 180 - 124 \\ e &= 56 \end{aligned}$$

solving for a given c

$$\begin{aligned} a + c &= 90 \\ a + 56 &= 90 \\ a &= 90 - 56 \\ a &= 34 \end{aligned}$$

solving for e given c

$$\begin{aligned} c &= 56 \\ e &= c \\ e &= 56 \end{aligned}$$

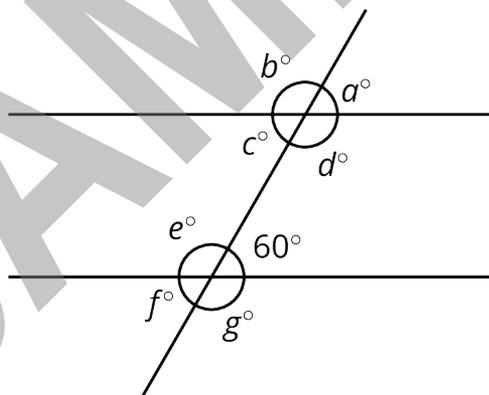
solving for b given that a and c are complementary

$$\begin{aligned} a + c &= 90 \\ a + b + c &= 180 \\ b + (a + c) &= 180 \\ b + 90 &= 180 \\ b &= 180 - 90 \\ b &= 90 \end{aligned}$$

Display the equations that represent different students' strategies side by side and have students contrast the different methods, for example, the difference between how two students worked the problem if one was given the measure of angle d first but the other was given the measure of angle c first.

Lesson Synthesis

Display the image of two parallel lines cut by a transversal. Tell students that in cases like this, translations and rotations can be particularly useful in figuring out angle measurements since they move angles to new positions, but the angle measure does not change.



Select students to point out examples of **alternate interior**, vertical, and supplementary angles in the picture. They should also be able to articulate which angles are congruent to one another and give an example of a rigid transformation that explains why.

In particular, make sure students can articulate:

- $c = 60$ because it is the measure of an angle forming an alternate interior angle with the given 60 degree angle.
- $e = d = 120$ because they are also alternate interior angles, each supplementary to a 60 degree angle.

- The rest of the angle measures can be found using vertical or supplementary angles.

12.5 All The Rest

Cool Down: 5 minutes

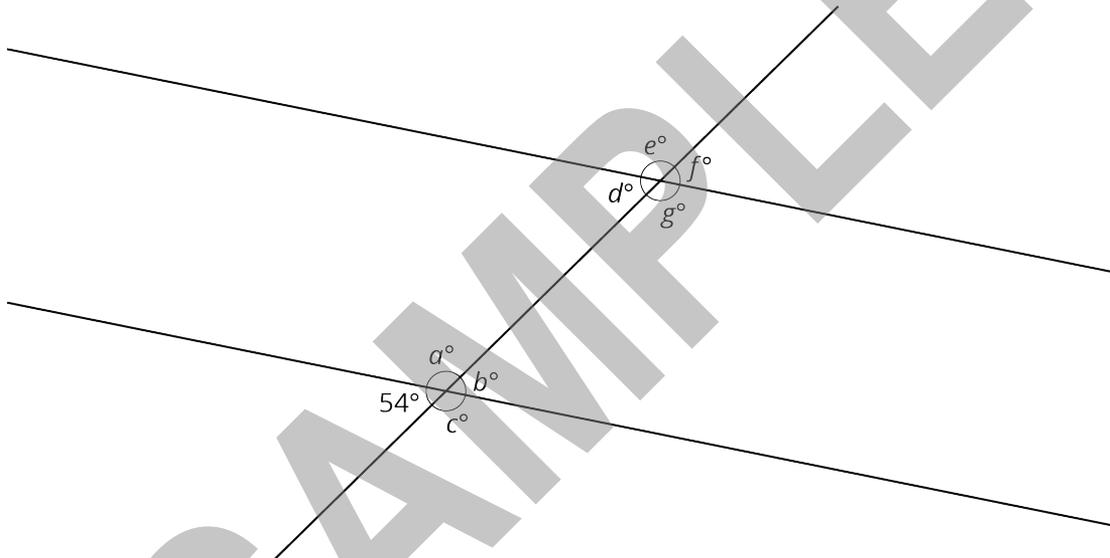
Students use what they have learned about vertical and alternate interior angles in this lesson and earlier lessons, applying it to a diagram in order to fill in angle measurements without needing to measure.

Addressing

- 8.G.A.5

Student Task Statement

The diagram shows two parallel lines cut by a transversal. One angle measure is shown.



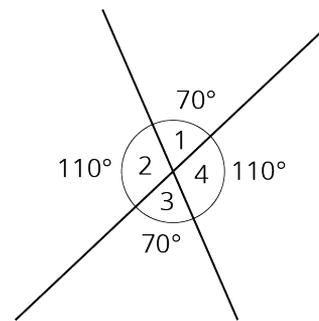
Find the values of a , b , c , d , e , f , and g .

Student Response

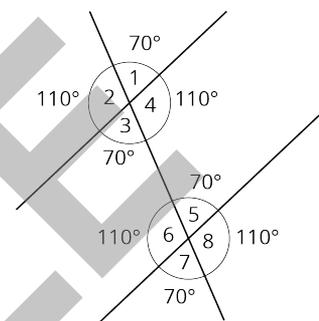
a : 126, b : 54, c : 126, d : 54, e : 126, f : 54, g : 126

Student Lesson Summary

If two angle measures add up to 90° , then we say the angles are **complementary**. If two angle measures add up to 180° , then we say the angles are **supplementary**. When two lines intersect, vertical angles are equal and adjacent angles are supplementary. For example, in this figure angles 1 and 3 are equal, angles 2 and 4 are equal, angles 1 and 4 are supplementary, and angles 2 and 3 are supplementary.



When two parallel lines are cut by another line, called a **transversal**, two pairs of **alternate interior angles** are created. ("Interior" means on the inside, or between, the two parallel lines.) For example, in this figure angles 3 and 5 are alternate interior angles and angles 4 and 6 are also alternate interior angles.



Alternate interior angles are equal because a 180° rotation around the midpoint of the segment that joins their vertices takes each angle to the other. Imagine a point M halfway between the two intersections—can you see how rotating 180° about M takes angle 3 to angle 5?

Using what we know about vertical angles, adjacent angles, and alternate interior angles, we can find the measures of any of the eight angles created by a transversal if we know just one of them. For example, starting with the fact that angle 1 is 70° we use vertical angles to see that angle 3 is 70° , then we use alternate interior angles to see that angle 5 is 70° , then we use the fact that angle 5 is supplementary to angle 8 to see that angle 8 is 110° since $180 - 70 = 110$. It turns out that there are only two different measures. In this example, angles 1, 3, 5, and 7 measure 70° , and angles 2, 4, 6, and 8 measure 110° .

Glossary

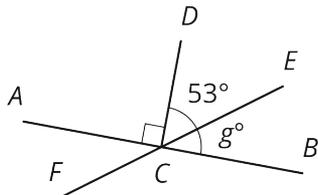
- alternate interior angles
- complementary
- supplementary
- transversal

Lesson 12 Practice Problems

Problem 1

Statement

Segments AB , EF , and CD intersect at point C , and angle ACD is a right angle. Find the value of g .



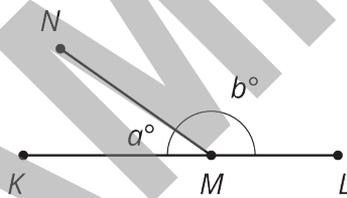
Solution

37

Problem 2

Statement

M is a point on line segment KL . NM is a line segment. Select **all** the equations that represent the relationship between the measures of the angles in the figure.



- A. $a = b$
- B. $a + b = 90$
- C. $b = 90 - a$
- D. $a + b = 180$
- E. $180 - a = b$
- F. $180 = b - a$

Solution

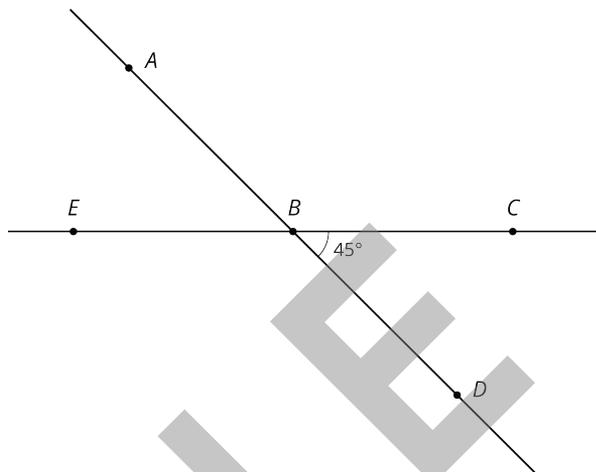
["D", "E"]

Problem 3

Statement

Use the diagram to find the measure of each angle.

- a. $m\angle ABC$
- b. $m\angle EBD$
- c. $m\angle ABE$



Solution

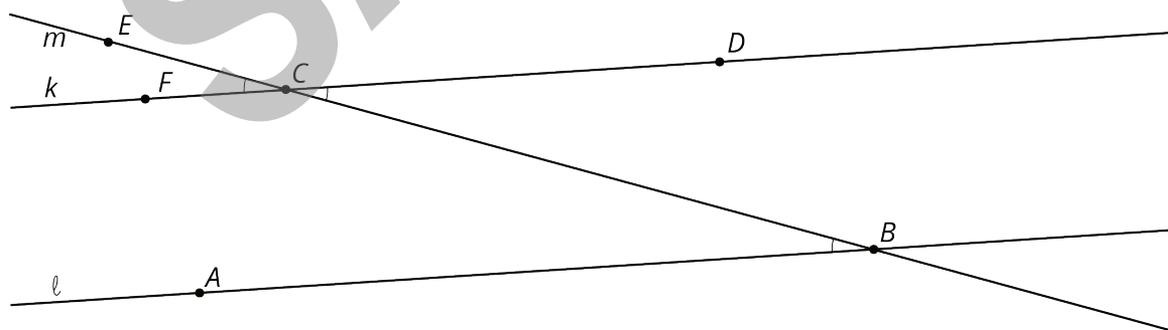
- a. 135 degrees
- b. 135 degrees
- c. 45 degrees

(From Unit 1, Lesson 8.)

Problem 4

Statement

Lines k and ℓ are parallel, and the measure of angle ABC is 19 degrees.



- a. Explain why the measure of angle ECF is 19 degrees. If you get stuck, consider translating line ℓ by moving B to C .
- b. What is the measure of angle BCD ? Explain.

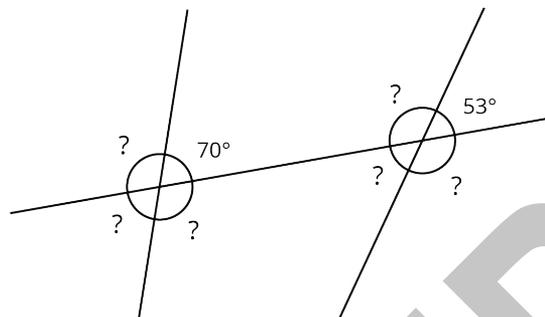
Solution

- If ℓ is translated so that B goes to C , then l goes to k because k is parallel to ℓ . Angle ABC matches up with angle FCE after this translation, so FCE (and ECF) is also a 19 degree angle.
- Angles ECF and BCD are congruent because they are vertical angles. Since angle ECF is a 19 degree angle, so is angle BCD .

Problem 5

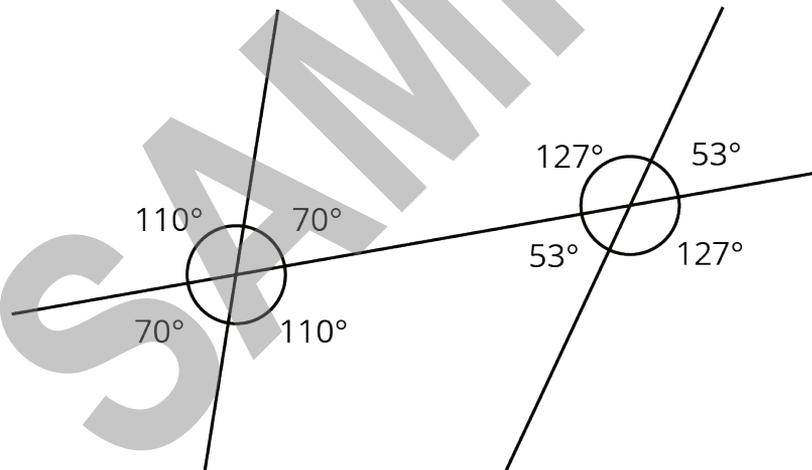
Statement

The diagram shows three lines with some marked angle measures.



Find the missing angle measures marked with question marks.

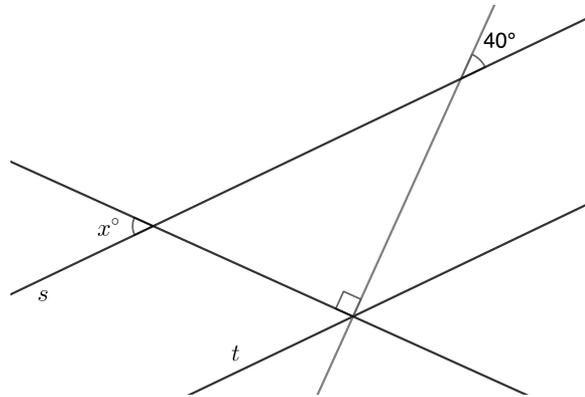
Solution



Problem 6

Statement

Lines s and t are parallel. Find the value of x .



Solution

$x = 50$. The measure of the given angle is 40 degrees. The corresponding angle on line t also measures 40 degrees. This angle is adjacent to the indicated 90-degree angle, on its right side. Similarly, the angle that measures x° corresponds to the angle that is adjacent to the indicated 90-degree angle, on its left side. This gives the equation $40 + 90 + x = 180$. x is 50 degrees, because $180 - (90 + 40) = 50$.

SAMPLE

Lesson 13: Adding the Angles in a Triangle

Goals

- Comprehend that a straight angle can be decomposed into 3 angles to construct a triangle.
- Justify (orally and in writing) that the sum of angles in a triangle is 180 degrees using properties of rigid motions.

Learning Targets

- If I know two of the angle measures in a triangle, I can find the third angle measure.

Lesson Narrative

In this lesson, the focus is on the interior angles of a triangle. What can we say about the three interior angles of a triangle? Do they have special properties?

The lesson opens with an optional activity looking at different types of triangles with a particular focus on the angle combinations of specific acute, right, and obtuse triangles. After being given a triangle, students form groups of 3 by identifying two other students with a triangle congruent to their own. After collecting some class data on all the triangles and their angles, they find that the sum of the angle measures in all the triangles turns out to be 180 degrees.

In the next activity, students observe that if a **straight angle** is decomposed into three angles, it appears that the three angles can be used to create a triangle. Together the activities provide evidence of a close connection between three positive numbers adding up to 180 and having a triangle with those three numbers as angle measures.

A new argument is needed to justify this relationship between three angles making a line and three angles being the angles of a triangle. This is the topic of the following lesson.

Alignments

Building On

- 7.G.A.2: Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

Addressing

- 8.G.A.2: Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
- 8.G.A.5: Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same

triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

Instructional Routines

- Group Presentations
- MLR7: Compare and Connect

Required Materials

Copies of blackline master Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Pre-printed slips, cut from copies of the blackline master

Required Preparation

Print copies of the Tear it Up blackline master. Prepare 1 copy for every group of 4 students. From the geometry toolkit, students will need scissors.

If you are doing the optional Find All Three activity, prepare 1 copy of the Find All Three blackline master for every 15 students. Cut these up ahead of time.

Student Learning Goals

Let's explore angles in triangles.

13.1 Can You Draw It?

Warm Up: 10 minutes

Students try to draw triangles satisfying different properties. They complete the table and then check with a partner whether or not they agree that the pictures are correct or that no such triangle can be drawn. The goals of this warm-up are:

- Reviewing different properties and types of triangles.
- Focusing on individual angle measures in triangles in preparation for studying their sum.

Note that we use the inclusive definition of isosceles triangle having at least two congruent sides. It is possible that in students' earlier experiences, they learned that an isosceles triangle has exactly two congruent sides. This issue may not even come up, but be aware that students may be working under a different definition of isosceles than what is written in the task statement.

Building On

- 7.G.A.2

Launch

Arrange students in groups of 2. Quiet work time for 3 minutes to complete the table followed by partner and whole-class discussion.

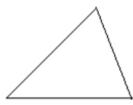
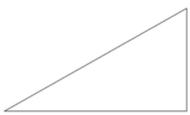
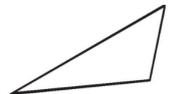
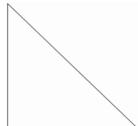
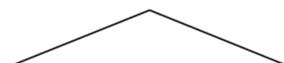
Student Task Statement

1. Complete the table by drawing a triangle in each cell that has the properties listed for its column and row. If you think you cannot draw a triangle with those properties, write "impossible" in the cell.
2. Share your drawings with a partner. Discuss your thinking. If you disagree, work to reach an agreement.

	acute (all angles acute)	right (has a right angle)	obtuse (has an obtuse angle)
scalene (side lengths all different)			
isosceles (at least two side lengths are equal)			
equilateral (three side lengths equal)			

Student Response

Sample response:

	acute (all angles acute)	right (has a right angle)	obtuse (has an obtuse angle)
scalene (side lengths all different)			
isosceles (at least two side lengths are equal)			
equilateral (three equal side lengths)		Impossible	Impossible

Activity Synthesis

Invite students to share a few triangles they were able to draw such as:

- Right and isosceles
- Equilateral and acute

Ask students to share which triangles they were unable to draw and why. For example, there is no right equilateral or obtuse equilateral triangle because the side opposite the right (or obtuse) angle is longer than either of the other two sides.

13.2 Find All Three

Optional: 15 minutes

This is a matching activity where each student receives a card showing a triangle and works to form a group of three. Each card has a triangle with the measure of only one of its angles given. Students use what they know about transformations and estimates of angle measures to find partners with triangles congruent to theirs. Each unique triangle's three interior angles are then displayed for all to see. Students notice that the sum of the measures of the angles in each triangle is 180 degrees.

During this activity, students can use MP7, thinking about applying rigid motions to their triangle to see if it might match up with another student's triangle. Or they may identify that their triangle is acute, right, or obtuse and use this as a criterion when they search for a congruent copy (also MP7).

You will need the Find All Three blackline master for this activity.

Addressing

- 8.G.A.2

- 8.G.A.5

Launch

Provide access to geometry toolkits. Distribute one card to each student, making sure that all three cards have been distributed for each triangle. (If the number of students in your class is not a multiple of three, it's okay for one or two students to take ownership over two cards showing congruent triangles.) Explain that there are two other students who have a triangle congruent to theirs that has been re-oriented in the coordinate plane through combinations of translations, rotations, and reflections. Instruct students to look at the triangle on their card and estimate the measures of the other two angles. With these estimates and their triangle in mind, students look for the two triangles congruent to theirs with one of the missing angles labeled.

Prepare and display a table for all to see with columns angle 1, angle 2, angle 3 and one row for each group of three students. It should look something like this:

student groups	angle 1	angle 2	angle 3

Once the three partners are together, they complete one row in the posted table for their triangle's angle measures. Whole-class discussion to follow.

Students might ask if they can use tracing paper to find congruent triangles. Ask how they would use it and listen for understanding of transformations to check for congruence. Respond that this seems to be a good idea.

Support for Students with Disabilities

Representation: Provide Access for Perception. Display or provide students with a physical copy of the written directions and read them aloud. Check for understanding by inviting students to rephrase directions in their own words. Consider keeping the display of directions visible throughout the activity.

Supports accessibility for: Language; Memory

Student Task Statement

Your teacher will give you a card with a picture of a triangle.

1. The measurement of one of the angles is labeled. Mentally estimate the measures of the other two angles.

2. Find two other students with triangles congruent to yours but with a different angle labeled. Confirm that the triangles are congruent, that each card has a different angle labeled, and that the angle measures make sense.
3. Enter the three angle measures for your triangle on the table your teacher has posted.

Student Response

The angle combinations are: 40, 50, 90; 40, 60, 80; 50, 50, 80; 20, 20, 140; 20, 40, 120

Activity Synthesis

Begin the discussion by asking students:

- "What were your thoughts as you set about to find your partners?"
- "How did you know that you found a correct partner?"

Expect students to discuss estimates for angle measures and their experience of how different transformations influence the position and appearance of a polygon.

Next look at the table of triangle angles and ask students:

- "Is there anything you notice about the combinations of the three angle measures?"
- "Is there something in common for each row?"

Guide students to notice that the sum in each row is the same, 180 degrees. Ask whether they think this is always be true for any triangle. Share with students that in the next activity, they will work towards considering whether this result is true for all triangles.

13.3 Tear It Up

25 minutes

In the optional activity, students found that the sum of the angles of all the triangles on the cards was 180 degrees and questioned if all triangles have the same angle sum. In this activity, students experiment with the converse: If we know the measures of three angles sum to 180 degrees, can these three angles be the interior angles in a triangle?

Students cut out three angles that form a line, and then try to use these three angles to make a triangle. Students also get to create their own three angles from a line and check whether they can construct a triangle with their angles.

Watch for students who successfully make triangles out of each set of angles and select them to share (both the finished product and how they worked to arrange the angles) during the discussion. Watch also for how students divide the blank line into angles. It is helpful if the rays all have about the same length as in the pre-made examples.

Addressing

- 8.G.A.5

Instructional Routines

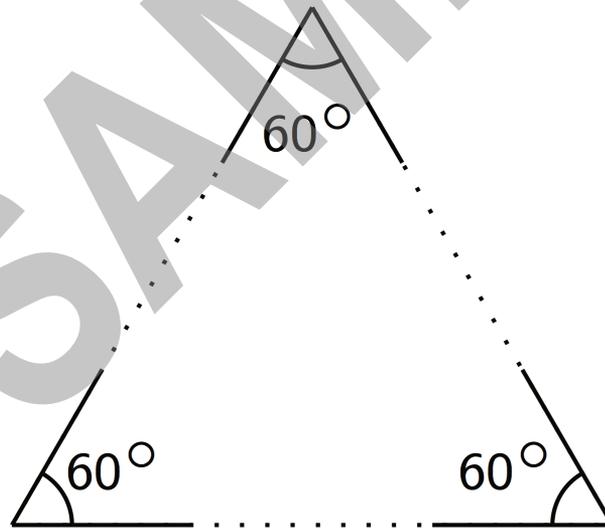
- Group Presentations
- MLR7: Compare and Connect

Launch

Arrange students in groups of 4. Provide access to geometry toolkits, especially scissors. Distribute 1 copy of the black line master to each group.

Instruct students to cut the four individual pictures out of the black line master. Each student will work with one of these. Instruct the student with the blank copy to use a straightedge to divide the line into three angles (different from the three angles that the other students in the group have). Demonstrate how to do this if needed.

If needed, you may wish to demonstrate “making a triangle” part of the activity so students understand the intent. With three cut-out 60 degree angles, for example, you can build an equilateral triangle. Here is a picture showing three 60 degree angles arranged so that they can be joined to form the three angles of an equilateral triangle. The students will need to arrange the angles carefully, and they may need to use a straightedge in order to add the dotted lines to join the angles and create a triangle.



Student Task Statement

Your teacher will give you a page with three sets of angles and a blank space. Cut out each set of three angles. Can you make a triangle from each set that has these same three angles?

Student Response

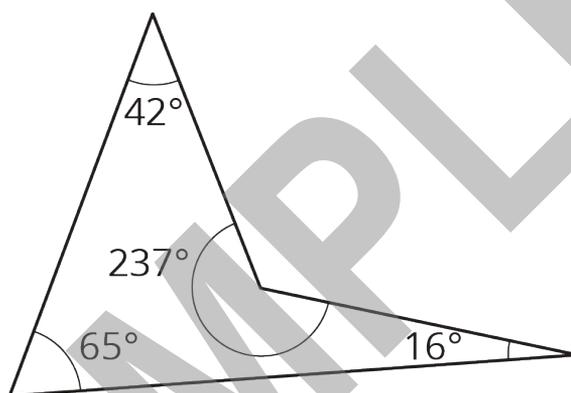
Answers vary. Sample response: We were all able to build triangles with the given sets of angles. One is a right triangle, one acute, and one obtuse. The three angles we chose also made a triangle.

Are You Ready for More?

1. Draw a quadrilateral. Cut it out, tear off its angles, and line them up. What do you notice?
2. Repeat this for several more quadrilaterals. Do you have a conjecture about the angles?

Student Response

The sum of interior angles in any quadrilateral is 360° . This is pretty clear with rectangles. Parallelograms have 2 pairs of equal supplementary angles, so they work too. In fact, it works for anything, even non-convex quadrilaterals.



Activity Synthesis

If time allows, have students do a "gallery walk" at the start of the discussion. Ask students to compare the triangle they made to the other triangles made from the same angles and be prepared to share what they noticed. (For example, students might notice that all the other triangles made with their angles looked pretty much the same, but were different sizes.) If students do not bring it up, direct students to notice that all of the "create three of your own angles" students were able to make a triangle, not just students with the ready-made angles.

Ask previously selected students to share their triangles and explain how they made the triangles. To make the triangles, some trial and error is needed. A basic method is to line up the line segments from two angles (to get one side of the triangle) and then try to place the third angle so that it lines up with the rays coming from the two angles already in place. Depending on the length of the rays, they may overlap, or the angles may need to be moved further apart. Ask questions to make sure that students see the important connection:

- "How do you know the three angles you were given sum to 180 degrees?" (They were adjacent to each other along a line.)
- "How do you know these can be the three angles of a triangle?" (We were able to make a triangle using these three angles.)

- "What do you know about the three angles of the triangle you made and why?" (Their measures sum to 180 because they were the same three angles that made a line.)

Ask students if they think they can make a triangle with *any* set of three angles that form a line and poll the class for a positive or negative response. Tell them that they will investigate this in the next lesson and emphasize that while experiments may lead us to believe this statement is true, the methods used are not very accurate and were only applied to a few sets of angles.

If time permits, perform a demonstration of the converse: Start with a triangle, tear off its three corners, and show that these three angles when placed adjacent each other sum to a line.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as "To arrange the angles, first, I ____ because..." or "I noticed ____ so I..."

Supports accessibility for: Language; Social-emotional skills

Support for English Language Learners

Representing, Conversing, Listening: MLR7 Compare and Connect. As students prepare their work for discussion, look for those who successfully construct triangles out of each set of angles and for those who successfully create their own three angles from a line and create triangles. Encourage students to explain how they worked to arrange the angles. Emphasize language used to make sense of strategies used to find that the sum of the angles in a triangle is 180 degrees.

Design Principle(s): Maximize Meta-awareness, Support sense-making

Lesson Synthesis

Some guiding questions for the discussion include:

- "What did we observe about the sum of the angles inside a triangle?" (The sum of the angles inside a triangle seem to always add up to 180 degrees.)
- "When you know two angles inside a triangle, how can you find the third angle?" (If all three angles add up to 180 degrees, then subtracting two of the angles from 180 will give the measure of the third angle.)
- "Are there pairs of angles that cannot be used to make a triangle?" (Yes. If the two angles are both bigger than or equal to 90 degrees, then you cannot make a triangle.)

Emphasize that we were able to see for multiple triangles that the sum of their angles is 180° and that using several sets of three angles adding to 180° we were able to build triangles with those angles. In the next lesson we will investigate and explain this interesting relationship.

13.4 Missing Angle Measures

Cool Down: 5 minutes

Students have experimented to see that the sum of the angles in a triangle is 180 degrees. While they will prove this in the next lesson, here they apply the concept in order to give examples of different kinds of triangles with one given angle measure.

Addressing

- 8.G.A.5

Launch

If needed, tell students to use their conjecture that the sum of the angles in a triangle is always 180 degrees as they work on these problems.

Student Task Statement

In triangle ABC , the measure of angle B is 50 degrees.

1. Give possible values for the measures of angles A and C if ABC is an acute triangle.
2. Give possible values for the measures of angles A and C if ABC is an obtuse triangle.
3. Give possible values for the measures of angles A and C if ABC is a right triangle.

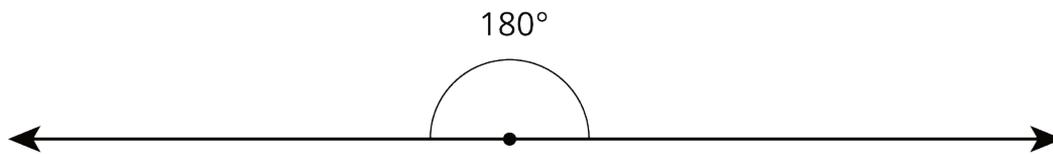
Student Response

Answers vary. Sample responses:

1. To make an acute triangle, the other two angles must measure less than 90 degrees (for example: 60, 70).
2. To make an obtuse triangle, one of the two angles must be greater than 90 degrees (100, 30).
3. There is only one way to make a right triangle (90, 40).

Student Lesson Summary

A 180° angle is called a **straight angle** because when it is made with two rays, they point in opposite directions and form a straight line.



If we experiment with angles in a triangle, we find that the sum of the measures of the three angles in each triangle is 180° —the same as a straight angle!

Through experimentation we find:

- If we add the three angles of a triangle physically by cutting them off and lining up the vertices and sides, then the three angles form a straight angle.
- If we have a line and two rays that form three angles added to make a straight angle, then there is a triangle with these three angles.



Glossary

- straight angle

Lesson 13 Practice Problems

Problem 1

Statement

In triangle ABC , the measure of angle A is 40° .

- Give possible measures for angles B and C if triangle ABC is isosceles.
- Give possible measures for angles B and C if triangle ABC is right.

Solution

- There are two possibilities: Angles B and C each measure 70° , or one of angles B and C measures 40° and the other measures 100° .
- One of angles B and C measures 50° , and the other angle measures 90° .

Problem 2

Statement

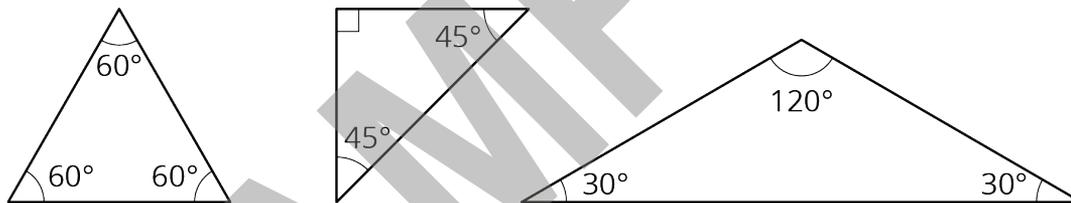
For each set of angles, decide if there is a triangle whose angles have these measures in degrees:

- a. 60, 60, 60
- b. 90, 90, 45
- c. 30, 40, 50
- d. 90, 45, 45
- e. 120, 30, 30

If you get stuck, consider making a line segment. Then use a protractor to measure angles with the first two angle measures.

Solution

Triangles can be made with the sets of angles in a, d, and e but not with b, and c.



Problem 3

Statement

Angle A in triangle ABC is obtuse. Can angle B or angle C be obtuse? Explain your reasoning.

Solution

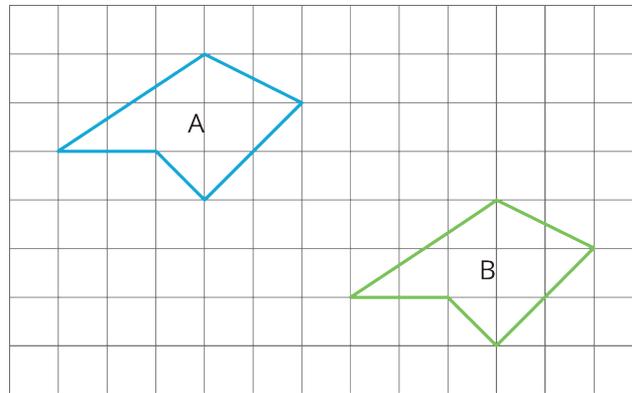
No, a triangle can not have two obtuse angles. If the obtuse angles were at vertices A and B , for example, then those angles do not meet at any point C .

Problem 4

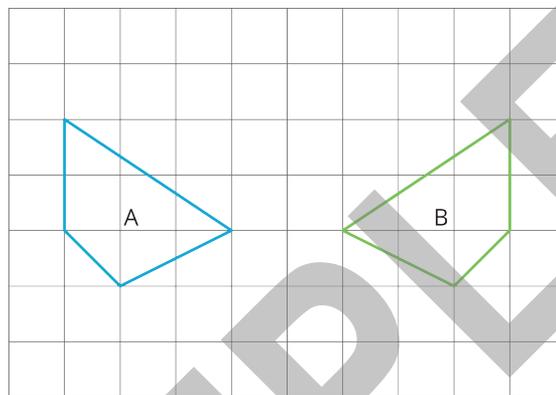
Statement

For each pair of polygons, describe the transformation that could be applied to Polygon A to get Polygon B.

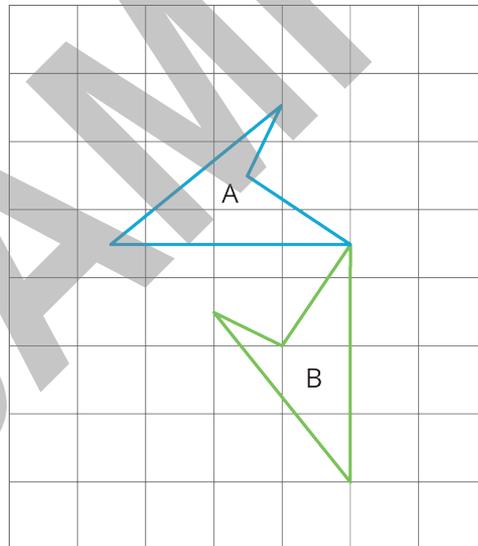
a.



b.



c.



Solution

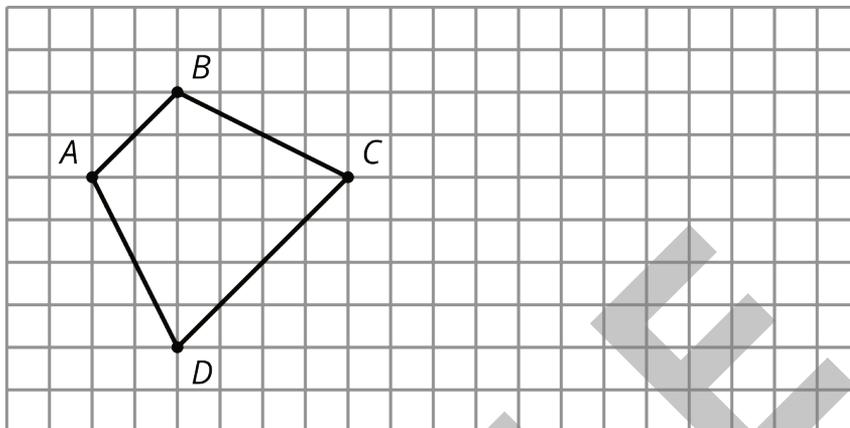
- Translation down 3 units and right 6 units
- Reflection with a vertical line of reflection halfway between the two polygons
- Rotation by 90 degrees counterclockwise with the vertex shared by the two polygons as the center of rotation

(From Unit 1, Lesson 3.)

Problem 5

Statement

On the grid, draw a scaled copy of quadrilateral $ABCD$ using a scale factor of $\frac{1}{2}$.



Solution

Answers vary. Each side is $\frac{1}{2}$ the length of the corresponding side on $ABCD$. For example, if $B'C'$ on the scaled copy corresponds to BC , then C' should be down 1 grid square and right 2 grid squares from B' . The same should be true for all other sides: this guarantees that corresponding angles will have the same measure.

(From Unit 1, Lesson 12.)

Lesson 14: Parallel Lines and the Angles in a Triangle

Goals

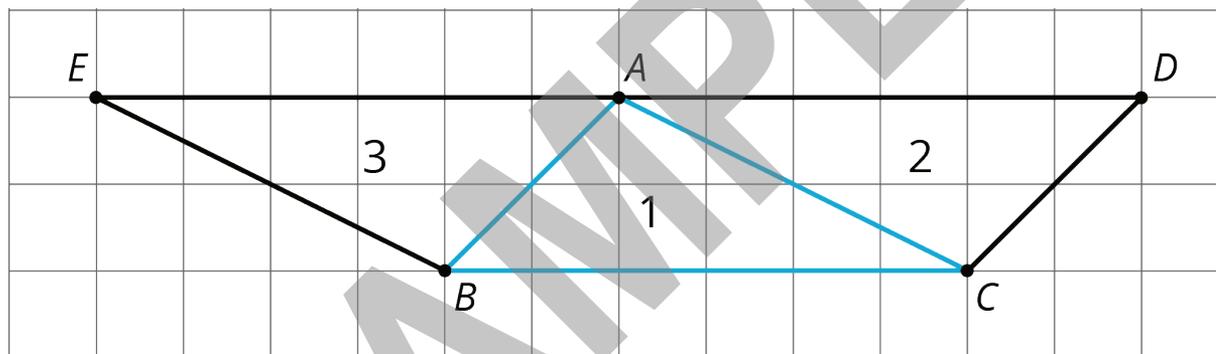
- Create diagrams using 180-degree rotations of triangles to justify (orally and in writing) that the measure of angles in a triangle sum up to 180 degrees.
- Generalize the Triangle Sum Theorem using rigid transformations or the congruence of alternate interior angles of parallel lines cut by a transversal.

Learning Targets

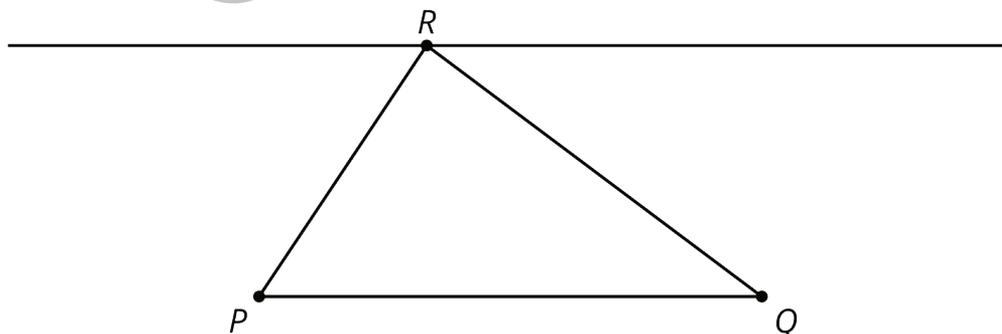
- I can explain using pictures why the sum of the angles in any triangle is 180 degrees.

Lesson Narrative

Earlier in this unit, students learned that when a 180° rotation is applied to a line ℓ , the resulting line is parallel to ℓ . Here is a picture students worked with earlier in the unit:



The picture was created by applying 180° rotations to $\triangle ABC$ with centers at the midpoints of segments AC and AB . Notice that E , A , and D all lie on the same grid line that is parallel to line BC . In this case, we have the structure of the grid to help see why this is true. This argument exhibits one aspect of MP7, using the structure of the grid to help explain why the three angles in this triangle add to 180 degrees.



In this lesson, students begin by examining the argument using grid lines described above. Then they examine a triangle off of a grid, PQR . Here an auxiliary line plays the role of the grid lines: the line parallel to line PQ through the opposite vertex R . The three angles sharing vertex R make a line and so they add to 180 degrees. Using what they have learned earlier in this unit (either congruent alternate interior angles for parallel lines cut by a transversal or applying rigid transformations explicitly), students argue that the sum of the angles in triangle PQR is the same as the sum of the angles meeting at vertex R . This shows that the sum of the angles in *any* triangle is 180 degrees. The idea of using an auxiliary line in a construction to solve a problem is explicitly called out in MP7.

Alignments

Building On

- 7.NS.A: Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.
- 8.G.A.1.b: Angles are taken to angles of the same measure.

Addressing

- 8.G.A.5: Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

Building Towards

- 8.G.B.6: Explain a proof of the Pythagorean Theorem and its converse.

Instructional Routines

- MLR8: Discussion Supports
- True or False

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's see why the angles in a triangle add to 180 degrees.

14.1 True or False: Computational Relationships

Warm Up: 5 minutes

This warm-up encourages students to reason algebraically about various computational relationships and patterns. While students may evaluate each side of the equation to determine if it is true or false, encourage students to think about the properties of arithmetic operations in their reasoning.

Seeing the structure that makes the equations true or false develops MP7.

Building On

- 7.NS.A

Instructional Routines

- True or False

Launch

Display one problem at a time. Tell students to give a signal when they have an answer and a strategy. After each problem, give students 1 minute of quiet think time and follow with a whole-class discussion.

Anticipated Misconceptions

In the first question, students may think you can round or adjust numbers in a subtraction problem in the same way as in addition problems. For example, when adding $62 + 28$, taking 2 from the 62 and adding it to the 28 does not change the sum. However, using that same strategy when subtracting, the distance between the numbers on the number line changes and the difference does not remain the same.

Student Task Statement

Is each equation true or false?

$$62 - 28 = 60 - 30$$

$$3 \cdot -8 = (2 \cdot -8) - 8$$

$$\frac{16}{-2} + \frac{24}{-2} = \frac{40}{-2}$$

Student Response

- False. Explanations vary. Possible response: Think about a number line. The difference between numbers is how far apart they are. 62 and 28 are further apart than 60 and 30.
- True. Explanations vary. Possible response: Rewrite $(3 \cdot -8)$ as $(2 \cdot -8) + (1 \cdot -8)$.
- True. Explanations vary. Possible response: Since $16 + 24 = 40$ both sides of the equation are equal to $\frac{40}{-2}$.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. Ask students how they decided upon a strategy. To involve more students in the conversation, consider asking:

- Do you agree or disagree? Why?
- Who can restate ___'s reasoning in a different way?
- Does anyone want to add on to ____'s reasoning?

After each true equation, ask students if they could rely on the reasoning used on the given problem to think about or solve other problems that are similar in type. After each false equation, ask students how we could make the equation true.

14.2 Angle Plus Two

15 minutes (there is a digital version of this activity)

In the previous lesson, students conjectured that the measures of the interior angles of a triangle add up to 180 degrees. The purpose of this activity is to explain this structure in some cases. Students apply 180° rotations to a triangle in order to calculate the sum of its three angles. They have applied these transformations earlier in the context of building shapes using rigid transformations. Here they exploit the structure of the coordinate grid to see in a particular case that the sum of the three angles in a triangle is 180 degrees. The next activity removes the grid lines and gives an argument that applies to all triangles.

Building On

- 8.G.A.1.b

Addressing

- 8.G.A.5

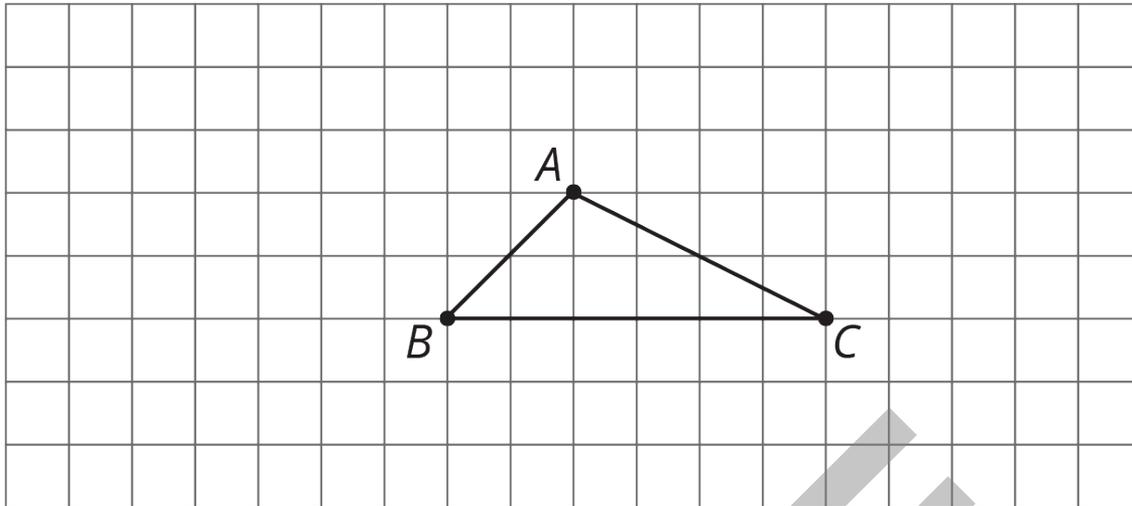
Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students in groups of 2–3. Provide access to geometry toolkits. Give 5 minutes work time building the diagram and measuring angles. Then allow for a short whole-class check-in about angle measurement error, and then provide time for students to complete the task.

Important note for classes using the digital activity: The applet measures angles in the standard direction, counterclockwise. Students need to select points in order. For example, to measure angle ACB in this triangle, students would select the angle measure tool and then click on A , then C , and then B . If they click on B , then C , and then A , they get the reflex angle.



Support for Students with Disabilities

Action and Expression: Internalize Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organizational skills in problem solving. For example, present 2–3 questions at a time and monitor students to ensure they are making progress throughout the activity.

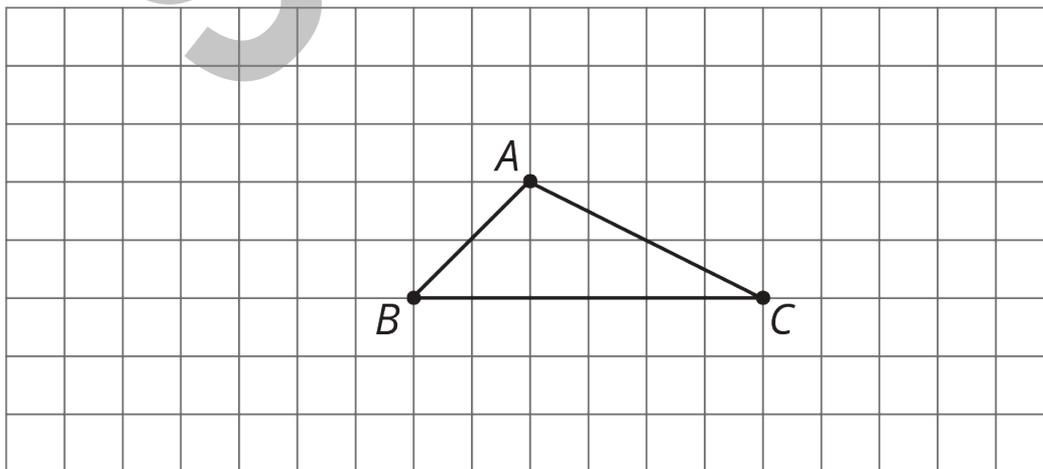
Supports accessibility for: Organization; Attention

Anticipated Misconceptions

Some students may have trouble with the rotations. If they struggle, remind them of similar work they did in a previous lesson. Help them with the first rotation, and allow them to do the second rotation on their own.

Student Task Statement

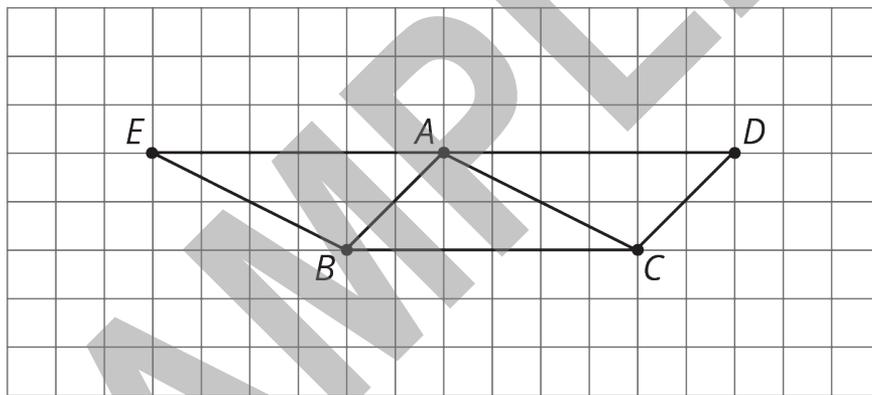
Here is triangle ABC .



1. Rotate triangle ABC 180° around the midpoint of side AC . Label the new vertex D .
2. Rotate triangle ABC 180° around the midpoint of side AB . Label the new vertex E .
3. Look at angles EAB , BAC , and CAD . Without measuring, write what you think is the sum of the measures of these angles. Explain or show your reasoning.
4. Is the measure of angle EAB equal to the measure of any angle in triangle ABC ? If so, which one? If not, how do you know?
5. Is the measure of angle CAD equal to the measure of any angle in triangle ABC ? If so, which one? If not, how do you know?
6. What is the sum of the measures of angles ABC , BAC , and ACB ?

Student Response

1. Rotating by 180° around the midpoint of segment AC , C and A trade places and B goes to the new point labeled D in the picture. Rotating 180° around the midpoint of segment AB , points A and B trade places and C goes to the new point labeled E in the picture.



2. They look like they will add to 180° , because they appear to form a straight angle and there are 180° in a straight angle.
3. Yes, angle ABC . When triangle ABC is rotated 180 degrees with center the midpoint of segment AB , $\angle ABC$ goes to $\angle EAB$.
4. Yes, angle ACB . When triangle ABC is rotated 180 degrees with center the midpoint of segment AC , $\angle ACB$ goes to $\angle CAD$.
5. The total measure of these angles should be 180° , because it is the same as the total measure of angles EAB , BAC , and CAD and these angles add up to 180° .

Activity Synthesis

Ask students how the grid lines helped to show that the sum of the angles in this triangle is 180 degrees. Important ideas to bring out include:

- A 180° rotation of a line BC (with center the midpoint of AB or the midpoint of AC) is parallel to line BC and lays upon the horizontal grid line that point A is on.

- The grid lines are parallel so the rotated angles lie on the (same) grid line.

Consider asking students, "Is it always true that the sum of the angles in a triangle is 180° ?" (Make sure students understand that the argument here does not apply to most triangles, since it relies heavily on the fact that BC lies on a grid line which means we know the images of BC , AE and DA , also lie on grid line that point A is on.)

It turns out that the key to showing the more general result lies in studying the rotations that were used to generate the three triangle picture. This investigation is the subject of the next activity.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to support whole-class discussion. For each observation that is shared, ask students to restate and/or revoice what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language about transformations.

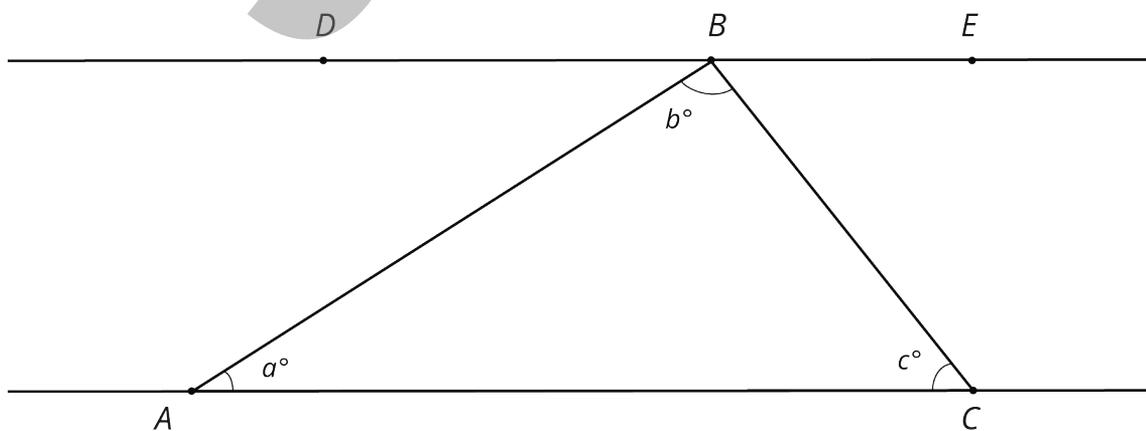
Design Principle(s): Support sense-making

14.3 Every Triangle in the World

15 minutes

The previous activity recalls the 180° rotations used to create an important diagram. This diagram shows that the sum of the angles in a triangle is 180° if the triangle happens to lie on a grid with a horizontal side. The purpose of this activity is to provide a complete argument, not depending on the grid, of why the sum of the three angles in a triangle is 180° .

In this activity, rather than building a complex shape from a triangle and its rotations, students begin with a triangle and a line parallel to the base through the opposite vertex:



In this image, lines AC and DE are parallel. The advantage to this situation is that we *know* that points D , B , and E all lie on a line. In order to calculate angles DBA and EBC , students can use either the rotation idea of the previous activity or congruence of alternate interior angles of parallel lines cut by a transversal. In either case, they need to analyze the given constraints and decide on a path to pursue to show the congruence of angles. Students then conclude from the fact that D , B , and E lie on the same line that $a + b + c = 180$.

There is a subtle distinction in the logic between this lesson and the previous. The previous lesson suggests that the sum of the angles in a triangle is 180° using direct measurements of a triangle on a grid. This activity *shows* that this is the case by using a generic triangle and reasoning about parallel lines cut by a transversal. But, in order to do so, we needed to know to draw the line parallel to line AC through B and that this idea came through experimenting with rotating triangles.

Launch

Keep students in the same groups. Tell students they'll be working on this activity *without* the geometry toolkit.

In case students have not seen this notation before, explain that $m\angle DBA$ is shorthand for "the measure of angle DBA ."

Begin with 5 minutes of quiet work time. Give groups time to compare their arguments, then have a whole-class discussion.

Support for Students with Disabilities

Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, "The angles in a triangle always add up to 180 degrees because...", "To show this, first I ____ because...", "That could/couldn't be true because...", "This method works/doesn't work because..."

Supports accessibility for: Language; Organization

Support for English Language Learners

Conversing, Representing, Writing: MLR5 Co-Craft Questions. This is the first time Math Language Routine 5 is suggested as a support in this course. In this routine, students are given a context or situation, often in the form of a problem stem with or without numerical values. Students develop mathematical questions that can be asked about the situation. A typical prompt is: “What mathematical questions could you ask about this situation?” The purpose of this routine is to allow students to make sense of a context before feeling pressure to produce answers, and to develop students’ awareness of the language used in mathematics problems.

Design Principle(s): Cultivate conversation; Support sense-making

How It Happens:

1. Ask students to keep their books or devices closed and display only the image and the task statement without revealing the questions that follow.
Ask, “What mathematical questions could you ask about this image?”
2. Give students 1 minute of individual time to write questions, and then 3 minutes to share ideas with a partner.
3. Invite each pair of students to contribute one written question to a poster, the whiteboard, or digital projection. Call on 2–3 pairs of students to present their question to the whole class, and invite the class to make comparisons among the questions. Listen for and amplify any questions involving the relationships between the sum of angles in a triangle and in a straight angle.
4. Reveal the three follow-up questions for this task and give students a couple of minutes to compare them to those on the display. Identify similarities and differences. Consider providing these prompts: “Which of your questions is most alike or different to the ones provided? Why?”, or “Is there a main mathematical concept that is present in both your questions and those provided? If so, describe it.”
5. Invite students to choose one student generated question (from the display) before continuing on to the task.

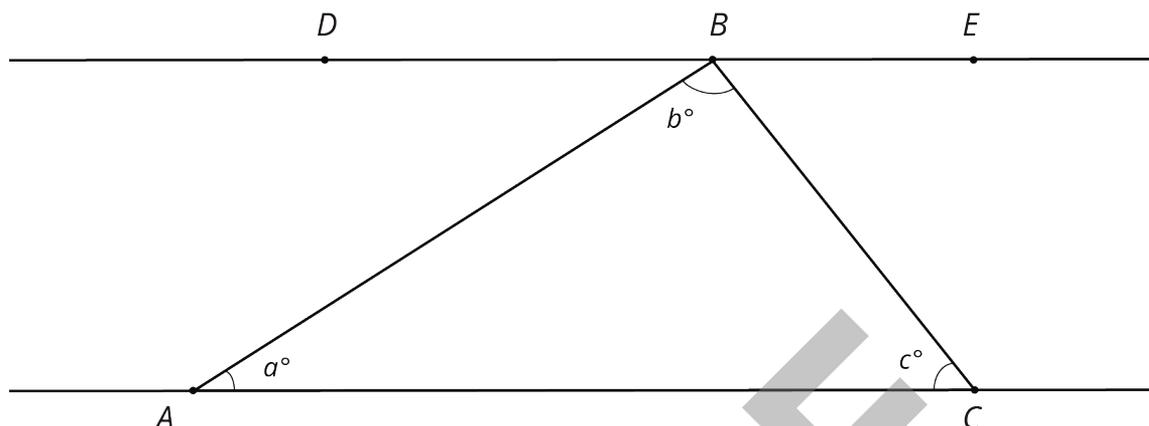
Anticipated Misconceptions

Some students may say that a , b , and c are the three angles in a triangle, so they add up to 180. Make sure that these students understand that the goal of this activity is to explain why this must be true. Encourage them to use their answer to the first question and think about what they know about different angles in the diagram.

For the last question students may not understand why their work in the previous question only shows $a + b + c = 180$ for one particular triangle. Consider drawing a different triangle (without the parallel line to one of the bases), labeling the three angle measures a , b , c , and asking the student why $a + b + c = 180$ for *this* triangle.

Student Task Statement

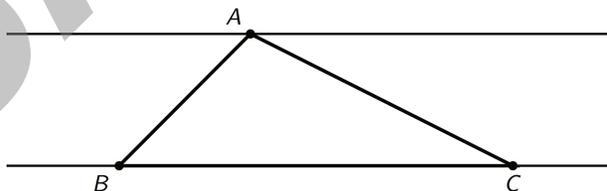
Here is $\triangle ABC$. Line DE is parallel to line AC .



1. What is $m\angle DBA + b + m\angle CBE$? Explain how you know.
2. Use your answer to explain why $a + b + c = 180$.
3. Explain why your argument will work for *any* triangle: that is, explain why the sum of the angle measures in *any* triangle is 180° .

Student Response

1. Angles DBA , ABC , and CBE make a line. So the sum of their angle measures is 180° .
2. Angles DBA and CAB are congruent because these are alternate interior angles for the parallel lines AC and DE with transversal AB . Angles EBC and BCA are congruent because these are alternate interior angles for the parallel lines AC and DE with transversal BC . Angles DBA , ABC , and CBE make a line, and so their angle measures add up to 180° . Then $a + b + c = 180$.
3. For any triangle, draw a line parallel to one side, containing the opposite vertex:



With this picture, use the same argument to show that the sum of the three angles of the triangle is 180° . This works for *every* triangle.

Are You Ready for More?

1. Using a ruler, create a few quadrilaterals. Use a protractor to measure the four angles inside the quadrilateral. What is the sum of these four angle measures?
2. Come up with an explanation for why anything you notice must be true (hint: draw one diagonal in each quadrilateral).

Student Response

1. The sum of the interior angle measures in any quadrilateral is 360° . Note that since protractors are imprecise, physical measurements may range by a few degrees away from this.
2. In any quadrilateral, you can draw a diagonal that partitions the quadrilateral into two triangles. The sum of the angle measures in each triangle is 180° , and the six angles in the two triangles comprise all of the angles in the quadrilateral.

Activity Synthesis

Ask students how this activity differs from the previous one, where $\triangle ABC$ had a horizontal side lying on a grid line. Emphasize that:

- This argument applies to *any* triangle ABC .
- The prior argument relies on having grid lines and having the base of the triangle lie on a grid line.
- This argument relies heavily on having the parallel line to AC through B drawn, something we can always add to a triangle.

The key inspiration in this activity is putting in the line DE through B parallel to AC . Once this line is drawn, previous results about parallel lines cut by a transversal allow us to see why the sum of the angles in a triangle is 180° . Tell students that the line DE is often called an ‘auxiliary construction’ because we are trying to show something about $\triangle ABC$ and this line happens to be very helpful in achieving that goal. It often takes experience and creativity to hit upon the right auxiliary construction when trying to prove things in mathematics.

14.4 Four Triangles Revisited

Optional: 5 minutes

This activity revisits a picture that students have seen earlier and that they will see again later when they investigate the Pythagorean Theorem. The four right triangles around the boundary make a quadrilateral inside which happens to be a square. Since the four triangles surrounding the inner quadrilateral are congruent by construction, this makes the inner shape a rhombus. The angle calculations that students make here justify why it is actually a square. The fact that quadrilateral $ACEG$ is a square is not asked for in the activity, but if any students notice this, encourage them to share their observations and reasoning.

Students are making use of structure (MP7) throughout this activity, most notably by:

- Using the fact that congruent triangles have congruent angles.
- Using the fact that the measures of angles that make a line add up to 180 degrees.

Building On

- 8.G.A.1.b

Addressing

- 8.G.A.5

Building Towards

- 8.G.B.6

Instructional Routines

- MLR8: Discussion Supports

Launch

Provide 3 minutes of work time followed by a whole-class discussion.

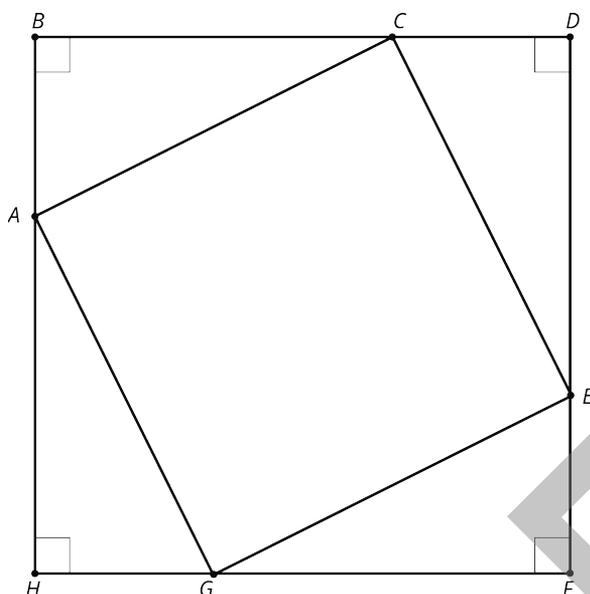
Support for Students with Disabilities

Action and Expression: Develop Expression and Communication. Maintain a display of important terms and information. During the launch, take time to review the following facts from previous lessons that students will need to access for this activity: congruent triangles have congruent angles, a straight angle is equal to 180 degrees, and the sum of the angles in a triangle add up to 180 degrees.

Supports accessibility for: Memory; Language

Student Task Statement

This diagram shows a square $BDFH$ that has been made by images of triangle ABC under rigid transformations.



Given that angle BAC measures 53 degrees, find as many other angle measures as you can.

Student Response

All other angles can be determined from the one given measure.

Angles ECD , GEF , and AGH all measure 53° because the measure of $\angle BAC$ is 53° . These three angles correspond to $\angle BAC$ under a rigid motion of $\triangle ABC$. To find angles ACB , CED , EGF , and GAH , notice that these are all congruent because they all correspond to $\angle ACB$ of $\triangle ABC$ under a rigid motion. Angle ACB measures 37° because the angles in a triangle add to 180° . One angle in $\triangle ABC$ measures 53° , and another measures 90° , so the third angle measures $180^\circ - 53^\circ - 90^\circ = 37^\circ$.

Angles ACE , CEG , EGA , and GAC all measure 90° . Here is an argument for $\angle ACE$: We know that angle ACD measures 37° and angle ECD measures 53° . So angle ACE must measure 90° because it makes a line together with $\angle ACD$ and $\angle ECD$.

Activity Synthesis

Display the image of the 4 triangles for all to see. Invite students to share how they calculated one of the other unknown angles in the image, adding to the image until all the unknown angles are filled in.

In wrapping up, note that angles ACE , CEG , EGA , and GAC are all right angles. It's not necessary to show or tell students that $ACEG$ is a square yet. This diagram will be used in establishing the Pythagorean Theorem later in the course, so it's a good place to end.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. To help students produce statements about finding the unknown angles in the diagram, provide sentence frames such as, “Knowing ____, helps me find ____ because . . .” or “Angle ____ corresponds to angle ____ because . . .”

Design Principle(s): Optimize output (for explanation); Support sense-making

Lesson Synthesis

Revisit the basic steps in the proof that shows that the sum of the angles in a triangle is 180° . Consider asking a student to make a triangle that you can display for all to see and add onto it showing each step.

- We had a triangle, and a line through one vertex parallel to the opposite side.
- We knew that the three angles with their vertices on the line summed to 180° .
- We knew that two of these angles were congruent to corresponding angles in the triangle, and the third one was inside the triangle.
- Therefore, the three angles in the triangle must also sum to 180° .

Tell students that this is one of the most useful results in geometry and they will get to use it again and again in the future.

14.5 Angle Sizes

Cool Down: 5 minutes

Students sketch different triangles and list angle possibilities. Knowing that the sum of the angles in a triangle is 180 degrees establishes that each angle in an equilateral triangle measures 60 degrees.

Addressing

- 8.G.A.5

Student Task Statement

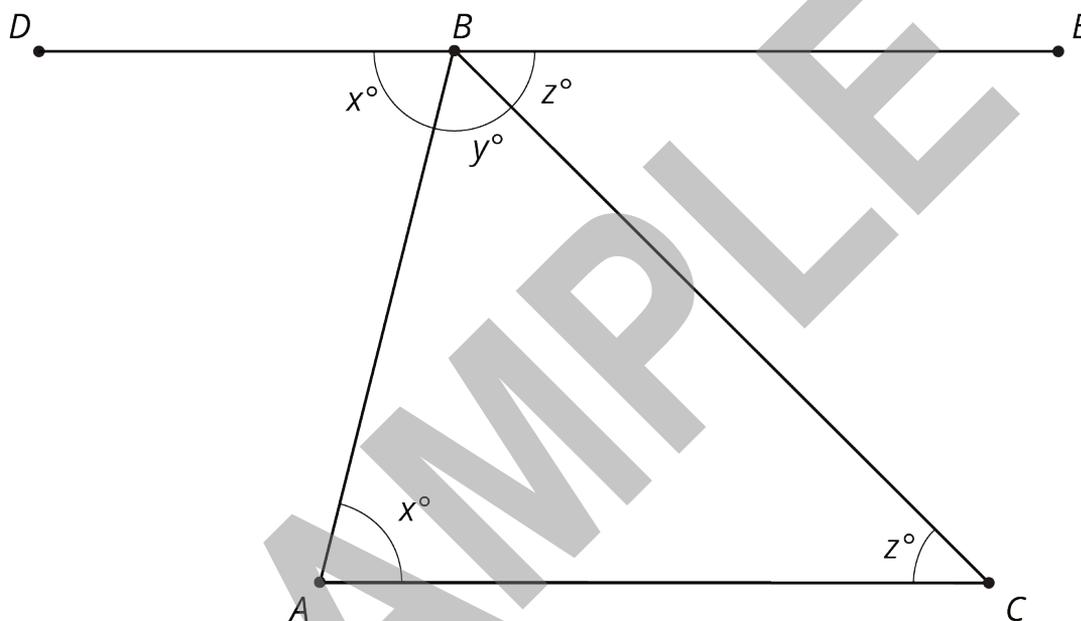
1. In an equilateral triangle, all side lengths are equal and all angle measures are equal. Sketch an equilateral triangle. What are the measures of its angles?
2. In an isosceles triangle, which is not equilateral, two side lengths are equal and two angle measures are equal. Sketch three different isosceles triangles.
3. List two different possibilities for the angle measures of an isosceles triangle.

Student Response

1. The three angle measures must add to 180° but must all be equal. Therefore each measure is 60° , since $60 = \frac{1}{3} \cdot 180$.
2. Answers vary. Triangles should each have two sides that are the same length but not three.
3. Answers vary. Sample responses: $70^\circ, 70^\circ, 40^\circ$ and $30^\circ, 30^\circ, 120^\circ$.

Student Lesson Summary

Using parallel lines and rotations, we can understand why the angles in a triangle always add to 180° . Here is triangle ABC . Line DE is parallel to AC and contains B .



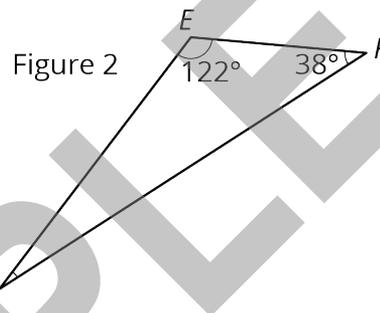
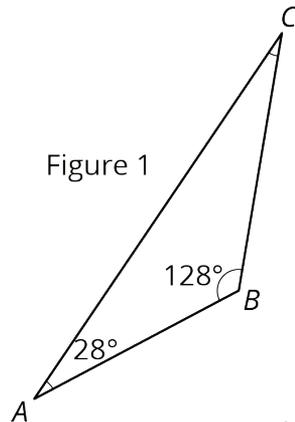
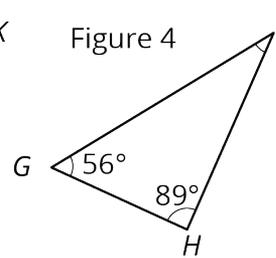
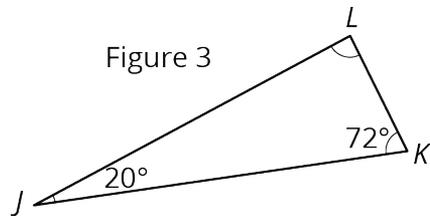
A 180° rotation of triangle ABC around the midpoint of AB interchanges angles A and DBA so they have the same measure: in the picture these angles are marked as x° . A 180° rotation of triangle ABC around the midpoint of BC interchanges angles C and CBE so they have the same measure: in the picture, these angles are marked as z° . Also, DBE is a straight line because 180° rotations take lines to parallel lines. So the three angles with vertex B make a line and they add up to 180° ($x + y + z = 180$). But x, y, z are the measures of the three angles in $\triangle ABC$ so the sum of the angles in a triangle is always 180° !

Lesson 14 Practice Problems

Problem 1

Statement

For each triangle, find the measure of the missing angle.



Solution

- a. 24 degrees ($24 + 28 + 128 = 180$)
- b. 20 degrees ($20 + 38 + 122 = 180$)
- c. 88 degrees ($88 + 20 + 72 = 180$)
- d. 35 degrees ($35 + 56 + 89 = 180$)

Problem 2

Statement

Is there a triangle with *two* right angles? Explain your reasoning.

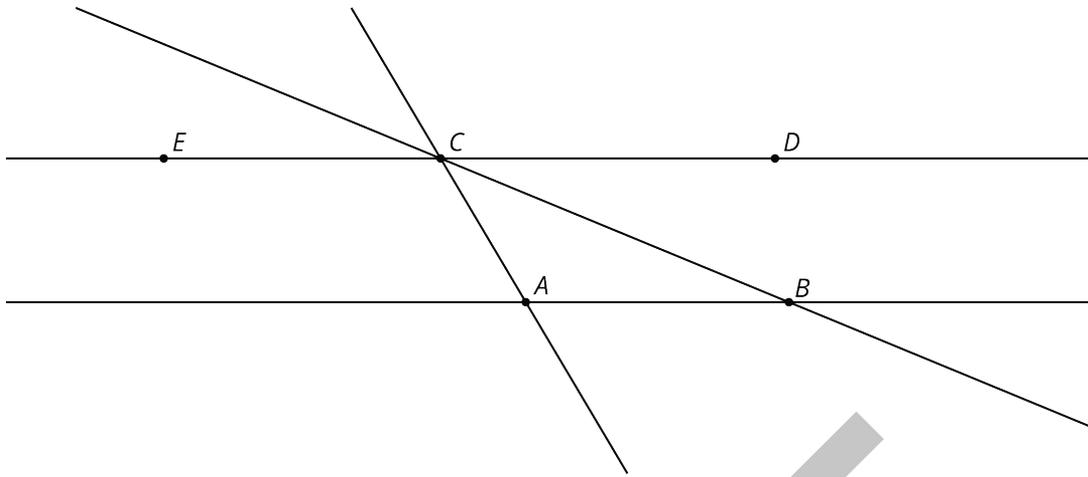
Solution

No, the three angles in a triangle add up to 180 degrees. Two right angles would already make 180 degrees, and so the third angle of the triangle would have to be 0 degrees—this is not possible.

Problem 3

Statement

In this diagram, lines AB and CD are parallel.



Angle ABC measures 35° and angle BAC measures 115° .

- What is $m\angle ACE$?
- What is $m\angle DCB$?
- What is $m\angle ACB$?

Solution

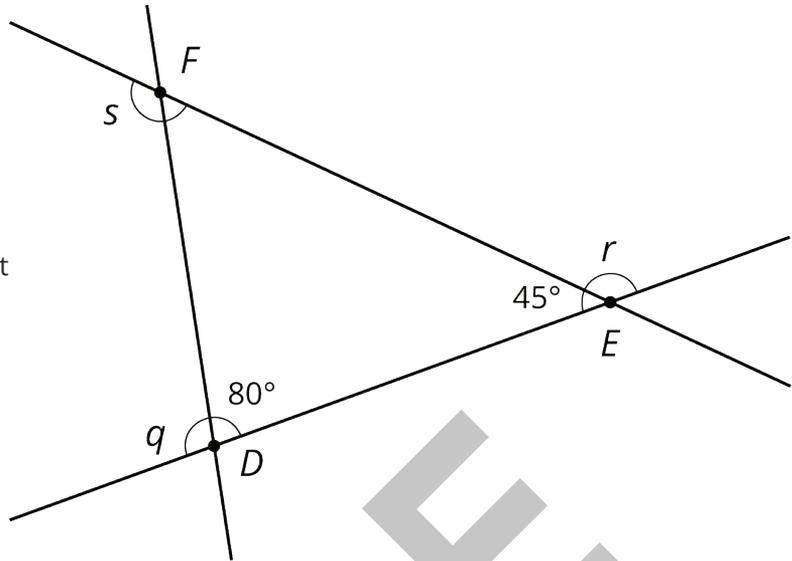
- 115°
- 35°
- 30°

Problem 4

Statement

Here is a diagram of triangle DEF .

- Find the measures of angles q , r , and s .
- Find the sum of the measures of angles q , r , and s .
- What do you notice about these three angles?



Solution

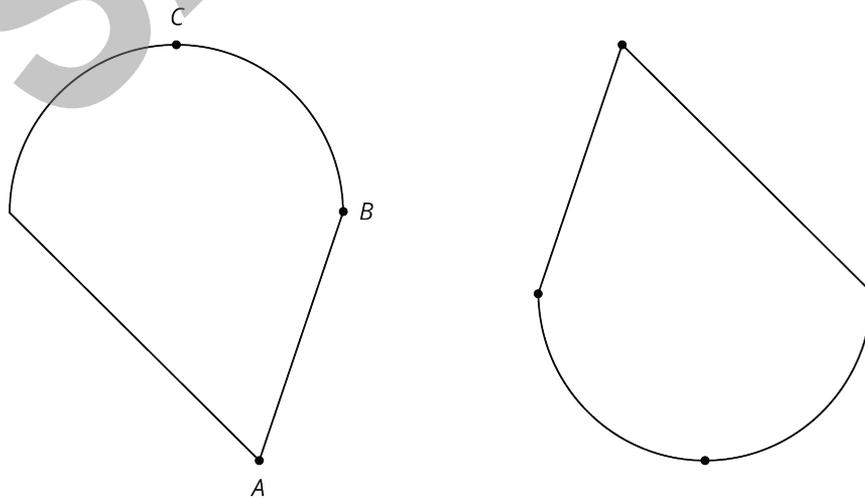
- $q = 100, r = 135, s = 125$
- $q + r + s = 360$
- Answers vary. Sample response: Those three angles together make one full revolution of a circle, or 360 degrees. If a person starts walking in the middle of segment DE toward E , then turns and walks to F , turns again and walks to D , and turns one more time to return to the starting place, they end up facing the same direction they started.

Problem 5

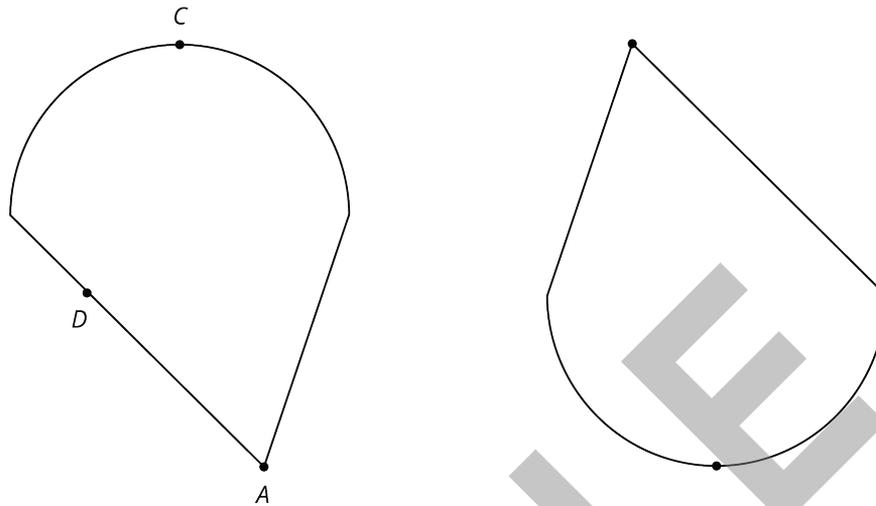
Statement

The two figures are congruent.

- Label the points A' , B' and C' that correspond to A , B , and C in the figure on the right.

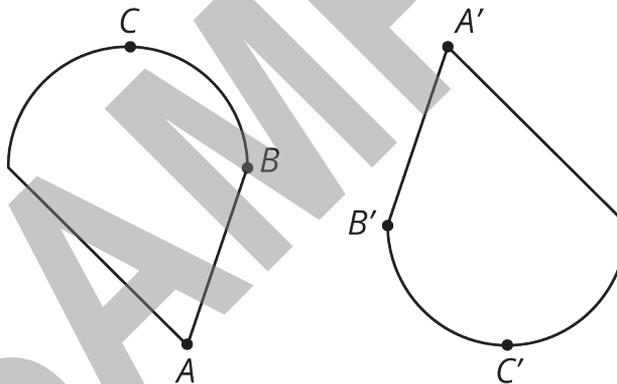


- b. If segment AB measures 2 cm, how long is segment $A'B'$? Explain.
- c. The point D is shown in addition to A and C . How can you find the point D' that corresponds to D ? Explain your reasoning.

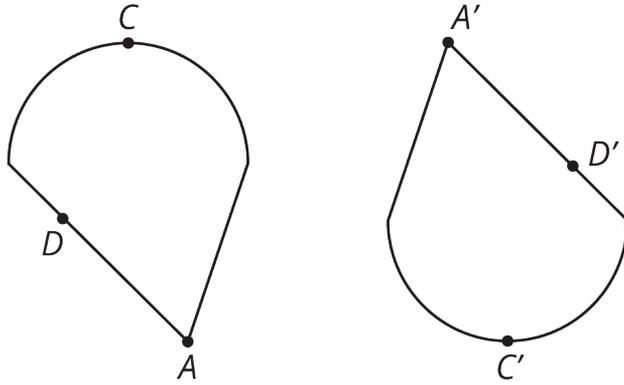


Solution

a.



- b. 2 cm. The shapes are congruent, and the corresponding segments of congruent shapes are congruent.
- c. Because the figures are congruent, the point D' will be on the corresponding side and will be the same distance from C' that D is from C . D can be found by looking for the point on the segment going down and to the right from A' that is the appropriate distance from C' .



(From Unit 1, Lesson 11.)

SAMPLE

Section: Drawing Polygons with Given Conditions

Lesson 15: Building Polygons

Goals

- Explain (in writing) how to use circles to locate the point where the sides of a triangle with known side lengths should meet.
- Recognize that four side lengths do not determine a unique quadrilateral, but that three side lengths can determine a unique triangle.
- Use manipulatives to justify when it is not possible to make a triangle with three given side lengths.
- Use manipulatives to show that there is a minimum and maximum length the third side of a triangle could be, given the other two side lengths.

Learning Targets

- I can show that the 3 side lengths that form a triangle cannot be rearranged to form a different triangle.
- I can show that the 4 side lengths that form a quadrilateral can be rearranged to form different quadrilaterals.
- I can show whether or not 3 side lengths will make a triangle.

Lesson Narrative

The goal of the lesson is to help students see that sometimes lots of different shapes can be made under given constraints about side lengths, and that at other times, with different constraints, it might be that only one shape is possible or that no shape is possible. In previous lessons, students have considered the requirements for two shapes to be congruent using sequences of transformations. The work here helps students expand their understanding of triangles and, in future lessons, how to identify congruent triangles from limited information about their sides and angles. A focus of the lesson is students using repeated reasoning with specific cases to formulate a general rule (MP8).

Throughout the lesson students work with either cardboard strips and metal fasteners or digital tools along with rulers and compasses. The purpose of the different tools is to help students move toward a mental understanding that does not depend on physical objects.

Alignments

Addressing

- 7.G.A.2: Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides,

noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

Instructional Routines

- MLR7: Compare and Connect
- Think Pair Share

Required Materials

Compasses

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty

paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Metal paper fasteners

brass brads

Pre-printed slips, cut from copies of the blackline master

Student Learning Goals

Let's build shapes.

15.1 Where Is Lin?

Warm Up: 5 minutes (there is a digital version of this activity)

The purpose of this warm-up is to remind students that when you have a fixed starting point, all the possible endpoints for a segment of a given length form a circle (centered around the starting point). The context of finding Lin's position in the playground helps make the geometric relationships more concrete for students. Since there are many possible distances between Lin and the swings (but not infinitely many), this activity serves as an introduction to formalizing rules about what lengths can and cannot be used to form a triangle.

Monitor for students who come up with different locations for Lin, as well as students who recognize that there are many possible locations, to share during the whole-class discussion.

Teacher Notes for IM 6–8 Math Accelerated

If students have yet to work with compasses this year, take time during the activity synthesis to demonstrate their use in preparation for future activities. Make sure students understand that a compass is useful not just for drawing circles, but also for transferring lengths in general.

Addressing

- 7.G.A.2

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. If necessary, remind students of the directions north, south, east, and west and their relative position on a map. Provide access to geometry toolkits. Give students 2 minutes of quiet work time, followed by a partner and whole-class discussion.

Students with access to the digital materials can explore the applet.

Support for Students with Disabilities

Representation: Develop Language and Symbols. Display or provide charts with symbols and meanings. Display a chart of a compass showing the directions north, south, east and west.

Supports accessibility for: Conceptual processing; Memory

Anticipated Misconceptions

Some students might assume that the swings, the slide, and Lin are all on a straight line, and that she must be 8 meters away. Ask these students if the problem tells us which direction Lin is from the slide.

Some students may confuse the type of compass discussed in the Launch and the type of compass discussed in the Activity Synthesis. Consider displaying a sample object or image of each of them and explain that the same name refers to two different tools.

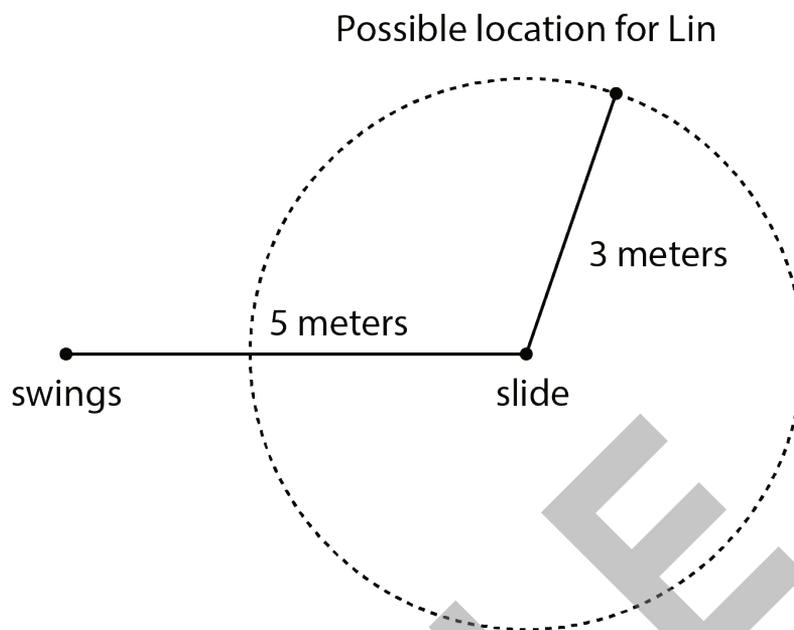
Student Task Statement

At a park, the slide is 5 meters east of the swings. Lin is standing 3 meters away from the slide.

1. Draw a diagram of the situation including a place where Lin could be.
2. How far away from the swings is Lin in your diagram?
3. Where are some other places Lin could be?

Student Response

1. Answers vary. See diagram.



2. There is no way to know for sure, because we don't know what direction Lin is from the slide. She could be anywhere between 2 and 8 meters away from the swings.
3. Lin could be at any position along a circle that is centered on the slide and has a radius of 3 meters.

Activity Synthesis

First, have students compare answers and share their reasoning with a partner until they reach an agreement.

Next, ask selected students to share their diagrams of where Lin is located. Discuss the following questions with the whole class:

- “Do we know for sure where Lin is?” (No, because we don't know what direction she is from the swings.)
- “What shape is made by all the possible locations where Lin could be?” (a circle)
- “What is the closest Lin could be to the swings?” (2 m)
- “What is the farthest Lin could be away from the swings?” (8 m)

Consider using the applet at <https://ggbm.at/qkHk6Tpj> to show all the locations where Lin could be. Based on their work with drawing circles in a previous unit, some students may suggest that a compass could be used to draw all the possible locations where Lin could be. Consider having a student demonstrate how this could be done. If not mentioned by students, it is *not* necessary for the teacher to bring it up at this point.

15.2 Building Diego's and Jada's Shapes

10 minutes (there is a digital version of this activity)

The purpose of this activity is to reinforce that some conditions define a unique polygon while others do not. Students build polygons given only a description of their side lengths. They articulate that this is not enough information to guarantee that a pair of quadrilaterals are identical copies. On the other hand, triangles have a special property that three specific side lengths result in a unique triangle. Students should notice that their recreation of Jada's triangle is rigid; the side lengths and angles are all fixed.

Monitor for students who try putting the side lengths together in different orders to build different polygons and invite them to share during the whole-class discussion.

Addressing

- 7.G.A.2

Instructional Routines

- MLR7: Compare and Connect

Launch

Arrange students in groups of 2. Remind students that a quadrilateral is a four-sided polygon. If necessary, demonstrate how to use the fasteners to connect the strips.

Distribute one set of strips and fasteners to each group. Provide access to geometry toolkit, including rulers and protractors. Encourage students to think about whether there are different shapes that would fulfill the given conditions. Give students 5–6 minutes of quiet work time followed by a whole-class discussion.

For classes using the digital materials, there is an applet for students to use to build polygons with the given side lengths. If necessary, demonstrate how to create a vertex by overlapping the endpoints of two segments. It may work best for positioning each segment to put the green endpoint in place first and then adjust the yellow endpoint as desired.

Support for Students with Disabilities

Representation: Internalize Comprehension. Begin the activity with concrete or familiar contexts. Remind students that a polygon is a closed figure with straight sides. Demonstrate how to use fasteners to connect the slips.

Supports accessibility for: Conceptual processing; Memory

Anticipated Misconceptions

Students may think that their triangle is different from Jada's because hers is "upside down." Ask the student to turn their triangle around and ask them if it is now a different triangle. While there is a

good debate to be had if they continue to insist they are different, let the students know that, for this unit, we will consider shapes that have been turned or flipped or moved as identical copies and thus “not different.”

Student Task Statement

1. Diego built a quadrilateral using side lengths of 4 in, 5 in, 6 in, and 9 in.
 - a. Build such a shape.
 - b. Is your shape an identical copy of Diego’s shape? Explain your reasoning.
2. Jada built a triangle using side lengths of 4 in, 5 in, and 8 in.
 - a. Build such a shape.
 - b. Is your shape an identical copy of Jada’s shape? Explain your reasoning.

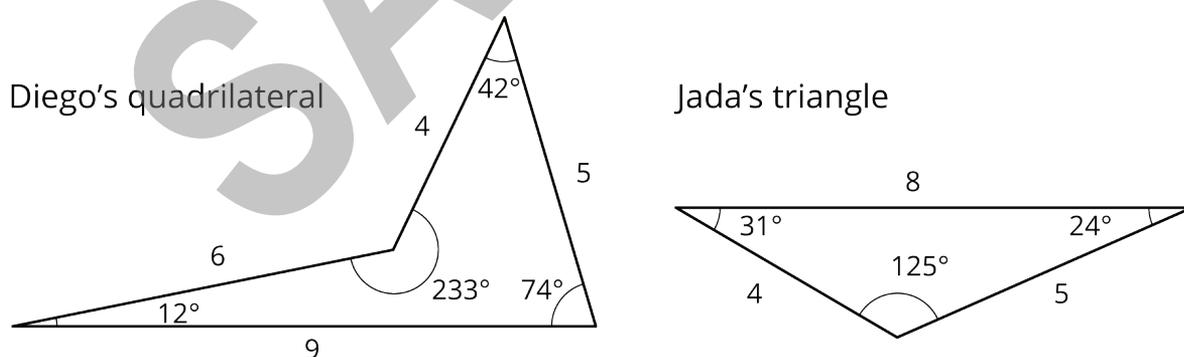
Student Response

1. No, my quadrilateral is probably not an identical copy of Diego’s quadrilateral, because it is floppy; there are lots of different angles I can use to make a quadrilateral with these side lengths.
2. Yes, my triangle should be an identical copy of Jada’s triangle, because it is not floppy; there is no way to change the angles to make a different triangle with these side lengths.

Activity Synthesis

Select previously identified students to share their constructions and explanations. Display each student’s example for all to see.

If desired, reveal Diego and Jada’s shapes and display for all to see along side students’ work.



Ask students:

- “Is this what you thought Jada and Diego’s shapes looked like?”
- “Which shape did you make an identical copy of?” (Jada’s triangle.)
- “Why did you not make an identical copy of Diego’s shape?” (Because you can make quadrilaterals with the same side lengths but different angle measure.)

Consider explaining to students how this finding is applied in construction projects. For stability, the internal structures of many buildings (and bridges) will include triangles. Rectangles or other polygons with more than three sides often include triangular supports on the inside, to make the construction more rigid and less floppy.

Support for English Language Learners

Speaking: MLR7 Compare and Connect. Use this routine to support whole-class discussion. Ask students to consider what is the same and what is different about their quadrilateral as compared to Diego's. Draw students' attention to the association between the order of side lengths in each quadrilateral and the angle measures in each shape. In this discussion, demonstrate the language used to make sense of the conditions required to make identical figures. Talking about and comparing the quadrilaterals will help draw students' attention to the conditions that define a unique polygon.

Design Principle(s): Maximize meta-awareness

15.3 Swinging the Sides Around

15 minutes (there is a digital version of this activity)

The purpose of this activity is to relate the process for *building* a triangle given 3 side lengths (using cardboard strips and metal fasteners) to the process for *drawing* a triangle given 3 side lengths (using a compass). Students use the cardboard strips as an informal compass for drawing all the possible locations where the given segments could end. They are reminded of their work with circles in a previous unit: that a circle is the set of all the points that are equally distant from a center point and that a compass is a useful tool, not just for drawing circles, but also for transferring lengths in general. This prepares them for using a compass to draw triangles in future lessons.

In this activity, students also consider what their drawing would look like if the two shorter sides were too short to make a triangle with the third given side length.

Left-handed students may find it easier to start with drawing the 3-inch circle on the left side of the 4-inch segment.

Teacher Notes for IM 6–8 Math Accelerated

In IM 6–8 Math Accelerated, the unit on circles is in the previous year instead of the previous unit as stated in the narrative. Remind students of their work with circles and that a circle is the set of all the points that are equally distant from a center point.

During the activity synthesis, use the term “congruent” instead of “identical copies”. After asking the question “What is the longest the third side could have been? And the shortest?”, if students do not bring up what happens when the third side length is less than 1 inch, invite them to consider this

case with their partner. If possible, demonstrate for students using the applet and the trace feature or, if time allows, have a student demonstrate or share a drawing they have made for the situation.

Addressing

- 7.G.A.2

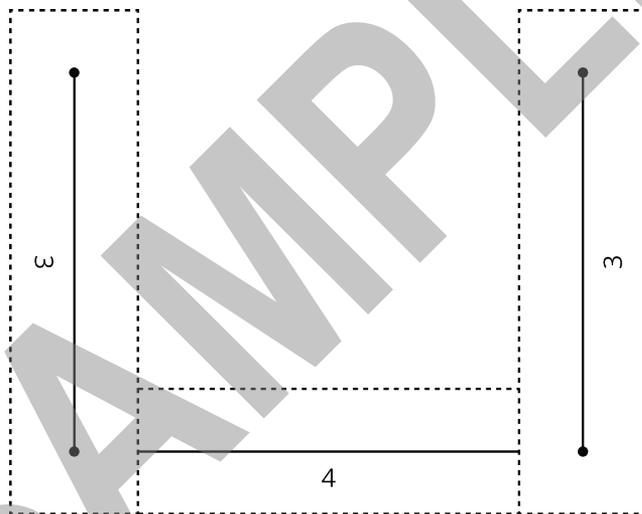
Instructional Routines

- MLR7: Compare and Connect

Launch

Arrange students in groups of 2. Distribute one copy of the blackline master to each group. Make sure each group has one complete set of strips and fasteners from the previous activity. Provide access to geometry toolkits and compasses.

Tell students to take one 4-inch piece and two 3-inch pieces and connect them so that the 4-inch piece is in between the 3-inch pieces as seen in the image. If necessary, display the image for all to see. Students should not connect the 3-inch pieces to each other.



Explain to students that the sheet distributed to them is the 4-inch segment that is mentioned in the task statement and they will be drawing on that sheet.



If using the digital version of the activity, students will be using the Trace feature to see the path of the point. It is accessed by right-clicking on the point.

Support for Students with Disabilities

Action and Expression: Internalize Executive Functions. Begin with a small-group or whole-class demonstration of how to use the cardboard strips as an informal compass for drawing all the possible locations where the given segments could end.

Supports accessibility for: Memory; Conceptual processing

Student Task Statement

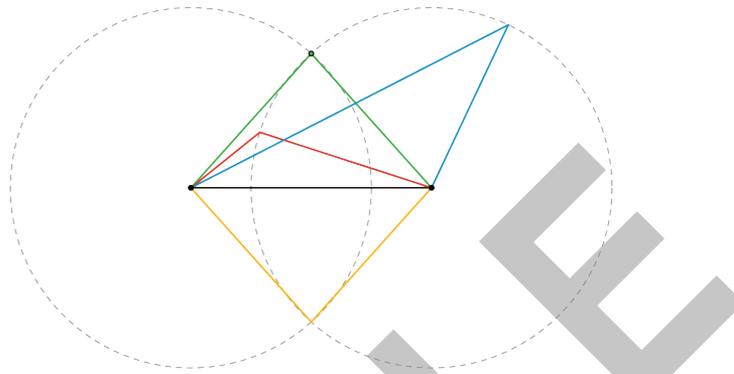
We'll explore a method for drawing a triangle that has three specific side lengths. Your teacher will give you a piece of paper showing a 4-inch segment as well as some instructions for which strips to use and how to connect them.

1. Follow these instructions to mark the possible endpoints of one side:
 - a. Put your 4-inch strip directly on top of the 4-inch segment on the piece of paper. Hold it in place.
 - b. For now, ignore the 3-inch strip on the left side. Rotate it so that it is out of the way.
 - c. In the 3-inch strip on the *right* side, put the tip of your pencil in the hole on the end that is not connected to anything. Use the pencil to move the strip around its hinge, drawing all the places where a 3-inch side could end.
 - d. Remove the connected strips from your paper.
2. What shape have you drawn while moving the 3-inch strip around? Why? Which tool in your geometry toolkit can do something similar?
3. Use your drawing to create two unique triangles, each with a base of length 4 inches and a side of length 3 inches. Use a different color to draw each triangle.
4. Reposition the strips on the paper so that the 4-inch strip is on top of the 4-inch segment again. In the 3-inch strip on the *left* side, put the tip of your pencil in the hole on the end that is not connected to anything. Use the pencil to move the strip around its hinge, drawing all the places where another 3-inch side could end.
5. Using a third color, draw a point where the two marks intersect. Using this third color, draw a triangle with side lengths of 4 inches, 3 inches, and 3 inches.

Student Response

1. Students draw the circle shown on the right side of the diagram.
2. A circle is all the points that are the same distance away from a center point. A compass can be used to draw this.

3. Answers vary. Possible response: Students draw the small red triangle and the large blue triangle shown in the diagram.
4. Students draw the circle shown on the left side of the diagram.
5. Answers vary. Possible response: Students draw the green or yellow triangle shown in the diagram.



Activity Synthesis

Display a 4-inch strip connected to two 3-inch strips, positioned parallel to each other as pictured in the Launch. To help students connect the process of building with cardboard strips to drawing on paper, ask questions like:

- “If you want to *build* a triangle with these side lengths, how do you know at what angle to position the cardboard strips?” (Turn the sides until their unattached endpoints are touching.)
- “If you want to *draw* a triangle with these side length, how can you know at what angle to draw the sides?” (Find the point where both circles intersect.)
- “We have seen with the cardboard strips that an unknown angle works like a hinge. How is that represented in your drawing?” (with a circle centered on the endpoint of one segment and a radius the length of the other segment)

Select students to share their drawings with the class. To reinforce the patterns that students noticed in the previous activity, consider asking questions like these:

- “How many different triangles could we draw when we had only traced a circle on one side? Why?” (Lots of different triangles, because we were only using two of the given side lengths.)
- “What is the longest the third side could have been? And the shortest?” (Less the 7 inches; More than 1 inch)
- “How many different triangles could we draw once we had traced a circle on each side?” (It looked like there were 2 different triangles, but they are identical copies, so there’s really only 1 unique triangle.)

Support for English Language Learners

Speaking, Listening, Representing: MLR7 Compare and Connect. Use this routine to help students compare their processes for building and drawing their triangles. Ask students “What is the same and what is different?” about the approaches. Draw students’ attention to the connections between building and drawing such as: opening the hinge between the cardboard strips and drawing the circle using the compass. These exchanges strengthen students’ language use and reasoning from concrete to representational approaches.

Design Principle(s): Maximize meta-awareness

Lesson Synthesis

Highlight some important takeaways from today’s lesson:

- “How is building a triangle with three given side lengths different from building a quadrilateral with four given side lengths?” (The triangle must be a specific one, but the quadrilateral might be a lot of different things by changing the angles.)
- “When you are given side lengths and asked to draw a triangle, how can you get started?” (Hold one length fixed and swing the other around in a circle.)
- “If you draw one side of the triangle with circles (of the correct radius for the other two side lengths) on each end, what does it look like when it is impossible to make a triangle?” (The two circles do not intersect, or they intersect at a point on the first line segment.)
- “If you draw one side of the triangle with circles on each end, and the circles do intersect, they may intersect twice. Why do we say there’s only one possible triangle instead of two?” (The two triangles are congruent.)

15.4 Finishing Elena’s Triangles

Cool Down: 5 minutes

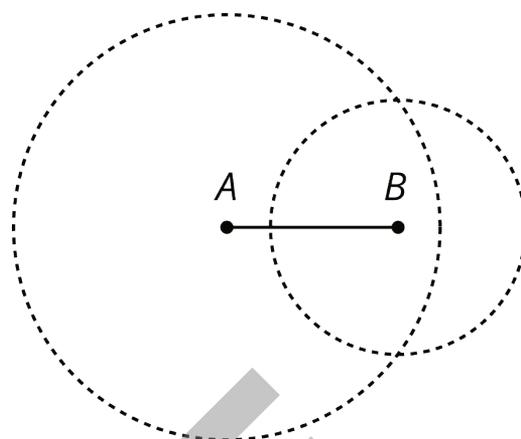
Addressing

- 7.G.A.2

Student Task Statement

Elena is trying to draw a triangle with side lengths 4 inches, 3 inches, and 5 inches.

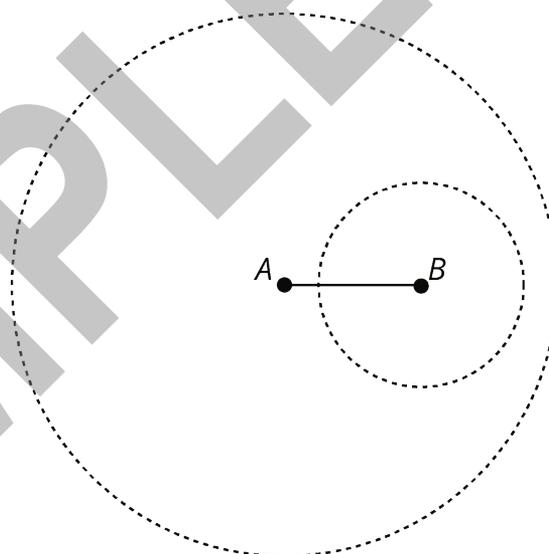
- She uses her ruler to draw a 4 inch line segment AB .
- She uses her compass to draw a circle around point B with radius 3 inches
- She draws another circle, around point A with radius 5 inches.



1. What should Elena do next? Explain and show how she can finish drawing the triangle.

Now Elena is trying to draw a triangle with side lengths 4 inches, 3 inches, and 8 inches.

- She uses her ruler to draw a 4 inch line segment AB .
- She uses her compass to draw a circle around point B with radius 3 inches.
- She draws another circle, around point A with radius 8 inches.



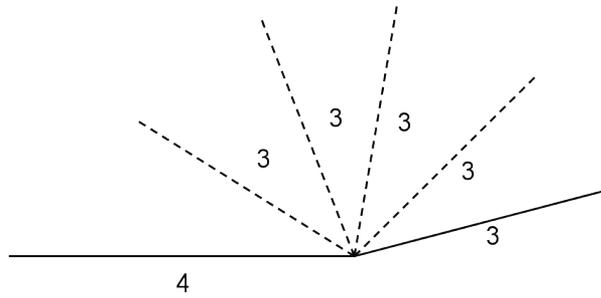
2. Explain what Elena's drawing means.

Student Response

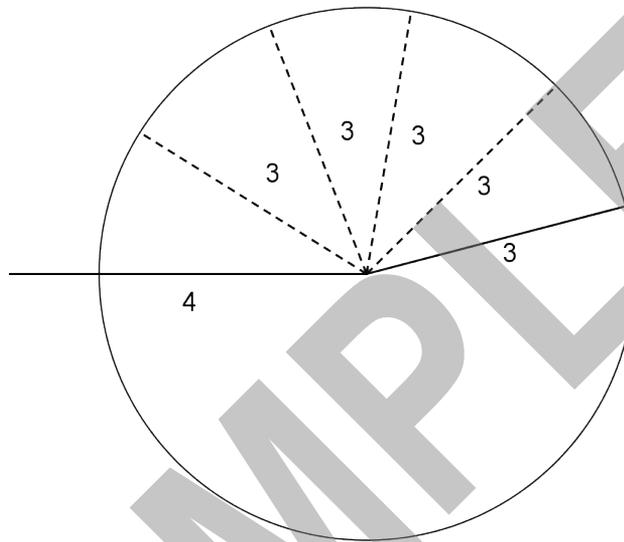
1. Elena should put a point where the two circles intersect and draw line segments connecting that point to points A and B to finish her triangle.
2. Elena's drawing means that there is no way to draw a triangle with these side lengths. The circles do not intersect, because the side lengths of 3 inches and 4 inches are too short to make a triangle with the third side of 8 inches.

Student Lesson Summary

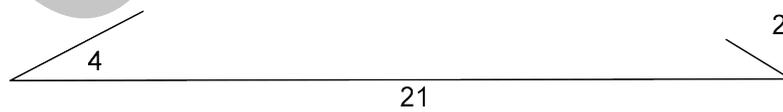
If we want to build a polygon with two given side lengths that share a vertex, we can think of them as being connected by a hinge that can be opened or closed:



All of the possible positions of the endpoint of the moving side form a circle:

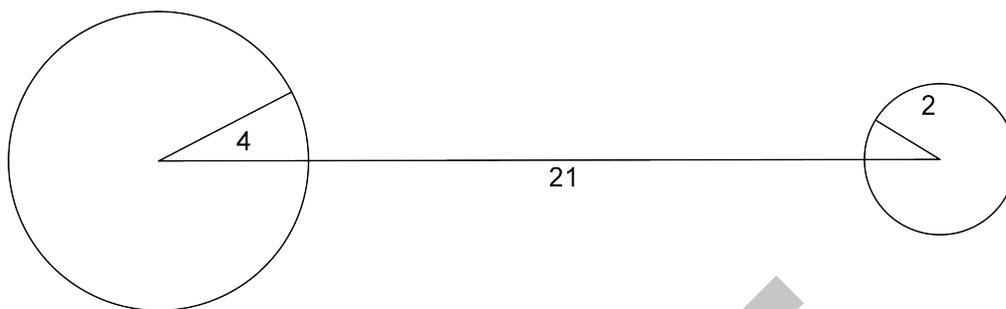


You may have noticed that sometimes it is not possible to build a polygon given a set of lengths. For example, if we have one really, really long segment and a bunch of short segments, we may not be able to connect them all up. Here's what happens if you try to make a triangle with side lengths 21, 4, and 2:



The short sides don't seem like they can meet up because they are too far away from each other.

If we draw circles of radius 4 and 2 on the endpoints of the side of length 21 to represent positions for the shorter sides, we can see that there are no places for the short sides that would allow them to meet up and form a triangle.



In general, the longest side length must be less than the sum of the other two side lengths. If not, we can't make a triangle!

If we *can* make a triangle with three given side lengths, it turns out that the measures of the corresponding angles will *always* be the same. For example, if two triangles have side lengths 3, 4, and 5, they will have the same corresponding angle measures.

Lesson 15 Practice Problems

Problem 1

Statement

A rectangle has side lengths of 6 units and 3 units. Could you make a quadrilateral that is not identical using the same four side lengths? If so, describe it.

Solution

Yes, you could make a parallelogram or a kite using the side lengths 3, 3, 6, and 6.

Problem 2

Statement

Come up with an example of three side lengths that can not possibly make a triangle, and explain how you know.

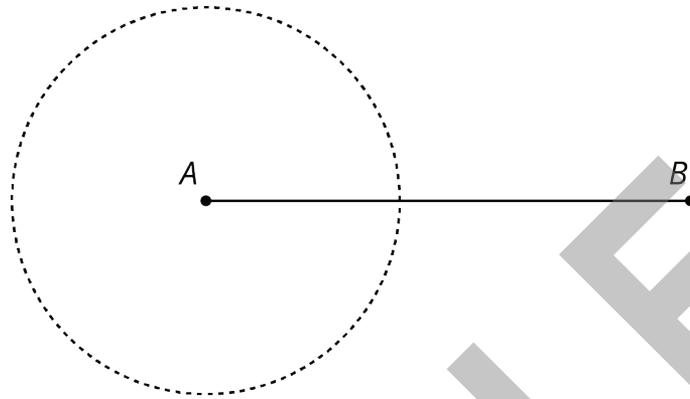
Solution

Answers vary. Sample response: the lengths 1 foot, 1 inch, and 1 inch can not possibly make a triangle, because if you attach the 1 inch lengths to either end of the 1 foot length, the 1 inch lengths are too short to connect at their other ends.

Problem 3

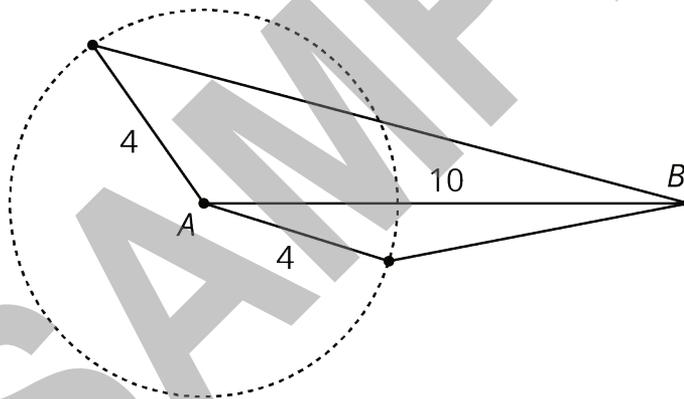
Statement

In the diagram, the length of segment AB is 10 units and the radius of the circle centered at A is 4 units. Use this to create two unique triangles, each with a side of length 10 and a side of length 4. Label the sides that have length 10 and 4.



Solution

Answers vary. Possible response:



Problem 4

Statement

Select **all** the sets of three side lengths that will make a triangle.

- A. 3, 4, 8
- B. 7, 6, 12
- C. 5, 11, 13
- D. 4, 6, 12
- E. 4, 6, 10

Solution

["B", "C"]

Problem 5

Statement

Based on signal strength, a person knows their lost phone is exactly 47 feet from the nearest cell tower. The person is currently standing 23 feet from the same cell tower. What is the closest the phone could be to the person? What is the furthest their phone could be from them?

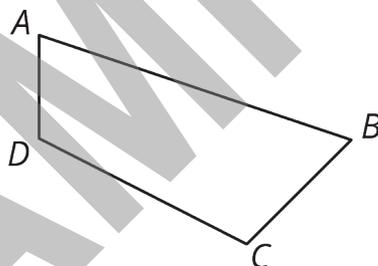
Solution

24 feet, 70 feet

Problem 6

Statement

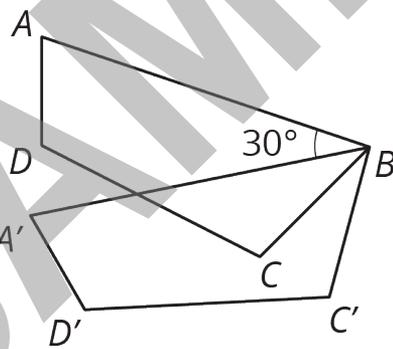
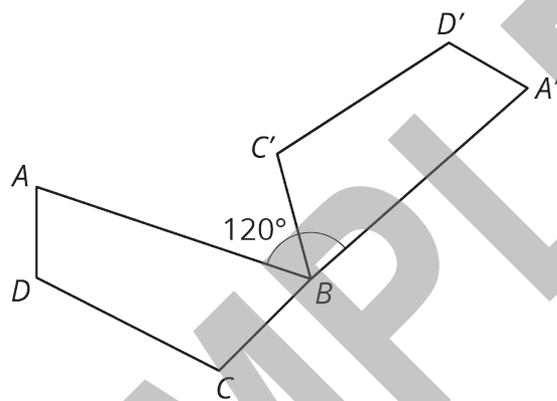
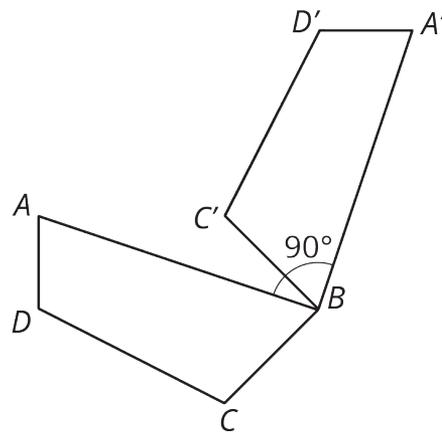
Here is quadrilateral $ABCD$.



Draw the image of quadrilateral $ABCD$ after each rotation using B as center.

- 90 degrees clockwise
- 120 degrees clockwise
- 30 degrees counterclockwise

Solution



(From Unit 1, Lesson 2.)

Lesson 16: Triangles with 3 Common Measures

Goals

- Describe, compare, and contrast (orally and in writing) triangles that share three common measures of angles or sides.
- Justify (orally and using other representations) whether triangles are identical copies or are “different” triangles.
- Recognize that examining which side lengths and angle measures are adjacent can help determine whether triangles are identical copies.

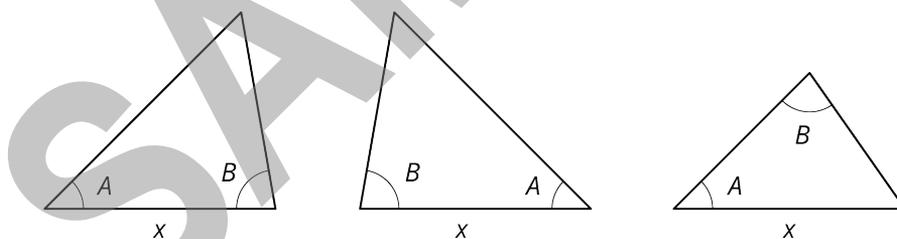
Learning Targets

- I understand that changing which sides and angles are next to each other can make different triangles.

Lesson Narrative

In this lesson, students examine sets of triangles in which all the triangles share 3 common measures of angles or sides. Students learn to recognize when triangles are “identical copies” that are oriented differently on the page, and when they are different triangles (meaning triangles that are not identical copies). This prepares them for trying to draw more than one triangle given 3 measures in the next lesson.

For example, suppose a triangle has angles that measure A and B and a side length that measures x . Here are 3 triangles that have these measures:



This example shows 2 “different triangles” (triangles that are not identical copies). The first two triangles are identical copies, so they are the same, but the third is not, so it is different than the other two.

Students see that the configuration of which sides and angles are adjacent to each other can help them decide whether triangles are identical copies or different triangles (not identical copies). In the example, the first two figures have angles A and B adjacent to side x . However, in the third figure angle B is no longer adjacent to side x . Here students can see that a good way to try to make a different triangle with the same 3 measures is to change which sides and angles are adjacent.

Students do not need to memorize how many different kinds of triangles are possible given different combinations of angles and sides, and they do not need to know criteria such as angle-side-angle for determining if two triangles are identical copies..

Teacher Notes for IM 6–8 Math Accelerated

In IM 6–8 Math Accelerated, this lesson occurs after students learn what it means for two shapes to be congruent and that the sum of the interior angles of a triangle is 180 degrees. Encourage students to use the language they learned about transformations to more precisely state why two shapes are or are not congruent instead of using the term “identical copies”. Where applicable, also modify the suggested questions to use with students, such as those in the lesson synthesis, to use “congruent” instead of “identical copies.”

For example, when discussing the warm-up it is appropriate for students to use words like “reflection” or “rotation” when describing how they noticed the first set of triangles are all congruent. Similarly, students will formally study scaled copies in a later unit, so they are not likely to use the language of scaled copies when discussing the second set of triangles in the warm-up.

Alignments

Addressing

- 7.G.A.2: Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

Instructional Routines

- MLR2: Collect and Display
- MLR8: Discussion Supports
- Think Pair Share

Required Materials

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Student Learning Goals

Let's contrast triangles.

16.1 3 Sides; 3 Angles

Warm Up: 10 minutes

The purpose of this warm-up is to begin looking at the different triangles that can be drawn when three measures are specified. The first set of triangles in this activity all share the same 3 side lengths. The second set of triangles all share the same 3 angle measures. Later in this lesson, students will look at sets of triangles that share some combination of side lengths and angle measures.

Addressing

- 7.G.A.2

Launch

Provide access to geometry toolkits. Give students 1 minute of quiet think time, followed by a whole-class discussion.

Anticipated Misconceptions

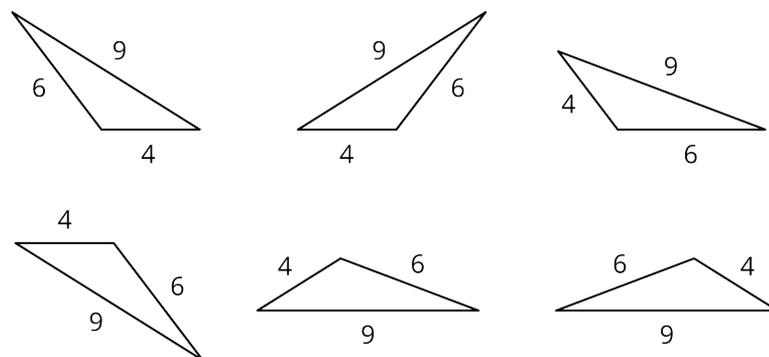
Some students may say that all the triangles in the second set are “the same shape.” This statement can result from two very different misconceptions. Listen to the students’ reasoning and explain as needed:

1. Just because they are all in the same category “triangles” doesn’t mean they are all the same shape. If we can take two shapes and position one exactly on top of the other, so all the sides and corners line up, then they are identical copies.
2. These triangles are scaled copies of each other, but that does not make them “the same” because their side lengths are still different. Only scaled copies made using a scale factor of 1 are identical copies.

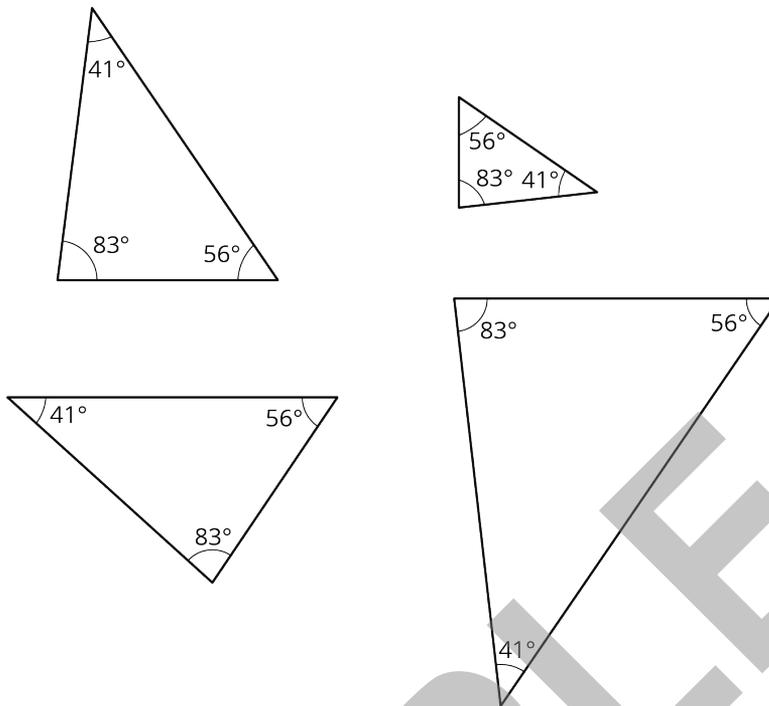
Student Task Statement

Examine each set of triangles. What do you notice? What is the same about the triangles in the set? What is different?

Set 1:



Set 2:



Student Response

1. All of the side lengths and angles are the same size. These triangles are identical copies. The triangles face different directions.
2. These triangles all have the same angles, but different side lengths. They could be scaled copies that are oriented differently.

Activity Synthesis

Invite students to share things they notice, things that are the same and things that are different about the triangles. Record and display these ideas for all to see.

If these discussion points do not come up in students' explanations make them explicit:

In the first set:

- All the triangles are identical copies, just in different orientations.
- They have the same 3 side lengths.
- They have the same 3 angle measures (can be checked with tracing paper or a protractor).

In the second set:

- The triangles are not identical copies.
 - Note: Students may recognize that these triangles are scaled copies of each other, since they have the same angle measures. However, this is the first time students have seen scaled copies in different orientations, and it is not essential to this lesson that students recognize that these triangles are scaled copies.

- They have the same 3 angle measures.
- They have different side lengths (can be checked with tracing paper or a ruler).

The goal is to make sure students understand that the second set has 3 different triangles (because they are different sizes) and the first set really only shows 1 triangle in many different orientations. Tracing paper may be helpful to convince students of this.

16.2 2 Sides and 1 Angle

15 minutes

In this activity, students examine different orientations of triangles that all share 2 sides lengths and one angle measure. They recognize that some of these triangles are identical copies and others are different triangles (not identical copies).

In the coming lessons, students are asked to draw their own triangles. On their own, students often have trouble thinking about triangles where the three given conditions are not included adjacent to one another. For example, when given two sides and an angle, many students will immediately think of putting the given angle between the two sides, but struggle with visualizing putting the angle anywhere else. This task is important for helping students view this as a viable option.

Addressing

- 7.G.A.2

Instructional Routines

- MLR2: Collect and Display
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time followed by time to discuss their explanations with a partner. Follow with a whole-class discussion. Provide access to geometry toolkits.

Support for Students with Disabilities

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. Provide students with a printed copy of the triangles for them to cut out and rearrange to determine the number of different triangles.

Supports accessibility for: Conceptual processing

Support for English Language Learners

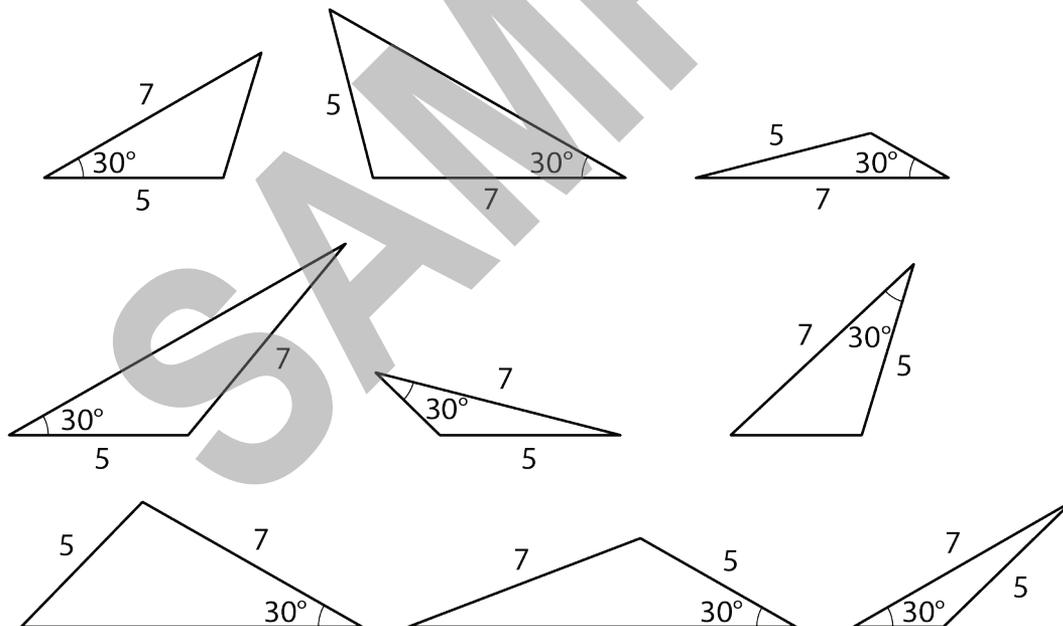
Conversing: MLR2 Collect and Display. As students discuss their explanations with a partner, listen for and collect vocabulary, gestures, and diagrams students use to identify and describe the similarities and differences between them. Capture student language that reflects a variety of ways to describe the differences between triangles and the relative position of sides and angles. Write the students' words on a visual display and update it throughout the remainder of the lesson. Remind students to borrow language from the display as needed. This will help students read and use mathematical language during their paired and whole-class discussions.
Design Principle(s): Optimize output (for explanation); Maximize meta-awareness.

Anticipated Misconceptions

Some students may say that there are 9 different triangles, because they do not recognize that some of them are identical copies oriented differently. Prompt them to use tracing paper to compare the triangles.

Student Task Statement

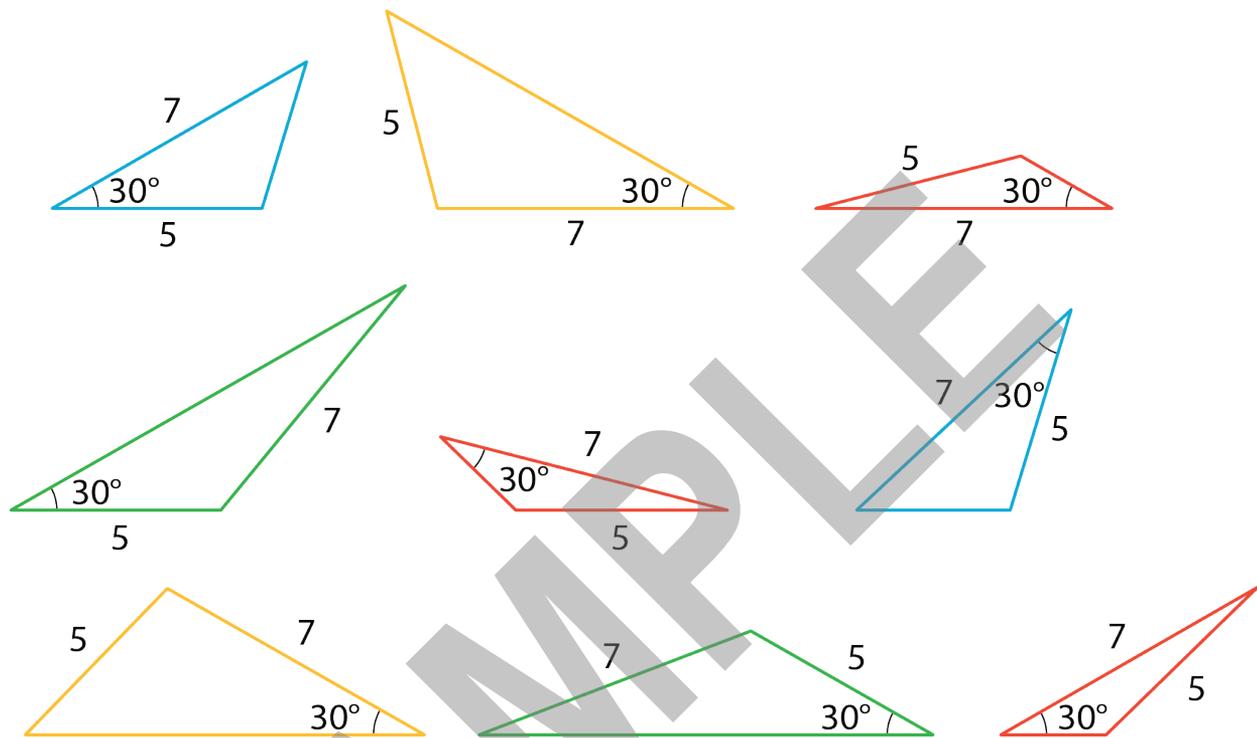
Examine this set of triangles.



1. What is the same about the triangles in the set? What is different?
2. How many different triangles are there? Explain or show your reasoning.

Student Response

1. All of the triangles have two sides that are the same length and one angle that is the same measure. The triangles are oriented differently and the two sides and one angle are in a different order.
2. There are 4 different triangles.



Sample Explanation:

- The triangles marked in blue have the common measurements in the order 7 cm, 30° , 5 cm.
- The triangles marked in green have the common measurements in the order 30° , 5 cm, 7 cm.
- The triangles marked in both yellow and red have the common measurements in the order 30° , 7 cm, 5 cm, but the yellow triangles are larger and the red triangles are smaller.

Activity Synthesis

Select students to share the similarities and differences between the triangles in the set.

Trace a few of the triangles from the set and show how you can turn, flip, or move some of them to line up while others cannot be lined up. Ask students what this means about all the triangles in the set (they are not all identical to each other). Explain that, "While there are certainly times when the position of a triangle is important ('I wouldn't want my roof upside down!'), for this unit in geometry, we will consider shapes the same if they are identical copies."

To highlight the differences among the triangles, ask students:

- “Is there only one possible triangle that could be created from the given conditions?” (No, there were 4.)
- “How would you explain what is different about these four triangles?” (Some have the 30° angle between the two sides of known length and others have the 30° angle next to the side of unknown length.)

Explain to students that it seems the order in which the conditions are included in the triangle (for example, is the angle between the two sides or not?) matters in creating different triangles. Emphasize that the three required pieces (2 sides and 1 angle) do not have to all be put next to one another. When they are asked to draw triangles with three or more conditions, they should consider the way in which the conditions are arranged in their drawing. For example, think about whether the given angle must go between the two sides or not.

16.3 2 Angles and 1 Side

10 minutes

This activity is similar to what students did in the previous activity; however, here the conditions given are 2 angles and 1 side.

Addressing

- 7.G.A.2

Instructional Routines

- MLR8: Discussion Supports

Launch

Keep students in the same groups. Tell students that this activity is similar to the previous one, and they should pay close attention to what they find different here. Provide access to geometry toolkits. Give students 2–3 minutes of quiet work time followed by time to discuss their explanations with a partner. Follow with a whole-class discussion.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, “Both ____ and ____ are alike because . . .”, “____ and ____ are different because . . .”

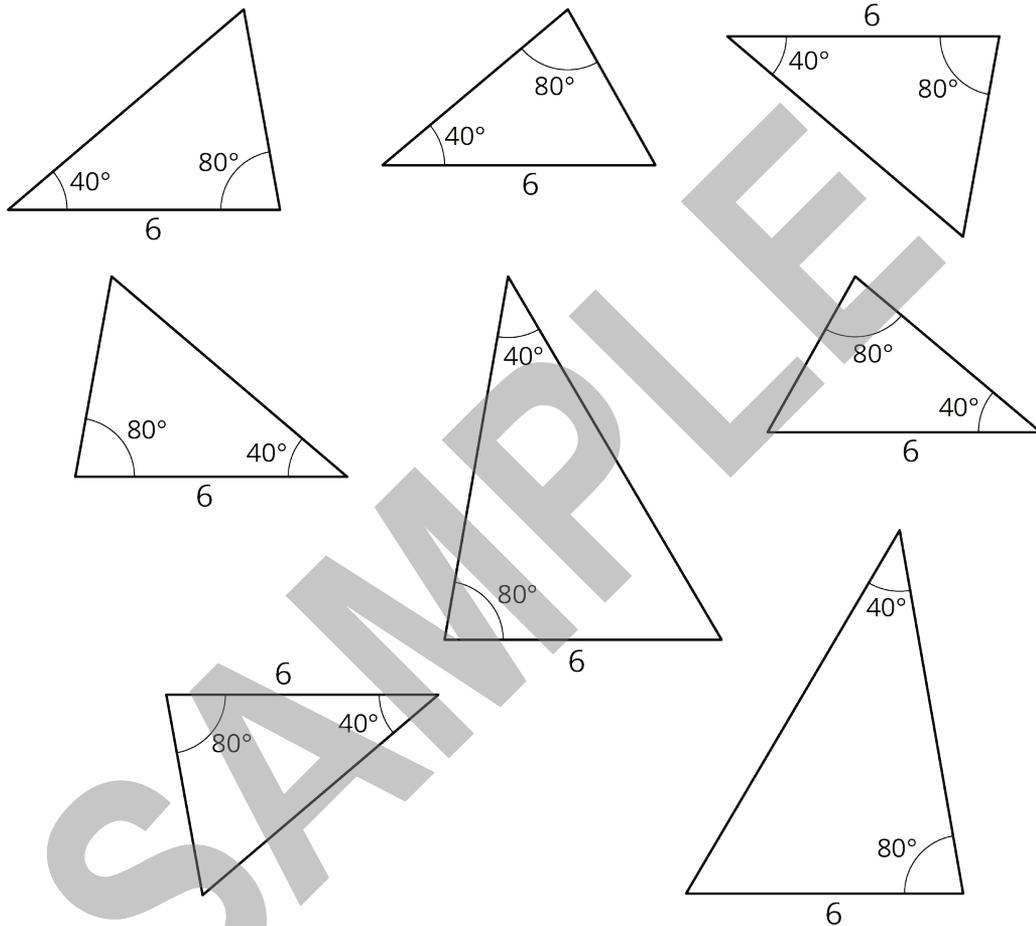
Supports accessibility for: Language; Social-emotional skills

Anticipated Misconceptions

Some students may say there are only 2 different triangles in this set, because they do not notice the slight size difference between the smaller two groups of triangles. Prompt them to look at where the 80° angle is located in comparison to the 6 cm side.

Student Task Statement

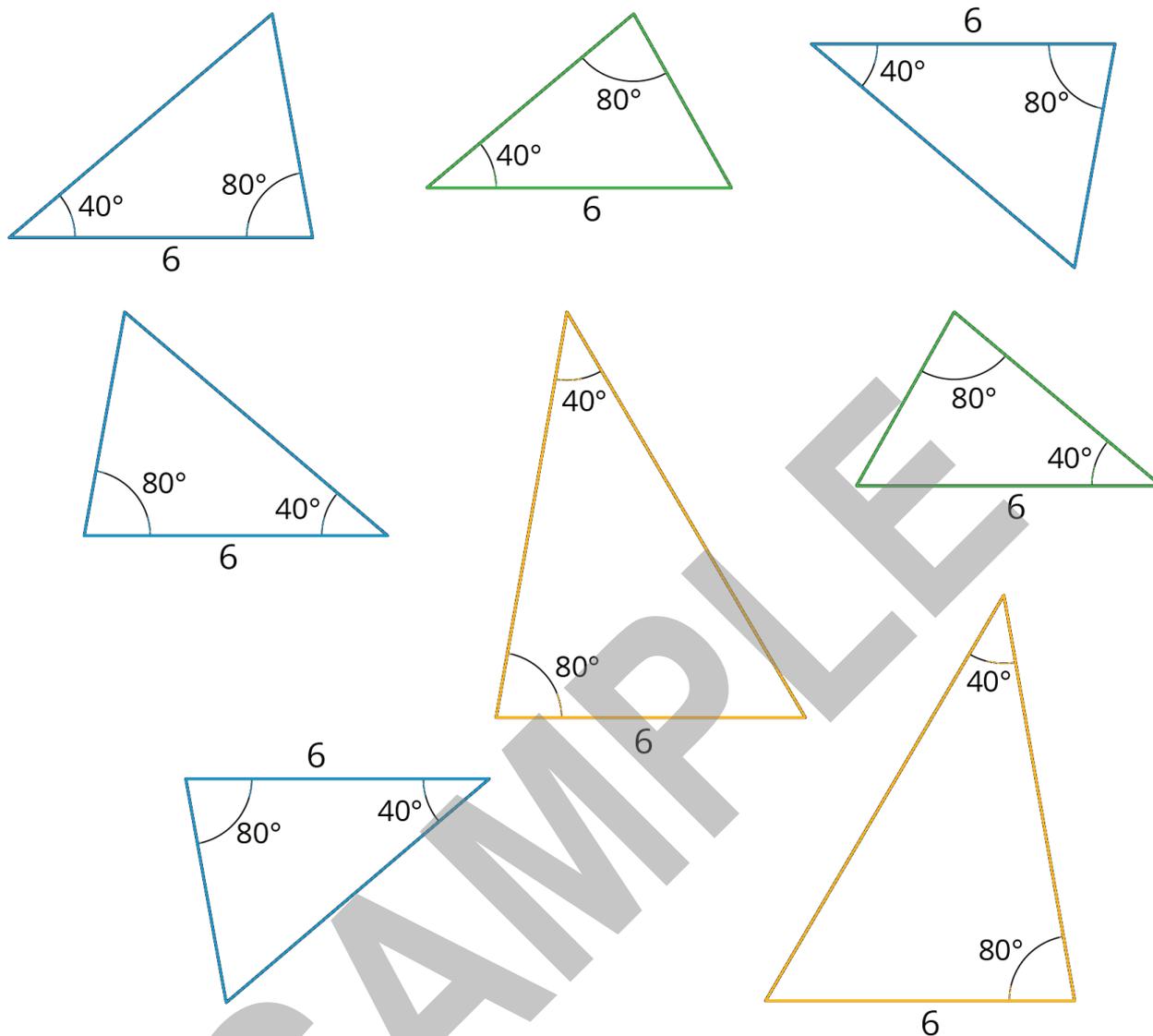
Examine this set of triangles.



1. What is the same about the triangles in the set? What is different?
2. How many different triangles are there? Explain or show your reasoning.

Student Response

1. All of the triangles have one side that is the same length and two angles that have the same measure. The triangles are oriented differently and the one side and two angles are in a different order.
2. There are 3 different triangles.



Sample explanation:

- The triangles marked in blue have the common measurements in the order 40° , 6 cm, 80° .
- The triangles marked in yellow have the common measurements in the order 40° , 80° , 6 cm.
- The triangles marked in green have the common measurements in the order 80° , 40° , 6 cm.

Activity Synthesis

Ask a few students to share how many different triangles they think are in this set. Select students to share the similarities and differences between triangles in the set. If necessary, trace a few triangles from the set and show how you can turn, flip, or move some of them to line up while others cannot be lined up. Ask students what this means about all the triangles in the set (they are not all identical to each other).

To highlight the differences among the triangles ask students:

- “What differences do you see between the triangles in this activity and the triangles in the previous activity?” (The given conditions here were 2 angles and 1 side, the previous activity was 2 sides and 1 angle.)
- “What similarities do you see between the triangles in this activity and the triangles in the previous activity?” (These triangles have all the same conditions but in a different order, and they made different triangles as was seen in the previous activity.)

If time permits, consider asking students to use a protractor to measure the unlabeled angle from each of the three different triangles. Discuss what they notice about the third angle. (It’s the same size in every triangle.)

Explain to students that here we see another example of different triangles that can be made using the same conditions (2 angles and 1 side) in different orders (side between the two angles, side next to the 40 degree angle and side next to the 80 degree angle). Tell them that in upcoming lessons we will continue to investigate what they noticed here with the addition of drawing the different triangles.

Support for English Language Learners

Speaking: MLR8 Discussion Supports. Use this routine to amplify mathematical uses of language to communicate about similar triangles. As students share the similarities and differences they noticed about the triangles, invite other students to press for details, challenge an idea, elaborate on an idea, or give an example of their own process. Revoice student ideas to demonstrate mathematical language when necessary. This will help students produce and make sense of the language needed to communicate their own ideas.

Design Principle(s): Optimize output (for explanation)

Lesson Synthesis

- For what we have done today, what does it mean for two triangles to be “different?” (They are not identical copies.)
- If you have a drawing of two triangles, how can you tell if they are identical copies? (If I trace one triangle and can move the tracing to perfectly line up with the other, then they are identical copies.)
- When trying to draw different triangles with the same set of conditions, what are some things to try? (Change the order of the conditions in the triangle.)

16.4 Comparing Andre and Noah’s Triangles

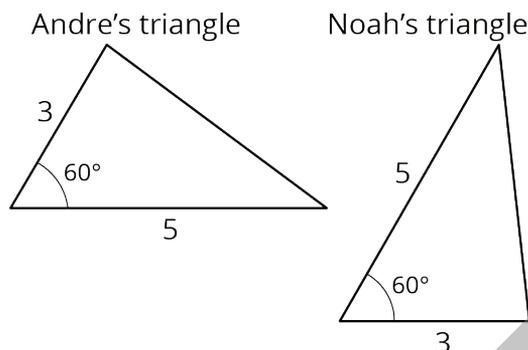
Cool Down: 5 minutes

Addressing

- 7.G.A.2

Student Task Statement

Andre and Noah each drew a triangle with side lengths of 5 cm and 3 cm and an angle that measures 60° , and then they showed each other their drawings.



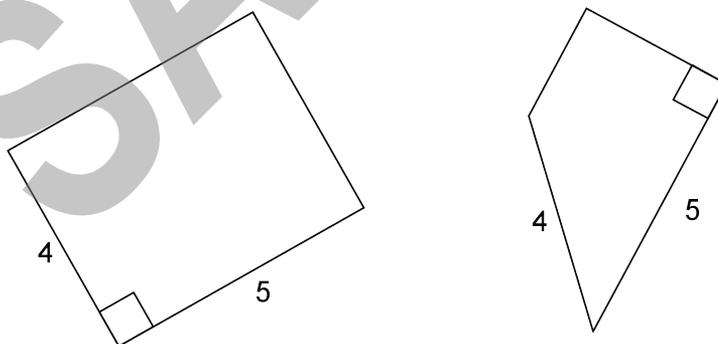
1. Did Andre and Noah draw different triangles? Explain your reasoning.
2. Explain what Andre and Noah would have to do to draw another triangle that is different from what either of them has already drawn.

Student Response

1. These are both the same triangle. In both cases, the 60° angle is between the 3 cm and 5 cm sides. If you trace one triangle, flip it and turn it, it can line up exactly with the other triangle.
2. To draw a different triangle, they should try putting the 60° angle next to the side of unknown length, instead of between the two known sides.

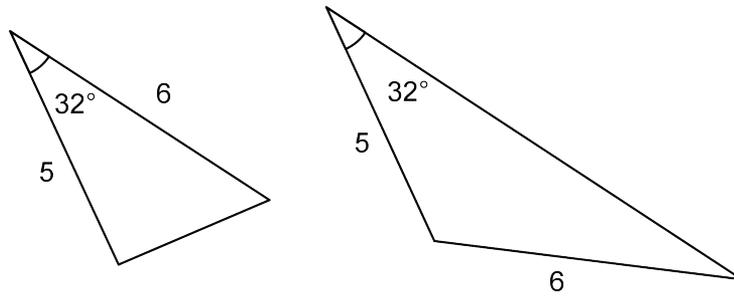
Student Lesson Summary

Both of these quadrilaterals have a right angle and side lengths 4 and 5:



However, in one case, the right angle is *between* the two given side lengths; in the other, it is not.

If we create two triangles with three equal measures, but these measures are not next to each other in the same order, that usually means the triangles are different. Here is an example:

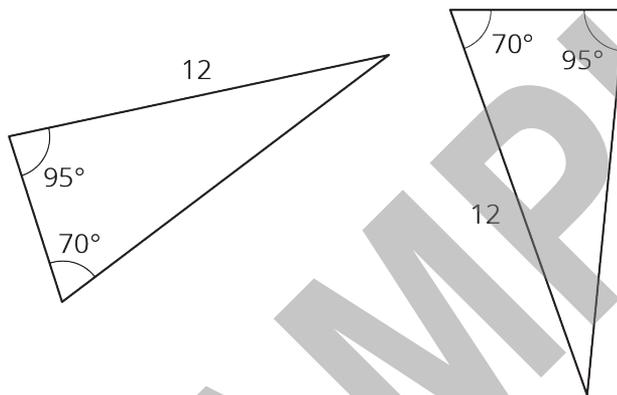


Lesson 16 Practice Problems

Problem 1

Statement

Are these two triangles identical? Explain how you know.



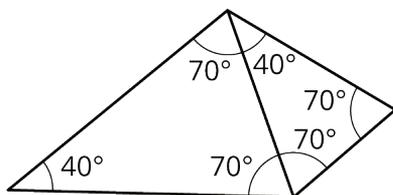
Solution

No, these two triangles are not identical. They have two of the same angle measures and one side length is the same, but the sides and angles are arranged differently in each triangle. In the triangle on the left, the side marked 12 is adjacent to the 95° angle. In the triangle on the right, the side marked 12 is adjacent to the 70° angle.

Problem 2

Statement

Are these triangles identical? Explain your reasoning.



Solution

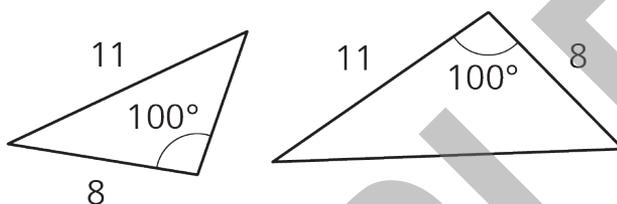
No, they are not identical. Although they have the same angle measurements, two of the side lengths are different.

Problem 3

Statement

Tyler claims that if two triangles each have a side length of 11 units and a side length of 8 units, and also an angle measuring 100° , they must be identical to each other. Do you agree? Explain your reasoning.

Solution



No, it is possible to build two different triangles with these measurements.

Problem 4

Statement

- Draw segment PQ .
- When PQ is rotated 180° around point R , the resulting segment is the same as PQ . Where could point R be located?

Solution

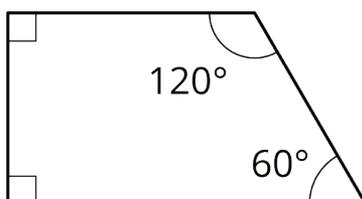
- Answers vary.
- R must be the midpoint of PQ .

(From Unit 1, Lesson 7.)

Problem 5

Statement

Here is trapezoid $ABCD$.





Using rigid transformations on the trapezoid, build a pattern. Describe some of the rigid transformations you used.

Solution

Answers vary. Sample response: clockwise rotations, centered at the vertex of the 60° angle, of 60° , 120° , 180° , 240° , and 300° make a “windmill” type figure with copies of the trapezoid.

(From Unit 1, Lesson 9.)

SAMPLE

Lesson 17: Drawing Triangles

Goals

- Draw triangles with two given angle measures and one side length, and describe (orally) how many different triangles could be drawn with the given conditions.
- Draw triangles with two given side lengths and one angle measure or three given angle measures, and describe (orally) how many different triangles could be drawn with the given conditions.
- Use drawings to justify (in writing) whether two given angle measures and one side length determine one unique triangle.

Learning Targets

- Given two angle measures and one side length, I can draw different triangles with these measurements or show that these measurements determine one unique triangle or no triangle.

Lesson Narrative

In the previous lesson, students were given collections of triangles and noticed that they shared angle and side measures, and that sometimes there was more than one type of triangle with the same measures. In this lesson, they build on that experience by drawing their own triangles with specified measures: a given angle, two given angles, and two given angles with a given side length (MP7). The purpose of this lesson is to give students experience using various tools to draw triangles with given conditions, and to help them see that sometimes the given conditions allow only one possible triangle, sometimes more than one, and sometimes none. Students are not expected to know rules about which conditions determine each possibility.

Alignments

Addressing

- 7.G.A.2: Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR5: Co-Craft Questions
- MLR7: Compare and Connect
- Think Pair Share

Required Materials

Compasses

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Required Preparation

To help students see how they can use a compass to draw different triangles with two of the same side lengths, you might choose to copy the How Many Can You Draw? blackline master for every student. This is optional.

Student Learning Goals

Let's see how many different triangles we can draw with certain measurements.

17.1 Using a Compass to Estimate Length

Warm Up: 5 minutes

The purpose of this warm-up is to remind students that a compass is useful for transferring a length in general, and not just for drawing circles. As students discuss answers with their partners, monitor for students who can clearly explain how they can use the compass to compare the length of the third side.

Addressing

- 7.G.A.2

Launch

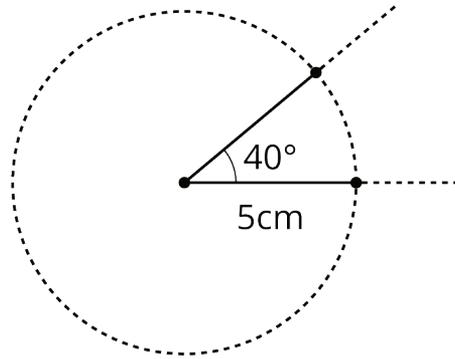
Arrange students in groups of 2. Give students 2 minutes of quiet work time followed by time to discuss their answers with their partner. Follow with a whole-class discussion. Provide access to geometry toolkits and compasses.

Student Task Statement

1. Draw a 40° angle.
2. Use a compass to make sure both sides of your angle have a length of 5 centimeters.
3. If you connect the ends of the sides you drew to make a triangle, is the third side longer or shorter than 5 centimeters? How can you use a compass to explain your answer?

Student Response

1. and 2.



2. The third side is shorter than 5 cm. I know this because I can use my compass that is set to a radius of 5 cm and place it at the end of one of the two sides. When I draw another circle, the end of the other side of the angle is inside, so the distance between the two ends must be less than 5 cm.

Activity Synthesis

Ask previously identified students to share their responses to the final question. Display their drawing of the angle for all to see. If not mentioned in students' explanations, demonstrate for all to see how to use the compass to estimate the length of the third side of the triangle.

17.2 How Many Can You Draw?

15 minutes (there is a digital version of this activity)

In this activity, students are asked to draw as many different triangles as they can with the given conditions. The purpose of this activity is to provide an opportunity for students to see the three main results for this unit: a situation in which only a unique triangle can be made, a situation in which it is impossible to create a triangle from the given conditions, and a situation in which multiple triangles can be created from the conditions.

Students are not expected to remember which conditions lead to which results, but should become more familiar with some methods for attempting to create different triangles. They will practice including various conditions into the triangles, including the conditions in different combinations, and recognizing when the resulting triangles are identical copies or not.

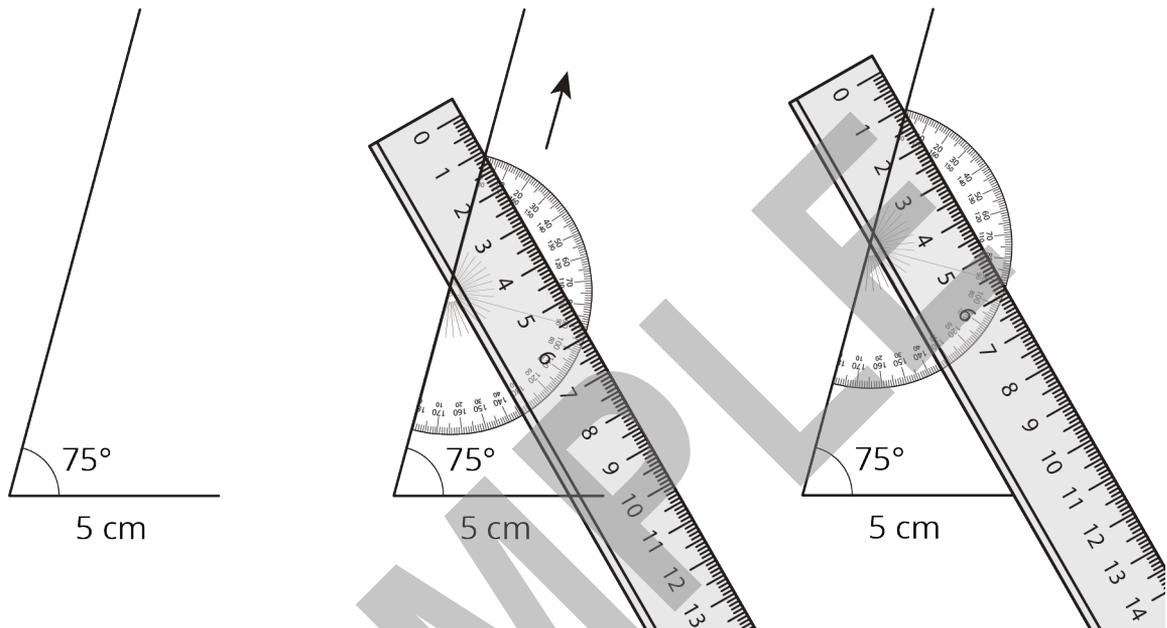
Teacher Notes for IM 6–8 Math Accelerated

The activity synthesis references a previous activity which is not included in IM 6–8 Math Accelerated. If time allows, demonstrate for students that one way to draw a triangle (Lin's method) with one angle measuring 60° , one angle measuring 90° , and one side measuring 4 cm is to:

1. Draw the 4 cm segment.
2. Draw the 60° angle on one end of the segment, with a very long ray.
3. Place a protractor along the ray.
4. Line up a ruler at the 90° measure on the protractor.

5. Keeping the ruler and protractor together, slide them along the ray until the edge of the ruler intersects with the other end of the 4 cm segment.
6. Keeping the ruler in place on the paper, remove the protractor from underneath.
7. Draw a line along the ruler from the ray to the segment.

Here is an image showing what this would look like for a triangle with one angle measuring 75° , one angle measuring 45° , and one side measuring 5 cm:



Addressing

- 7.G.A.2

Instructional Routines

- MLR7: Compare and Connect

Launch

Keep students in same groups. Tell students they must try at least two different times to draw a triangle with the measurements given in each problem. Give students 5 minutes of quiet work time followed by time to discuss their different triangles with a partner. Follow with a whole-class discussion. Provide access to geometry toolkits.

Support for Students with Disabilities

Action and Expression: Internalize Executive Functions. Provide students with a printed graphic organizer to categorize the different triangles by condition.

Supports accessibility for: Language; Organization

Anticipated Misconceptions

Some students may draw two different orientations of the same triangle for the third set of conditions, with the 4 cm side in between the 60° and 90° angles. Prompt them to use tracing paper to check whether their two triangles are really different (not identical copies).

Some students may say the third set of measurements determines one unique triangle, because they assume the side length must go between the two given angle measures. Remind them of the discussion about Lin's triangle in the previous activity.

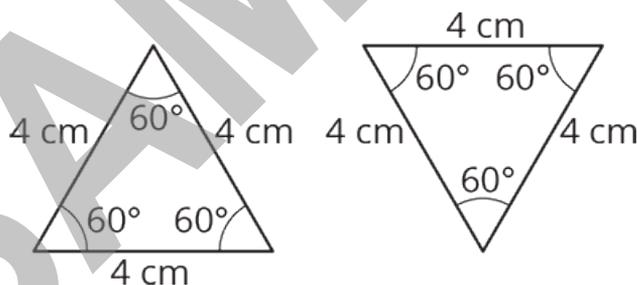
Student Task Statement

1. Draw as many different triangles as you can with each of these sets of measurements:
 - a. Two angles measure 60° , and one side measures 4 cm.
 - b. Two angles measure 90° , and one side measures 4 cm.
 - c. One angle measures 60° , one angle measures 90° , and one side measures 4 cm.
2. Which of these sets of measurements determine one unique triangle? Explain or show your reasoning.

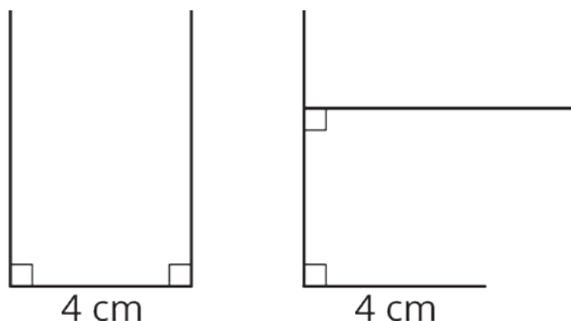
Student Response

1. Answers vary. Sample responses:

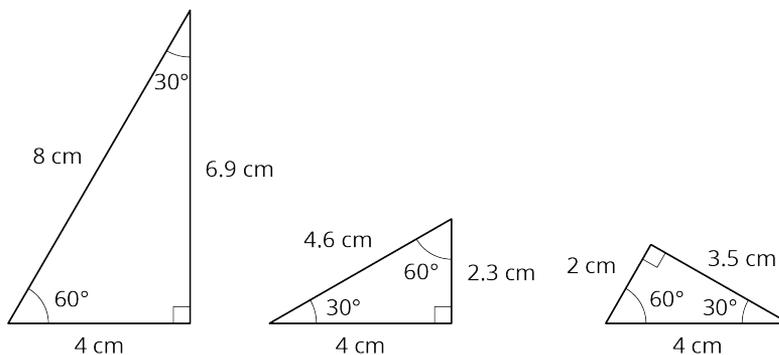
- a. Two orientations of the same triangle.



- b. Two attempts to draw a triangle with two 90° angles and a 4 cm side. There is no possible triangle with these conditions.

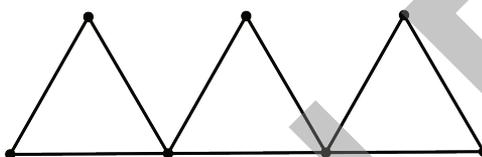


- c. Three different triangles can be made with the conditions.



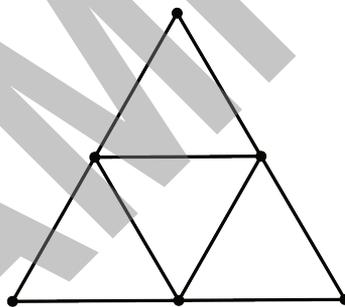
2. Only the first set of measurements determine a unique triangle.

Are You Ready for More?



In the diagram, 9 toothpicks are used to make three equilateral triangles. Figure out a way to move only 3 of the toothpicks so that the diagram has exactly 5 equilateral triangles.

Student Response



There are four small equilateral triangles and one large one.

Activity Synthesis

Ask students to indicate how many *different* triangles (triangles that are not identical copies) they could draw for each set of conditions. Select students to share their drawings and reasoning about the uniqueness of each problem. Discuss methods students used to try to think about other triangles that might fit the conditions.

Consider asking some of the following questions:

- “Which conditions produced a unique triangle?” (the first set of conditions)
- “Were there conditions that produced more than one triangle?” (the third set of conditions)
- “Were there conditions you could not draw a triangle for?” (the second set of conditions)

- “Why could you not draw a triangle for the second set of conditions?” (because two sides are parallel and will never intersect)

If not mentioned by students, explain to students that for the third set of conditions it is possible that all students drew identical copies using the 4 cm length as the side between the 60° and 90° angles. Consider asking them to think of the previous activity and to try to draw the triangle the way Lin would.

In grade 7, students do not need to know that the angles within a triangle sum to 180° . Tell them that next year they will learn more about why these different conditions determine different numbers of triangles.

Support for English Language Learners

Speaking: MLR7 Compare and Connect. Use this routine to compare and contrast the different ways students reasoned about the uniqueness of the constructed triangles. Ask students to consider what is the same and what is different about the triangles produced for each condition. Draw students’ attention to the association between the conditions given and the ability to construct unique, many, or no triangles. In this discussion, model the language used to make sense of the conditions that resulted in the three different scenarios. These exchanges strengthen students’ mathematical language use and supports them to compare geometric shapes.

Design Principle(s): Maximize meta-awareness; Support sense-making

17.3 Revisiting How Many Can You Draw?

15 minutes (there is a digital version of this activity)

Students continue to practice drawing triangles from given conditions and categorizing their results. This activity focuses on the inclusion of a single angle and two sides and then on three angle conditions. They do not need to memorize which conditions result in unique triangles, but should begin to notice how some conditions (such as the equal side lengths) result in certain requirements for the completed triangle.

Addressing

- 7.G.A.2

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR5: Co-Craft Questions
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to geometry toolkits and compasses. Remind students of the activity in a previous lesson where they used the strips and fasteners to draw triangles on their paper. Ask what other tool also helps you find all the points that are a certain distance from a center point (a compass). Tell students that they should attempt to create a triangle with the given specifications. If they can create one, they should attempt to either create at least one more triangle not congruent to the first or justify to themselves why there is only one. If they cannot create any, they should show some valid attempts to include as many pieces as they can and be ready to explain why they cannot include the remaining conditions.

Give students 7–8 minutes of partner work time, followed by a whole-class discussion.

If using the digital lesson, students should still try to create a triangle with the given specifications. If they can create one, they should attempt to either create at least one more or justify to themselves why there is only one. If they cannot create any, they should be ready to explain some of their attempts and why they cannot include the remaining conditions.

Support for English Language Learners

Reading, Speaking: MLR5 Co-craft Questions. Display just the statement: “One angle measures 40 degrees, one side measures 4 cm, and one side measures 5 cm.” Invite students to write down possible mathematical questions that could be asked with this information. Ask students to compare the questions generated with a partner before selecting 1–2 groups to share their questions with the class. Listen for the ways the given conditions are used or referenced in students’ questions. This routine will help students to understand the context of this problem before they are asked to create a drawing.

Design Principle(s): Maximize meta-awareness; Cultivate conversation

Anticipated Misconceptions

Some students may draw two different orientations of the same triangle for the first set of conditions, with the 40° angle in between the 4 cm and 5 cm sides. Prompt them to use tracing paper to check whether their two triangles are really different (not identical copies).

If students struggle to create more than one triangle from the first set of conditions, prompt them to write down the order they already used for their measurements and then brainstorm other possible orders they could use.

If students struggle to get started, remind them of the technique shared at the end of the previous activity of using the protractor and a ruler to make an angle that can move along a line.

Student Task Statement

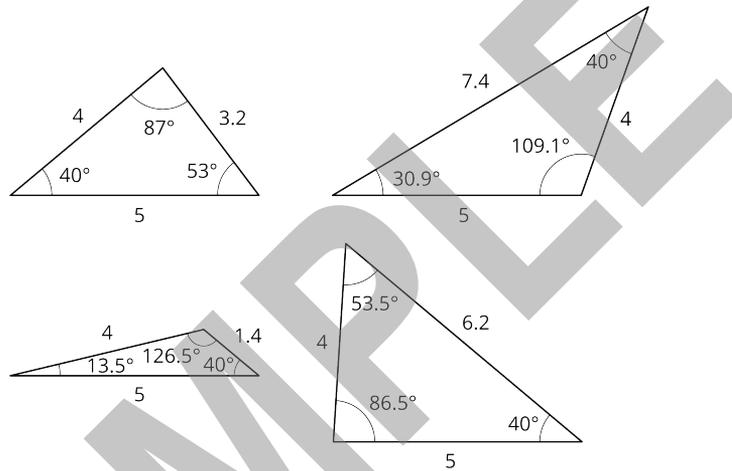
1. Draw as many different triangles as you can with this set of measurements.
 - a. One angle measures 40° , one side measures 4 cm, and one side measures 5 cm.

- b. Do these measurements determine one unique triangle? How do you know?
2. Draw as many different triangles as you can with each of these sets of angle measurements. Do either of these sets of measurements determine one unique triangle? Explain how do you know.
- One angle measures 50° , one measures 60° , and one measures 70° .
 - One angle measures 50° , one measures 60° , and one measures 100° .

Student Response

1. Drawings vary. Sample responses:

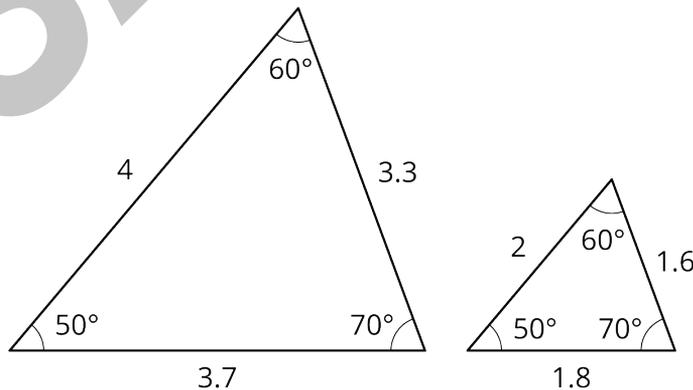
a. There are 4 different triangles that can be drawn from the given conditions.



b. No, I was able to draw several triangles with these measurements.

2. Sample response:

a. There are many possible triangles that can be made with these angle measurements, because we don't know any of the side lengths. All the possible triangles are larger or smaller versions of each other.



b. There is no way to draw a triangle with these angle measurements, because once the first two angles are drawn, the third angle is already set.

■ If I start with 50° and 60° , then the third angle is always 70° . I can't make it 100° .

- If I start with 50° and 100° , then the third angle is always 30° . I can't make it 60° .
- If I start with 60° and 100° , then the third angle is always 20° . I can't make it 50° .

Are You Ready for More?

Using *only* a compass and the edge of a blank index card, draw a perfectly equilateral triangle. (Note! The tools are part of the challenge! You may not use a protractor! You may not use a ruler!)

Student Response

Draw a line segment and label the endpoints A and B . Open compass so that the radius is the same length as the line segment you drew. Place pointy end of ruler on A , swing arc. Put pointy end on B , swing arc. Intersection of arcs is the other vertex of the equilateral triangle.

Activity Synthesis

Select students to share how many different triangles they were able to draw with each set of conditions.

If not brought up in student explanations, point out that for the first problem, one possible order for the measurements (40° , 5 cm, 4 cm) can result in two different triangles (the bottom two in the solution image). One way to show this is to draw a 5 cm segment and then use a compass to draw a circle with a 4 cm radius centered on the segment's left endpoint. Next, draw a ray at a 40° angle centered on the segment's right endpoint. Notice that this ray intersects the circle twice. Each one of these points could be the third vertex of the triangle. While it is helpful for students to notice this interesting aspect of their drawing, it is not important for students to learn rules about the number of possible triangles given different sets of conditions.

If students do not touch on the scaled triangles created by the same three angles, ask, "Why is there more than one triangle that can be made with the measurements in the first problem?" (Because there are no side lengths mentioned, so we can create smaller or larger copies of the triangles with the same angles but with shorter or longer side lengths.) Students begin their study of scaled copies in the next unit, so their work on this question is a sneak peak of the work ahead.

MLR 1 (Stronger and Clearer Each Time): Before discussing the second set of conditions as a whole class, have student pairs share their reasoning for why there were no triangles that could be drawn with the given measures, with two different partners in a row. Have students practice using mathematical language to be as clear as possible when sharing with the class, when and if they are called upon.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Break the class into small group discussion groups and then invite a representative from each group to report back to the whole class.

Supports accessibility for: Attention; Social-emotional skills

Lesson Synthesis

Here are some questions for discussion:

- “How was a compass useful in drawing triangles today?” (It helps find all the points a certain distance away.)
- “What strategies did you use to include two given side lengths and a given angle?” (Draw one of the side lengths and use a protractor to draw the angle at one end, then use a compass to finish the picture.)
- “What strategies did you use to include three given angles?” (Draw one angle then use a protractor and ruler to slide along one side of the first angle.)

17.4 Finishing Noah’s Triangle

Cool Down: 5 minutes

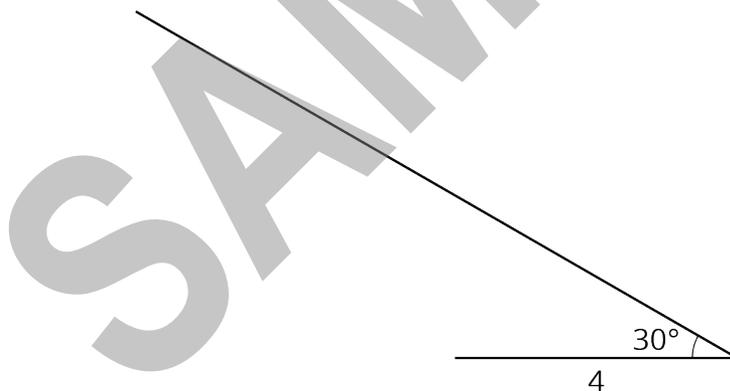
Addressing

- 7.G.A.2

Student Task Statement

Noah is trying to draw a triangle with a 30° angle and side lengths of 4 cm and 6 cm.

- He uses his ruler to draw a 4 cm line segment.
- He uses his protractor to draw a 30° angle on one end of the line segment.



1. What should Noah do next? Explain and show how he can finish drawing the triangle.
2. Is there a different triangle Noah could draw that would answer the question? Explain or show your reasoning.

Student Response

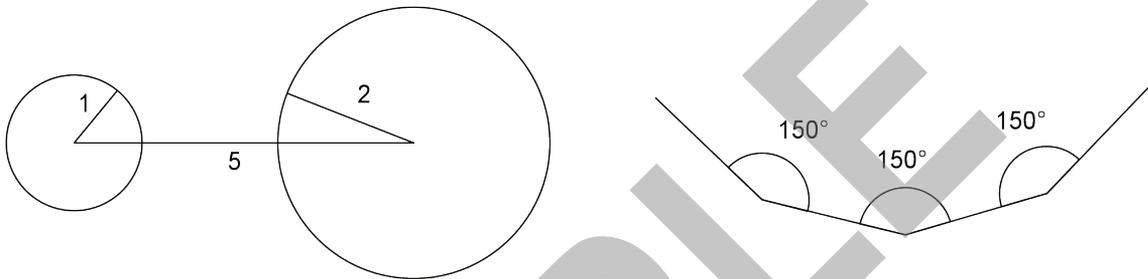
1. Noah should use a compass to draw a circle with radius 6 cm and center at one end of the 4 cm side. He should then draw segments connecting both ends of the 4 cm side to the point where the circle and ray cross to complete the triangle.

2. Yes. Noah could try beginning with the same setup he has already drawn again, but this time center the circle on the other end of the 4 cm side. He could also start with the 6 cm side drawn instead of the 4 cm side and follow the same process.

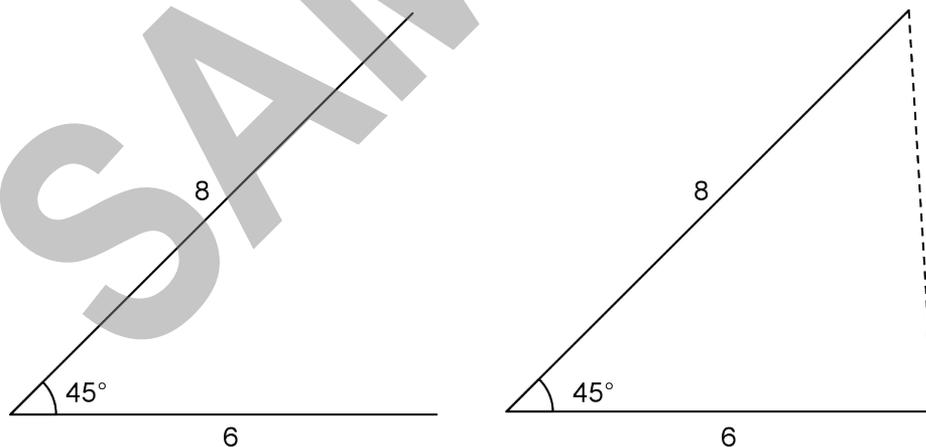
Student Lesson Summary

A triangle has six measures: three side lengths and three angle measures.

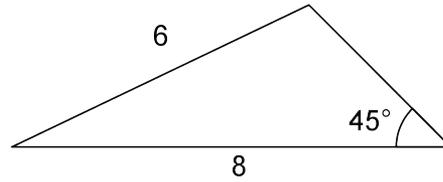
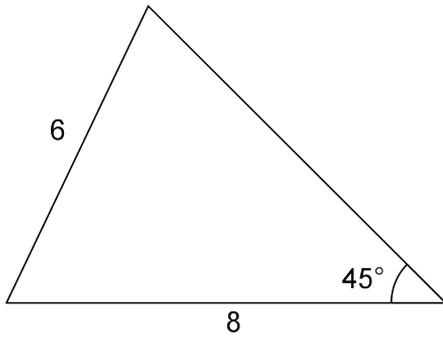
If we are given three measures, then sometimes, there is no triangle that can be made. For example, there is no triangle with side lengths 1, 2, 5, and there is no triangle with all three angles measuring 150° .



Sometimes, only one triangle can be made. By this we mean that any triangle we make will be the same, having the same six measures. For example, if a triangle can be made with three given side lengths, then the corresponding angles will have the same measures. Another example is shown here: an angle measuring 45° between two side lengths of 6 and 8 units. With this information, one unique triangle can be made.



Sometimes, two or more different triangles can be made with three given measures. For example, here are two different triangles that can be made with an angle measuring 45° and side lengths 6 and 8. Notice the angle is not between the given sides.



Three pieces of information about a triangle's side lengths and angle measures may determine no triangles, one unique triangle, or more than one triangle. It depends on the information.

Lesson 17 Practice Problems

Problem 1

Statement

Use a protractor to try to draw each triangle. Which of these three triangles is impossible to draw?

- A triangle where one angle measures 20° and another angle measures 45°
- A triangle where one angle measures 120° and another angle measures 50°
- A triangle where one angle measures 90° and another angle measures 100°

Solution

It is impossible to draw a triangle where one angle measures 90° and another angle measures 100° .

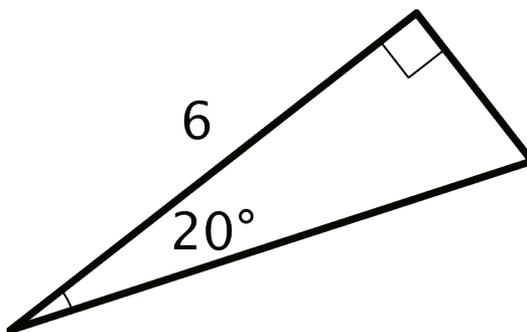
Problem 2

Statement

A triangle has an angle measuring 90° , an angle measuring 20° , and a side that is 6 units long. The 6-unit side is in between the 90° and 20° angles.

- Sketch this triangle and label your sketch with the given measures.
- How many unique triangles can you draw like this?

Solution



- a.
- b. There is only one triangle that fits this description, so long as the 6-unit side is between the two given angles.

Problem 3

Statement

A triangle has sides of length 7 cm, 4 cm, and 5 cm. How many unique triangles can be drawn that fit that description? Explain or show your reasoning.

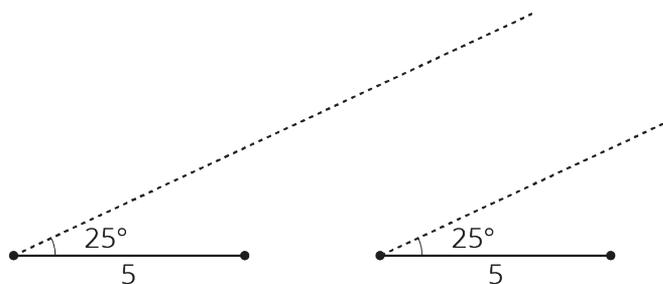
Solution

You can only draw one unique triangle with those same 3 measures. If you start by drawing the 7 cm side and then draw circles of radii 4 cm and 5 cm at each endpoint, the circles will cross at two places. Connecting the endpoints of the 7 cm side to those crossing points will produce two identical triangles, each having side lengths 7 cm, 4 cm, and 5 cm. There are no other points that could be the third vertex of the triangle.

Problem 4

Statement

A triangle has one side that is 5 units long and an adjacent angle that measures 25° . The two other angles in the triangle measure 90° and 65° . Complete the two diagrams to create two *different* triangles with these measurements.



Solution

Answers vary.

Problem 5

Statement

Is it possible to make a triangle that has angles measuring 90 degrees, 30 degrees, and 100 degrees? If so, draw an example. If not, explain your reasoning.

Solution

No, if you try to draw a triangle that has a 90 degree angle on the end of a side and a 100 degree angle on the other end of the same side, there is no way to make the other two sides meet to form a triangle.

SAMPLE

Section: Let's Put It to Work

Lesson 18: Rotate and Tessellate

Goals

- Create tessellations and designs with rotational symmetry using rigid transformations.
- Explain (orally and in writing) the rigid transformations needed to move a tessellation or design with rotational symmetry onto itself.

Learning Targets

- I can repeatedly use rigid transformations to make interesting repeating patterns of figures.
- I can use properties of angle sums to reason about how figures will fit together.

Lesson Narrative

In this unit, students have learned how to name different types of rigid motions of the plane and have studied how to move different figures (lines, line segments, polygons, and more complex shapes). They have also used rigid motions to define what it means for figures to be congruent and have used rigid motions to investigate the sum of the angles in a triangle. In this lesson, students use the language of transformations to produce, describe, and investigate patterns in the plane. This is a direct extension of earlier work with triangles

- three triangles were arranged in the plane to show that the sum of the angles in a triangle is 180 degrees
- four copies of a triangle were arranged in a large square, cutting out a smaller square in the middle

Here the focus is more creative. Students will examine and create different patterns of shapes, including *tessellations* (patterns that fill the entire plane), and complex designs that exhibit rotational symmetry (that is, the design is congruent to itself by several rotations). Depending on the time available, students might work on both activities or choose one of the two.

As with many activities in this lesson, MP7 is central as students use the structure of a given set of polygons to produce a tessellation. The side lengths and angles of the polygons are constraints and through experimenting and abstract reasoning students discover a repeating pattern (MP2).

Teacher Notes for IM 6–8 Math Accelerated

This lesson is optional. Use after Lesson 17.

Alignments

Building On

- 4.MD.C: Geometric measurement: understand concepts of angle and measure angles.

- 7.G.B.5: Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

Addressing

- 8.G.A: Understand congruence and similarity using physical models, transparencies, or geometry software.

Instructional Routines

- Group Presentations
- MLR8: Discussion Supports

Required Materials

Blank paper

Copies of blackline master

Geometry toolkits

For grade 6: tracing paper, graph paper, colored pencils, scissors, and an index card to use as a straightedge or to mark right angles.

For grades 7 and 8: everything in grade 6, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Notes: (1) "Tracing paper" is easiest to use when it's a smaller size. Commercially-available "patty paper" is 5 inches by 5 inches and ideal for this. If using larger sheets of tracing paper, consider cutting them down for student use. (2) When compasses are required in grades 6-8 they are listed as a separate Required Material.

Graph paper

Isometric graph paper

Required Preparation

Print the Deducing Angle Measures blackline master. Prepare 1 copy for every 2 students. Cut the copies in half, so that there are enough copies for each student to receive a half-sheet. If possible, make these copies on cardstock so that students will have an easier time tracing shapes after they cut them out. If available, pattern blocks also work well for this.

Students may benefit from using graph paper and isometric graph paper, but these materials are optional.

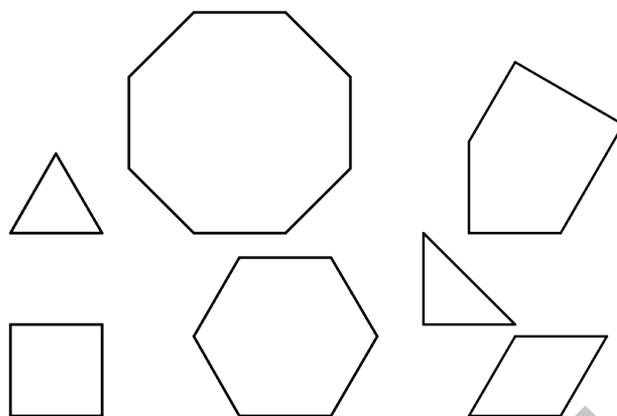
Student Learning Goals

Let's make complex patterns using transformations.

18.1 Deducing Angle Measures

Warm Up: 10 minutes

Throughout this lesson, students build different patterns with copies of some polygons. In this activity, they make some copies of each polygon and arrange them in a circle. They calculate some of the angles of the polygons while also gaining an intuition for how the polygons fit together. Here are the figures included in the blackline master:



Students might use a protractor to measure angles, but the measures of all angles can also be deduced. In the first question in the task, students are instructed to fit copies of an equilateral triangle around a single vertex. Six copies fit, leading them to deduce that each angle measures 60° because $360 \div 6 = 60$. For the other shapes, they can reason about angles that sum to 360° , angles that sum to a line, and angles that sum to a known angle.

Building On

- 4.MD.C
- 7.G.B.5

Launch

Provide access to geometry toolkits. Distribute one half-sheet (that contains 7 shapes) to each student. It may be desirable to demonstrate how to use tracing paper to position and trace copies of the triangle around a single vertex, as described in the first question.

Anticipated Misconceptions

When deducing angle measures, it is important to know that angles "all the way around" a vertex sum to 360° . It is also important to know that angles that make a line when adjacent sum to 180° . Monitor for students who need to be reminded of these facts.

Student Task Statement

Your teacher will give you some shapes.

1. How many copies of the equilateral triangle can you fit together around a single vertex, so that the triangles' edges have no gaps or overlaps? What is the measure of each angle in these triangles?
2. What are the measures of the angles in the
 - a. square?
 - b. hexagon?
 - c. parallelogram?

- d. right triangle?
- e. octagon?
- f. pentagon?

Student Response

1. 6, 60°
2. Measures of angles:
 - a. Square: 90°
 - b. Hexagon: 120°
 - c. Parallelogram: 120° and 60°
 - d. Right triangle: 45° and 90°
 - e. Octagon: 135°
 - f. Pentagon: 90° , 120° , and 150°

Activity Synthesis

For the remainder of the lesson, it is not so important that the degree measures of the angles are known, so don't dwell on the answers. Select a few students who deduced angles' measures by fitting pieces together to present their work. Make sure students see lots of examples of shapes fitting together like puzzle pieces.

Recall from the previous lesson that the 3 congruent angles in an equilateral triangle make a line or 180-degree angle, so it makes sense that 6 copies of this angle make a full circle.

18.2 Tessellate This

35 minutes

Each classroom activity in this lesson (this one, creating a tessellation, and the next one, creating a design with rotational symmetry) could easily take an entire class period or more. Consider letting students choose to pursue one of the two activities.

A tessellation of the plane is a regular repeating pattern of one or more shapes that covers the entire plane. Some of the most familiar examples of tessellations are seen in bathroom and kitchen tiles. Tiles (for flooring, ceiling, bathrooms, kitchens) are often composed of copies of the same shapes because they need to fit together and extend in a regular pattern to cover a large surface.

Addressing

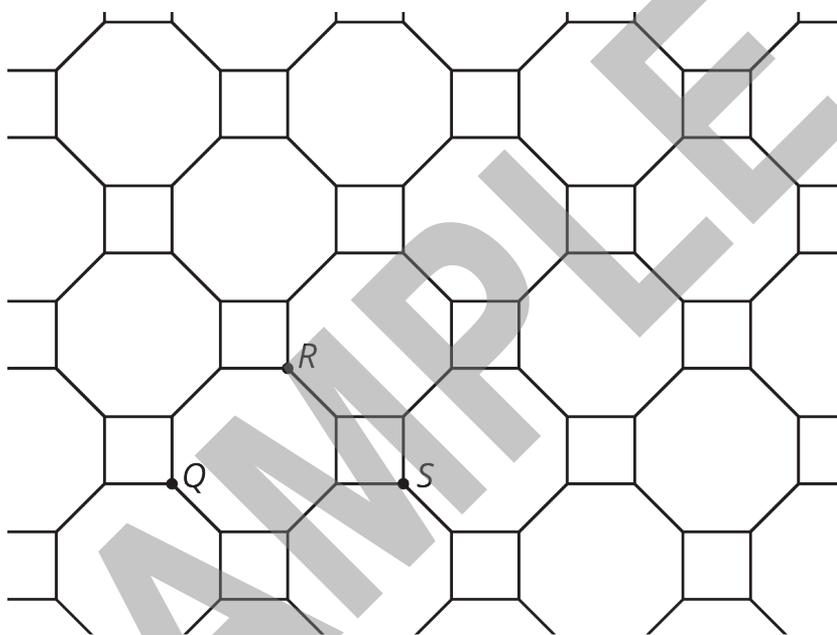
- 8.G.A

Instructional Routines

- Group Presentations

Launch

Share with students a definition of tessellation, like, “A tessellation of the plane is a regular repeating pattern of one or more shapes that covers the entire plane.” Consider showing several examples of tessellations. A true tessellation covers the entire plane: While this is impossible to show, we can identify a pattern that keeps going forever in all directions. This is important when we think about tessellations and symmetry. One definition of symmetry is, “You can pick it up and put it down a different way and it looks exactly the same.” In a tessellation, you can perform a translation and the image looks exactly the same. In the example of this tiling, the translation that takes point Q to point R results in a figure that looks exactly the same as the one you started with. So does the translation that takes S to Q . Describing one of these translations shows that this figure has translational symmetry.



Provide access to geometry toolkits. Suggest to students that if they cut out a shape, it is easy to make many copies of the shape by tracing it. Encourage students to use the shapes from the previous activity (or pattern blocks if available) and experiment putting them together. They do not need to use all of the shapes, so if students are struggling, suggest that they try using copies of a couple of the simpler shapes.

Support for Students with Disabilities

Representation: Internalize Comprehension. Provide a range of examples and counterexamples of shapes to use in creating a tessellation. For example, show 1–2 shapes that do not quite fit together that create gaps or overlaps. Then show 1–2 shapes that correctly create a tessellation. Consider providing step-by-step directions for students to create a shape that will make a repeating pattern.

Supports accessibility for: Conceptual processing

Anticipated Misconceptions

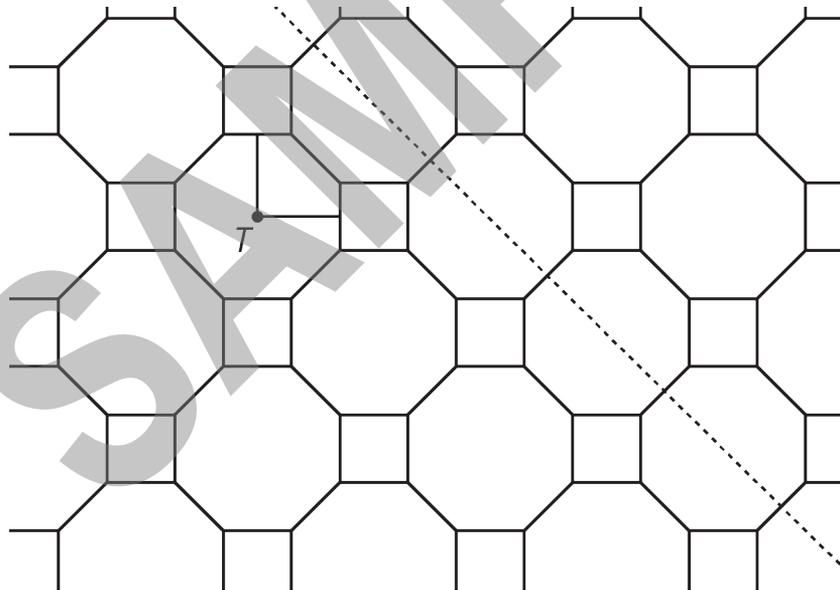
Watch out for students who choose shapes that almost-but-don't-quite fit together. Reiterate that the pattern has to keep going forever—often small gaps or overlaps become more obvious when you try to continue the pattern.

Student Task Statement

1. Design your own **tessellation**. You will need to decide which shapes you want to use and make copies. Remember that a tessellation is a repeating pattern that goes on forever to fill up the entire plane.
2. Find a partner and trade pictures. Describe a transformation of your partner's picture that takes the pattern to itself. How many different transformations can you find that take the pattern to itself? Consider translations, reflections, and rotations.
3. If there's time, color and decorate your tessellation.

Student Response

1. Answers vary.
2. Answers vary. For example, in the tessellation given previously, we could reflect across the dashed line, or rotate 90 degrees clockwise around the point marked T .



Activity Synthesis

Invite students to share their designs and also describe a transformation that takes the design to itself. Consider decorating your room with their finished products.

18.3 Rotate That

35 minutes

Each classroom activity in this lesson (the previous one, creating a tessellation, and this one, creating a design with rotational symmetry) could easily take an entire class period or more. Consider letting students choose to pursue one of the two activities.

In this activity, using their geometry toolkits, students can make their own design that has rotational symmetry. They then share designs and find the different rotations (and possibly reflections) that make the shape match up with itself.

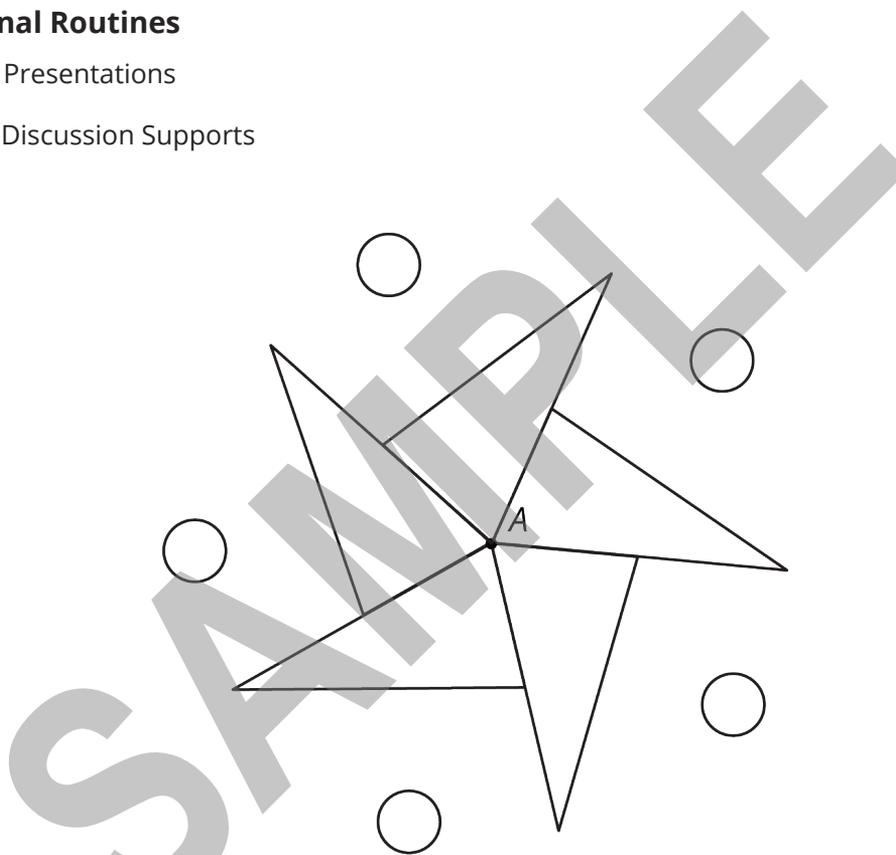
Addressing

- 8.G.A

Instructional Routines

- Group Presentations
- MLR8: Discussion Supports

Launch



Ask students what transformation they could perform on the figure so that it matches up with its original position. There are a number of rotations using A as the center that would work: 72° or any multiple of 72° . Make sure students understand that the 5 triangles in this pattern are congruent and that $5 \cdot 72 = 360$: This is why multiples of 72° with center A match this figure up with itself. They need to be careful in selecting angles for the shapes in their pattern. If they struggle, consider asking them to use pattern tiles or copies of the shapes from the previous activity to help build a pattern.

If possible, show students several examples of figures that have rotational symmetry.

Provide access to geometry toolkits. If possible, provide access to square graph paper or isometric graph paper.

Support for Students with Disabilities

Action and Expression: Provide Access for Physical Action. Provide students with access to square graph paper, isometric graph paper, and/or pattern blocks or shape cut-outs for making a design with rotational symmetry.

Supports accessibility for: Visual-spatial processing; Organization

Anticipated Misconceptions

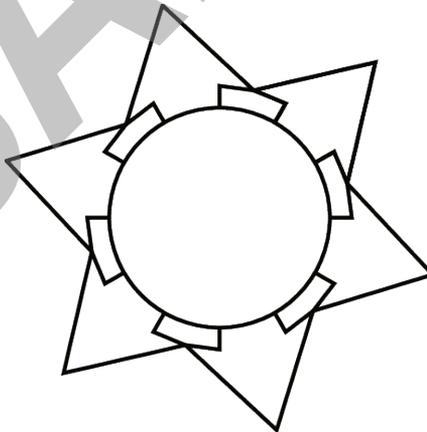
Before now, students may think that reflection symmetry is the only kind of symmetry. Because of this, they may create a design that has reflection symmetry but not rotational symmetry. Steer students in the right direction by asking them to perform a rotation that takes the figure to itself. Acknowledge that reflection symmetry is a type of symmetry, but the task here is to create a design with rotational symmetry.

Student Task Statement

1. Make a design with rotational symmetry.
2. Find a partner who has also made a design. Exchange designs and find a transformation of your partner's design that takes it to itself. Consider rotations, reflections, and translations.
3. If there's time, color and decorate your design.

Student Response

Answers vary. An example shape is below.



Activity Synthesis

Invite students to share their designs and also describe a transformation that takes the design to itself. Consider decorating your room with their finished products.

Support for English Language Learners

Speaking, Listening: MLR8 Discussion Supports. Use this routine to support whole-class discussion. For each design and transformation that is shared, ask students to restate what they heard using precise mathematical language. Consider providing students time to restate what they heard to a partner before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This provides more students with an opportunity to produce language as they interpret the reasoning of others.

Design Principle(s): Support sense-making

Glossary

- tessellation

SAMPLE

SAMPLE



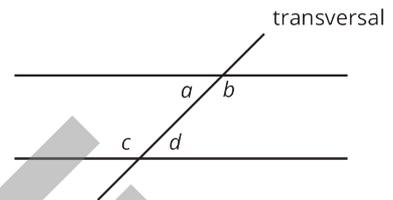
Glossary

alternate interior angles

Interior angles are angles that are made by a transversal crossing two parallel lines. They are the angles that lie between the parallel lines, not outside them.

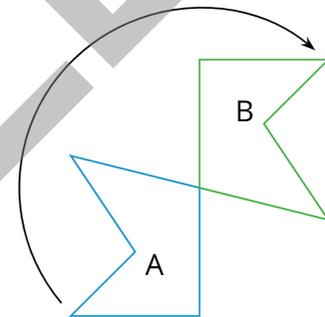
If two interior angles lie on opposite sides of the transversal they are called alternate interior angles.

In the figure, a and d are alternate interior angles, and b and c are also alternate interior angles.



clockwise

Clockwise means to turn in the same direction as the hands of a clock. The top turns to the right. This diagram shows Figure A turned clockwise to make Figure B.



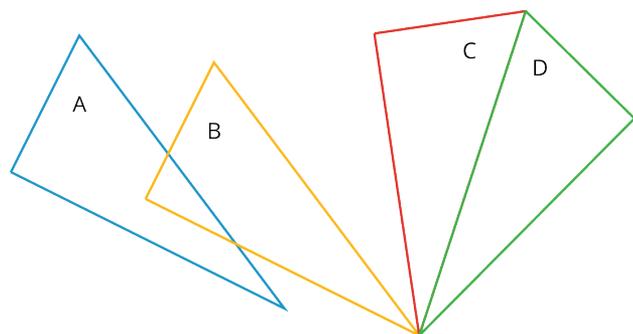
complementary

Two angles are complementary to each other if their measures add up to 90° . The two acute angles in a right triangle are complementary to each other.

congruent

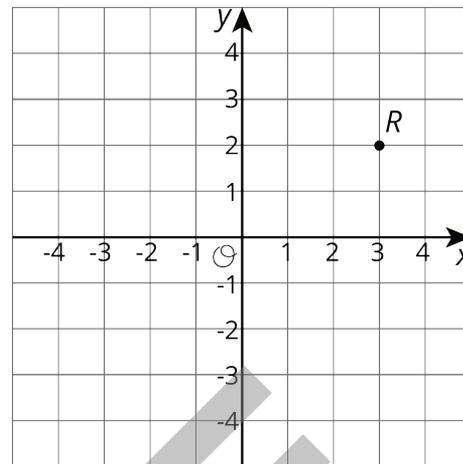
One figure is congruent to another if it can be moved with translations, rotations, and reflections to fit exactly over the other.

In the figure, Triangle A is congruent to Triangles B, C, and D. A translation takes Triangle A to Triangle B, a rotation takes Triangle B to Triangle C, and a reflection takes Triangle C to Triangle D.



coordinate plane

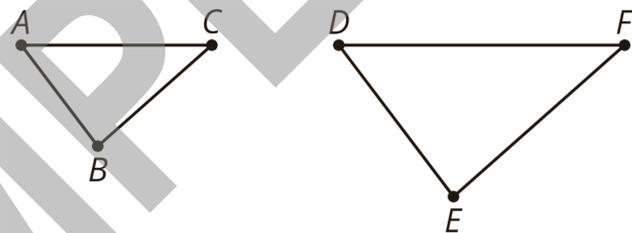
The coordinate plane is a system for telling where points are. For example, point R is located at $(3, 2)$ on the coordinate plane, because it is three units to the right and two units up.



corresponding

When part of an original figure matches up with part of a copy, we call them corresponding parts. These could be points, segments, angles, or distances.

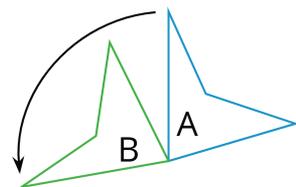
For example, point B in the first triangle corresponds to point E in the second triangle. Segment AC corresponds to segment DF .



counterclockwise

Counterclockwise means to turn opposite of the way the hands of a clock turn. The top turns to the left.

This diagram shows Figure A turned counterclockwise to make Figure B.

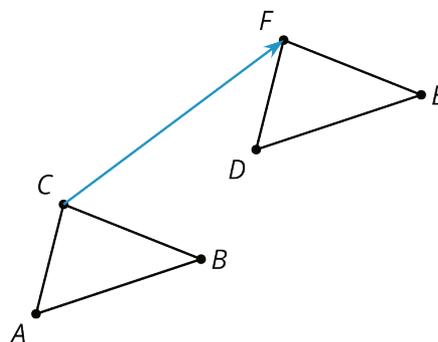


image

Translations, rotations, and reflections move objects in the plane. Points, segments, and other parts of the original all have corresponding parts on the "moved object." The moved object is called the image.

For example, here is triangle ABC and a translation to the right and up which is labeled DEF .

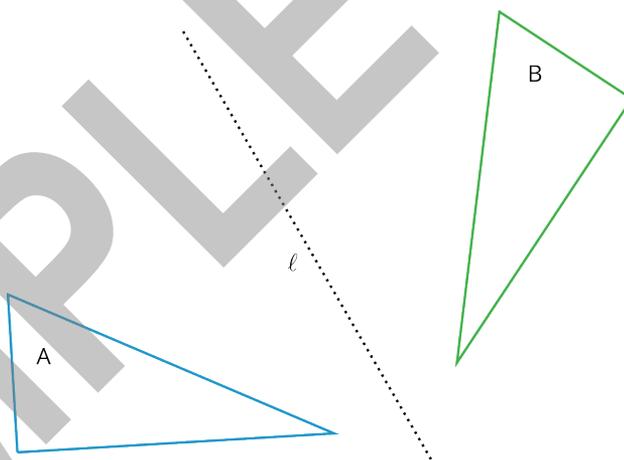
Point F in the image corresponds to point C , segment EF in the image corresponds to segment BD , and angle DEF corresponds to angle ABC .



reflection

A reflection across a line moves every point on a figure to a point directly on the opposite side of the line. The new point is the same distance from the line as it was in the original figure.

This diagram shows a reflection of A over line ℓ that makes the mirror image B .



right angle

When you divide a straight angle into two angles with equal measures, each of the two angles is a right angle. For example, the four corners of a square are right angles.

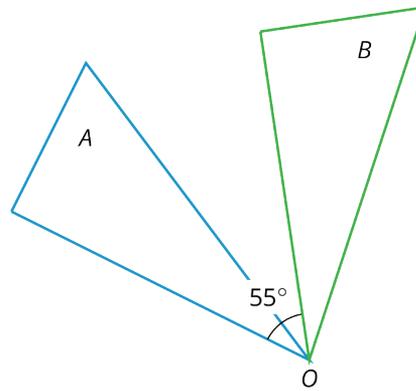
rigid transformation

A rigid transformation is a move that does not change any measurements of a figure. Translations, rotations, and reflections are rigid transformations, as is any sequence of these.

rotation

A rotation moves every point on a figure around a center by a given angle in a specific direction.

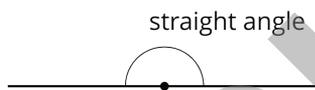
This diagram shows Triangle A rotated around center O by 55 degrees clockwise to get Triangle B.



sequence of transformations

straight angle

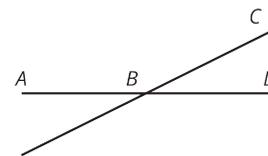
A straight angle is an angle that forms a straight line. It measures 180 degrees.



supplementary

Two angles are supplementary to each other if their measures add up to 180° .

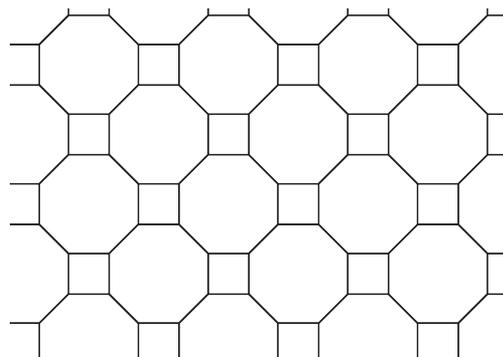
For example, angle ABC is supplementary to angle CBD , because they add up to a straight angle, which has measure 180° .



tessellation

A tessellation is a repeating pattern of one or more shapes. The sides of the shapes fit together perfectly and do not overlap. The pattern goes on forever in all directions.

This diagram shows part of a tessellation.



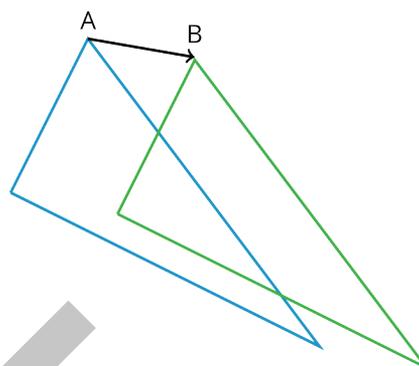
transformation

A transformation is a translation, rotation, reflection, or dilation, or combination of these. There is also a more general concept of a transformation of the plane that is not discussed in grade 8.

translation

A translation moves every point in a figure a given distance in a given direction.

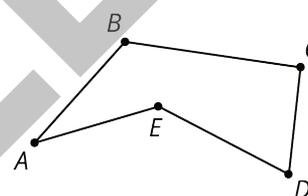
This diagram shows a translation of Figure A to Figure B using the direction and distance given by the arrow.



transversal vertex

A vertex is a point where two or more edges meet. When we have more than one vertex, we call them vertices.

The vertices in this polygon are labeled A , B , C , D , and E .

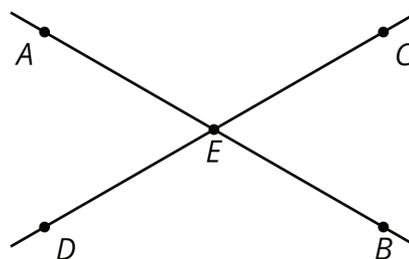


vertical angles

Vertical angles are opposite angles that share the same vertex. They are formed by a pair of intersecting lines. Their angle measures are equal.

For example, angles AEC and DEB are vertical angles. If angle AEC measures 120° , then angle DEB must also measure 120° .

Angles AED and BEC are another pair of vertical angles.



Notes

SAMPLE

Notes

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